Algebra I : Rings of Fractions.

R comm. ring, $0 \neq D \subseteq R$

s.t. D doesn't contain 0 or any 0-divisors and closed under multiplication

Thm $\exists$ comm. ring w/ 1 Q s.t. ~~$Q \supseteq R$~~

- $R \subseteq Q$
- $D \subseteq Q^\times$ (elts of D are units in Q)
- every elt of Q is rd^{-1} for $r \in R, d \in D$.

  If $D = R \setminus \{0\}$, then Q is a field

- Q is the smallest ring where elts of D become units

universal property: S comm. ring w/ 1

If ~~φ~~ $\varphi : R \hookrightarrow S$ injective

~~$\varphi(d)$ a~~ $\varphi(D) \subseteq S^\times$

Then $\exists$ inj. $\Phi : Q \to S$, $\Phi|_R = \varphi$:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & \nearrow_{\Phi} & \\ Q & & \end{array}$$

That is, any ring cont. a copy of R, elts of D units contains a copy of Q.

Pf: $\tilde{F} = \{(r, d) \mid r \in R, d \in D\}$.

rel^n on $\tilde{F}$: $(r,d) \sim (s,e) \Leftrightarrow re = sd$.

Clearly reflexive, symmetric.

Transitivity: $(r,d) \sim (s,e) \Rightarrow re - sd = 0 \Rightarrow ref - sdf = 0$

$(s,e) \sim (t,f) \Rightarrow sf - te = 0 \Rightarrow \underset{\text{add}}{\underline{sfd - ted = 0}}$

$\Rightarrow ref - ted = 0 \Rightarrow rf = td$

(e not 0 or 0-divisor) $\Rightarrow (r,d) \sim (t,f)$.

Equiv. class of (r, δ) denoted $\frac{r}{\delta}$.

$\mathbb{Q}$ as a set: set of equiv. classes $\frac{r}{\delta}$.

Since D is closed under multiplication,

$\frac{r}{\delta} = \frac{re}{\delta e} \Leftrightarrow (r,\delta) \sim (re, \delta e) \Leftrightarrow r\delta e = r\delta e$

Ring structure! $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ Pause

need to Check: well-defined

$(\mathbb{Q}, +)$ is an abelian gp, identity $= \frac{0}{d}$, and inverse of $\frac{a}{\delta}$ is $\frac{-a}{\delta}$.

$\cdot$ is assoc, dist, and commutative

$\mathbb{Q}$ has 1 ($\frac{d}{d}$ for any $d \in D$)

All easy. All elts of D have mult. inverses: $d = \frac{de}{e}$ $\Rightarrow \frac{1}{d} = \frac{e}{ed}$.

Also skip uniqueness / universal prop.

Def: Q is called the ring of fractions of D w.r.t. R, denoted $D^{-1}R$.

- If R is a domain and $D = R \setminus \{0\}$, Q is the field of fractions / quotient field of R.

Example:

- R a field $\Rightarrow$ field of fractions $= R$.
- $FF(\mathbb{Z}) = \mathbb{Q}$

  $FF(\mathcal{O}_D) = \mathbb{Q}(\sqrt{D})$

- $FF(2\mathbb{Z}) = \mathbb{Q}$ (a unit arises anyway)
- $FF(\mathbb{Z}[x]) = \mathbb{Q}(x)$ ~~rat~~ ratl fn $/\mathbb{Q}$.

R (domain) $FF(R[x]) = FF(R)(x)$. ratl fn $/ FF(R)$.