Algebra 1 Lecture 5: Gp Actions

Recall: An action is: $G \times A \to A$

1). $g'(g \cdot a) = (g'g) \cdot a \quad \forall g', g, a$

2). $e \cdot a = a . \quad \forall a.$

- Going to be gone. (Extra OH)
- If not on Piazza, ask me!

Observation:

$\exists$ natural bijection

$$\{\text{Actions of } G \text{ on } A\} \longleftrightarrow \{\text{hom. } G \to S_A\},$$

action $\longrightarrow$ permutation repⁿ

This is like repⁿˢ of groups into GL_n.

Corr: Given an action,

$\forall g \in G$, Set $\sigma_g : A \to A$,
$a \mapsto g \cdot a.$

Claim: σ_g is a perm. (ie, $\sigma_g \in S_A$.)

and the map $G \to S_A$ is a hom.
$g \mapsto \sigma_g$

Pf: $(\sigma_{g^{-1}} \circ \sigma_g)(a) = \sigma_{g^{-1}}(\sigma_g(a)) = g^{-1} \cdot (g \cdot a)),$

$\underset{(1)}{=} (g^{-1}g) \cdot a = e \cdot a \underset{(2)}{=} a.$

$\sigma_{g^{-1}} \circ \sigma_g = id_A.$

- For $\sigma_g \circ \sigma_{g^{-1}}$ ($g^{-1} \in G$ too) so σ_g is a bijection $\Rightarrow \sigma_g \in S_A$.

σ: $\sigma_{g_1 g_2} = \sigma_{g_1} \circ \sigma_{g_2}.$

Indeed: $\sigma_{g_1 g_2}(a) = (g_1 g_2) \cdot a \underset{(1)}{=} g_1 \cdot (g_2 \cdot a) = \sigma_{g_1}(\sigma_{g_2}(a)).$

Conversely:

If $\varphi: G \to S_A$ is a hom, Recover

$g \cdot a := \varphi(g)(a)$ is ~~acted~~ an action;

~~$g_1 \cdot (g_2 \cdot a) = \varphi(g_1)(\varphi(g_2)(a))$~~ ———— have

$(g_1 g_2) \cdot a = \varphi(g_1 g_2)(a) = \left[\varphi(g_1)\varphi(g_2)\right](a) = \varphi(g_1)\left(\varphi(g_2)(a)\right) = g_1 \cdot (g_2 \cdot a)$ ✓] how mult. works in S_A

$e \cdot a = \varphi(e)(a) = id_A(a) = a$ ✓

Ex: • Trivial action:

$g \cdot a = a \quad \forall g, a.$

$\boxed{\text{Ask the class}}$

Corr. to $G \to S_A$

$g \mapsto id_A \quad \forall g \in G.$

• S_A acts on A:

$\sigma \cdot a = \sigma(a)$. Think S_n

perm rep$^{\underline{n}}$: ~~$g \mapsto \sigma_g : a \mapsto \sigma(g)(a)$~~

$g \mapsto \sigma_g : \underset{\underset{g}{\parallel}}{a} \mapsto g(a) \quad \forall a \in A.$

is the identity hom. $S_A \to S_A$.

• $D_{2n} \circlearrowright \{1, \ldots, n\}$ by labelling the vertices of n-gon.

• $GL_n(\mathbb{R}) \circlearrowright \mathbb{R}^n$:

$A \cdot v = Av$,

• $G \circlearrowright G$ on itself

$g \cdot h = gh$. (mult. on left) (next time)

• $G \circlearrowright G$ by conj.

$g \cdot h = ghg^{-1}$

(soon)

(2)

Kernel: $\{g \in G \mid g\cdot a = a \quad \forall a \in A\}$.

faithful: $\ker = \{e\}$.

The kernel is the kernel of the perm. rep^{n} $G \to S_A$

($\{g \in G \mid \sigma_g(a) = g\cdot a = a \ \forall a \in A\}$).

$\Rightarrow$ the kernel is a normal subgp of G.

~~Lemma~~:

$\Rightarrow$ An action is faithful $\Leftrightarrow$ the perm. rep^{n} is injective.

($\Leftrightarrow \ker = \{e\}$)

Cor: An action is faithful iff

distinct elts of G induce distinct perms of A.

Ex: The trivial action of $|G| > 1$ is never faithful.

$G \xrightarrow[\ker = G]{} \{id\} \subseteq S_A$.

Ex: $D_{2n} \circlearrowleft \{1, \dots, n\}$ is faithful.

(geometry tells us something!). (kind of by def^{n})

Ex: The left action of G on itself is faithful.

If $g\cdot h = h \quad \forall h$, then $g = e$

even for some h!

Recall: $G_a = \text{stab}(a) \leq G$.

the orbit of $a \in A$ is $\{g\cdot a, g \in G\} = G\cdot a$.

