Algebra 1 Lecture 6: Cayley's Thm & Burnside's Lemma.

We saw:

Any gp. acts on itself by left mult.

$$g \cdot a := g\,a \quad \forall g, a \in G.$$

If $|G| = n$, we can label the elts of G as $1, \dots, n$.

i.e. $G = \{g_1, \dots, g_n\}$,

$\forall g \in G$, $\sigma_g \in S_n$

$$\sigma_g(i) = j \iff g\,g_i = g_j.$$

Ex: $G = \{1, a, b, c\}$ Klein Vierergruppe $(\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$.

(squared anything is 1, prod. of two distinct non-ident. = 3rd non-ident.)

compute permutation

σ_a: $\sigma_a(1) = 2$ as $a \cdot 1 = a1 = a$
$\sigma_a(2) = 1$ as $a \cdot a = aa = 1$
$\sigma_a(3) = 4$ as $a \cdot b = ab = c$
$\sigma_a(4) = 3$ as $a \cdot c = ac = b$.

$\rightsquigarrow \sigma_a = (12)(34) \in S_4$.

Permutation repn:

$1 \mapsto id.$
$a \mapsto \sigma_a = (12)(34)$
$b \mapsto \sigma_b = (13)(24)$
$c \mapsto \sigma_c = (14)(23)$.

$\rightsquigarrow$ gives $G \to S_4$.

More general action: $H \leq G$. $A = G/H$ set of left cosets.

Action: $g \cdot aH = gaH$.

$H = \{e\}$ recovers the above action.

If $[G:H]=m$, label the left cosets of H by $1,\dots,m$.

So list left cosets by $a_1H,\dots,a_mH$.

σ_g : determined by

$$\sigma_g(i)=j \iff g a_i H = a_j H.$$

Ex: $G=D_8$, $H=\langle s\rangle$

Left cosets: $1H, rH, r^2H, r^3H$.

label: 1 2 3 4

Find σ_s:

$sH=H \Rightarrow 1\mapsto 1$

$srH=r^3H \Rightarrow 2\mapsto 4$

$sr^2H=r^2H \Rightarrow 3\mapsto 3$

$sr^3H=rH \Rightarrow 4\to 2$

$\Rightarrow \sigma_s=(24)$.

[e.g.: $srH = r^{-1}sH = r^{-1}H = r^3H$]

Thm: G a gp. $H\le G$. G acts on $\{\text{left cosets of } H\} = A$.

1). G acts transitively

2). $\mathrm{Stab}(H)=H$.

3). the kernel is $\bigcap_{x\in G} xHx^{-1}$, the largest normal subgp of G contained in H.

Pf: 1). Let $aH, bH\in A$.

Then $(ba^{-1})\cdot(aH)=bH$.

2). $\mathrm{Stab}(H)=\{g\in G \mid gH=H\}=H$. today

3). kernel of action $= \ker(\pi_H) = \{g\in G \mid gxH = xH \ \forall x\in G\}$

$\boxed{\pi_H: G\to S_A \text{ perm. rep}}$

$= \{g\in G \mid (x^{-1}gx)\in H \ \forall x\in G\}$

$= \{g\in G \mid g\in xHx^{-1} \ \forall x\in G\}$

(Q) $= \bigcap xHx^{-1}$

sr^3sr

Finally, $\ker \pi_H \trianglelefteq G$ (it is a kernel...)

and $\ker \pi_H \leq H$.

Thus, if $\underset{H}{\overset{\cap}{N}} \trianglelefteq G$, then $N = xNx^{-1} \leq xHx^{-1}$

$\Rightarrow N \leq \ker \pi_H$ ∎

Corollary: (Cayley's Thm)

Every gp. is $\cong$ to a subgp. of a symmetric gp.

If $|G| = n$, we can take $G \cong$ subgp of S_n.

Pf: If $H = \{e\}$, (G acts on itself by the comment above)

then we get a hom.

$\pi : G \to S_G$.

$\ker(\pi) \leq H = \{1\}$

So, by the 1st iso theorem

$G \xrightarrow{\sim} \pi(G) \leq S_G$.

This is very interesting for historical reasons

(this is how people first studied gps in S_n).

But it's not so useful in practice, as $n!$ is huge.

Cor: If $|G| = n$, ~~p|G|~~ ~~is~~ p is the smallest prime divisor of a

any index p subgp. is normal.

Pf: $\mathcal{S}$ $H \leq G$, $[G:H] = p$. π_H = perm. rep^{n} above. $K := \ker(\pi_H)$.

~~k~~ $:= [H:K] = k$. Then $[G:K] = [G:H] \cdot [H:K] = pk$. Above thm $\Rightarrow G/K \hookrightarrow S_p$

Lagrange $\Rightarrow$ $pk = |G/K| \mid p!$

$\Rightarrow k \mid \frac{p!}{p} = (p-1)!$ All primes dividing $(p-1)!$ are $< p$,
and by the minimality assumption on p,
every prime divisor of k is $\geq p$. But then $k=1$,
$\Rightarrow H = K \trianglelefteq G$ $\blacksquare$