The orbit of a is the equiv. class of a under the equiv. rel^n:

$$a \sim b \iff a = g\cdot b, \text{ some } g \in G.$$
(think: like cosets)

This is an equiv. rel^n:

$a \sim a$: $a = e\cdot a$ ✓

$a \sim b$: $a = g\cdot b \Rightarrow g^{-1}a = b \Rightarrow b \sim a$ ✓

$a \sim b, b \sim c \Rightarrow a = g\cdot b,\ b = h\cdot c \Rightarrow a = g\cdot(h\cdot c) = (gh)\cdot c \Rightarrow a \sim c.$

Thus, the orbits of elts. of a under G partition G.

~~Eff.~~ Fund. Rel^n: $|G| < \infty$. Orbit-Stabilizer

$$\forall a \in A,\ |G\cdot a|\cdot|Stab(a)| = |G|.$$

[More generally, ~~assum~~ using axiom of choice, there is a bijection on $(G\cdot a)\times Stab(a) \to G$ for any G]

(using Lagrange's)

Pf: We find a bijection: $G\cdot a \longrightarrow G/Stab(a)$ ← cosets (go the other way so well-defined) (may not be a gp.)

$g\cdot a \longmapsto g\,Stab(a)$ (only natural!)

Surj: any $g\cdot Stab(a)$ mapped to by $g\cdot a$.

inj: $g\,Stab(a) = h\,Stab(a) \Rightarrow \cancel{gh^{-1}}\ h^{-1}g \in Stab(a)$

$\cancel{\Rightarrow (gh^{-1})\cdot a = a} \Rightarrow \cancel{h^{-1}g(a)}$

$\Rightarrow (h^{-1}g)\cdot a = a \Rightarrow g\cdot a = h\cdot a$ $\Rightarrow$ bijection!

Action is <u>transitive</u> if only one orbit.

<u>Ex</u>: $SL_2(\mathbb{Z}) \circlearrowright \mathbb{Q} \cup \{\infty\}$.

$$\gamma \cdot x = \frac{ax+b}{cx+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} x = \begin{pmatrix} aa'+bc' & ab'+bd' \\ a'c+c'd & b'c+dd' \end{pmatrix} x$$

$$= \frac{(aa'+bc')x + ab'+bd'}{(a'c+c'd)x + b'c+dd'} = \frac{a\left(\frac{a'x+b'}{c'x+d'}\right)+b}{c\left(\frac{a'x+b'}{c'x+d'}\right)+d} \checkmark$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot x = \frac{x+1}{1} = x \checkmark$ (best to use generators S, T). $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

stays in $\mathbb{Q} \cup \{\infty\}$.

$$\gamma \cdot \infty = \frac{a}{c}.$$

~~$n \neq 0$: $\frac{a\frac{m}{n}+b}{c\frac{m}{n}+b} = \frac{am+b}{cm+b} \in \begin{cases} \mathbb{Q} & cm+b \neq 0 \\ \{\infty\} & cm+b = 0 \end{cases}$~~

Check. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot 0 = 0 \Rightarrow b = 0$

$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \cdot \infty = \infty \Rightarrow c = 0 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.

this is <u>transitive</u>.

given any $\frac{a}{c}$, want elt. sends $\infty \to \frac{a}{c}$. $(a,c) = 1$

solve $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1$. possible by Bezout. Call this matrix $\gamma_{\frac{a}{c}}$

So given $\frac{a}{c}, \frac{m}{n}$,

"Connect anything to ∞".

$$\frac{a}{c} \longrightarrow \frac{m}{n} \quad \text{via } \gamma_{\frac{m}{n}}\left(\gamma_{\frac{a}{c}}\right)^{-1}$$

$$\frac{a}{c} \xrightarrow{(\gamma_{\frac{a}{c}})^{-1}} \infty \xrightarrow{\gamma_{\frac{m}{n}}} \frac{m}{n}$$

Not faithful: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as I_2. $-\gamma$ acts as γ. $-\gamma$ acts as in the kernel, only $\pm I$ is in the kernel, so $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$ acts faithfully.

<u>Ex</u>: S_3 acts on $\{1,2,3,4\}$.

orbits: $\{1,2,3\}, \{4\}$

kernel: $\{e\}$

Stab: $\{1\}: \{e, (2\,3)\}$

$\{2\}: \{e, (1,3)\}$

$\{3\}: \{e, (1,2)\}$

$\{4\}: S_3$.

$2 \cdot 3 = 2 \cdot 3$

$= 2 \cdot 3$

$= 1 \cdot 6 = 6 = |S_3|$

Not the Burnside: Average # of fixed pts $= \frac{1}{6}(2+2+2+6) = 2 = \#$ of orbits

Orb.-stab.(S)