Algebra 1 Lecture 3: Normalizers, Centralizers, Isomorphism thm.

Last time: $Z(G)$ = set of elts commuting w/ everything.

center

More generally: Centralizer: $\emptyset \neq A \subseteq G$

$$C_G(A) = \{g \mid gag^{-1} = a \ \forall a \in A\}$$

Note: $gag^{-1} = a \iff ga = ag$. Set of elts commuting w/ each A.
Fixed under conjugation

Prop: $C_G(A) \leq G$

Pf: $e \in C_G(A) \Rightarrow C_G(A) \neq \emptyset$.

~~$(gag^{-1})^{-1} = \ldots$~~ $g \in C_G(A) \Rightarrow$ $gag^{-1} = a \Rightarrow a = g^{-1}ag \Rightarrow g^{-1} \in C_G(A)$

Closed under $(\cdot)^{-1}$

$g, h \in C_G(A) \Rightarrow ghah^{-1}g^{-1} = gag^{-1} = a \Rightarrow gh \in C_G(A)$ ∎

Notation: $C_G(a) = C_G(\{a\})$.

Note: $\langle a \rangle \leq C_G(a)$.

Also, $C_G(G) = Z(G)$, so in particular $Z(G) \leq G$.
(but we already saw $Z(G) \trianglelefteq G$)

Defn: The normalizer of A in G is

$$N_G(A) = \{g \in G \mid gAg^{-1} = A\}.$$

It is a subgroup too (exercise).

This is a weaker condition, so $C_G(A) \leq N_G(A)$.

Good Ex: $H, K \leq G$. $H \leq N_G(K) \Rightarrow HK \leq G$.
In particular, if $K \trianglelefteq G$ (i.e., $N_G(K) = G$), then $HK \leq G$

Pf: $HK \le G \overset{\text{(last time)}}{\Longleftrightarrow} HK = KH$.

Now, if $H \le N_G(K)$,

then $hk = \underbrace{hkh^{-1}}_{\in K} h \in KH$,

$kh = h \underbrace{h^{-1}kh}_{\in K} \in HK \Rightarrow HK = KH$

In terms of actions: (more on these later)

Def$^{\underline{n}}$: A gp. action of G on a set A is a f$^{\underline{n}}$ $G \times A \to A$, $(g,a) \mapsto g \cdot a$
s.t.
1) $g_1(g_2 \cdot a) = (g_1 g_2) \cdot a \quad \forall g_1, g_2 \in G, a \in A$
2) $e \cdot a = a \quad \forall a \in A$

compatible w/ gp. action structure.

~~G acts on g by~~ conjugation action:

$g \cdot h = ghg^{-1}$.

Ex: Check this is a gp. action:

e.g.: $(g_1 g_2) \cdot h = g_1 g_2 h g_2^{-1} g_1^{-1} = g_1 (g_2 \cdot h) g_1^{-1} = g_1 \cdot (g_2 \cdot h)$

$e \cdot h = ehe^{-1} = h$.

Stabilizers: $G \circlearrowright A$. "G acts on A"

for $a \in A$, $G_a = \{g \in G \mid g \cdot a = a\}$ fixes a.

Ex: $G_s \le G$.

kernel: $\{g \in G \mid g \cdot s = s \ \forall s \in S\}$. acts trivially

Ex) This is a subgp.

Ex: $G \circlearrowright G$ by conj.

kernel $= Z(G)$.

Ex: $N_G(A)$ acts on A by conjugation (by def$^{\underline{n}}$ of N_G, $N_G(A) \times A \to A$ stays in set.)

kernel is $C_G(A)$.

Ex: $G \supset 2^6$ conjugation.

stab: $G_A = N_{AG}(A)$.

Cyclic grps. Already saw them, a few more facts;

They are all $\cong \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. Generators aren't unique. obvious $\cong$) Send one gen to another, extending multiplicatively (conjugation)

Ex. 2: $|x| = n < \infty$, then $|x^a| = \frac{n}{(n,a)}$.

Pf. (gone a bit fast on this already) Set $y = x^a$, $(n,a) = d$, $n = db$, $a = dc$

as $d = (n,a)$, $(b,c) = 1$.

we want $|y| = b$.

$$y^b = x^{ab} = x^{dcb} = x^{nc} = 1^c = 1. \Rightarrow |y| \mid b.$$

Say $|y| = k$. Then $x^{ak} = y^k = 1 \Rightarrow n \mid ak$, or $db \mid dck$

$\Rightarrow b \mid ck$. As $(b,c) = 1$, $b \mid k$. So $b \mid k$ and $k \mid b \Rightarrow b = k$ ∎

Ex 1: If $x^m = 1$, then $|x| \mid m$.

Pf: Use Euclidean alg. (like in our pf that all cyclics are $\cong$)

$|x| = n$: $m = nq + r$, $1 = x^m = x^{nq+r} = x^r$, $r < n \Rightarrow r = 0 \Rightarrow n \mid m$.

So if $G = \langle g \rangle$, $(a,n) = 1$, then $G = \langle g^a \rangle$

$\Rightarrow$ # of generators is $\varphi(n)$.

Exercise: Subgps, quotients of cyclic gps are cyclic.

Well ordering / Euclidean division

more generally:

Prop Groups gen. by sets. (saw roughly already)

$A \subseteq G$ $\langle A \rangle = \{a_1^{\varepsilon_1} \cdots a_k^{\varepsilon_k} \quad a_i \in A, \ \varepsilon_i = \pm 1\}$.

This is the smallest subgp. containing A.

<u>Lemma</u>: $\langle A \rangle = \bigcap_{\substack{A \leq H \\ H \leq G}} H =: \bar{A}$

Pf: We have seen that both are subgroups.

$A \subseteq \langle A \rangle$ (a is a single elt. in a product) $a = a^1$.

$\Rightarrow \bar{A} \leq \langle A \rangle$ (def of intersection).

Now $\bar{A}$ is a gp. containing A, and closed under products, inverses

$\Rightarrow \langle A \rangle \subseteq \bar{A} \Rightarrow \langle A \rangle = \bar{A}$.

cut?

video?

Ex: On HW, you will see $N = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$ $yx = xy[x,y]$ ...

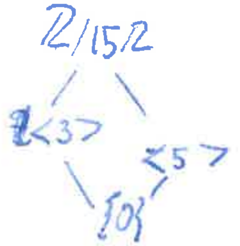
the commutator subgp. ... This is the ~~largest~~ smallest subgp. such that G/N, the abelianization is abelian (the largest ab. quotient as the iso thms will show)

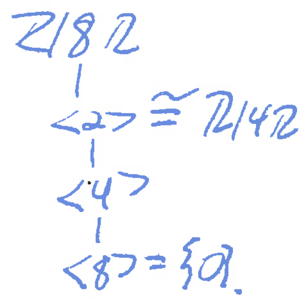
Lattice of subgps:

Start here

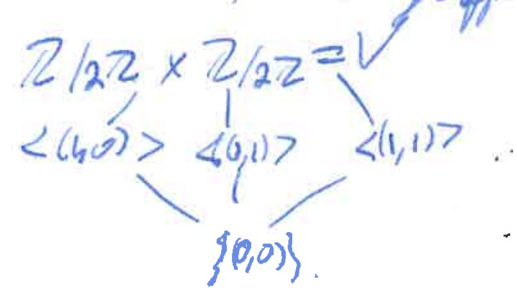
Useful notation/pictures. Make a Poset of subgps of G. $H \leq G$ iff ... $H \leq G$!

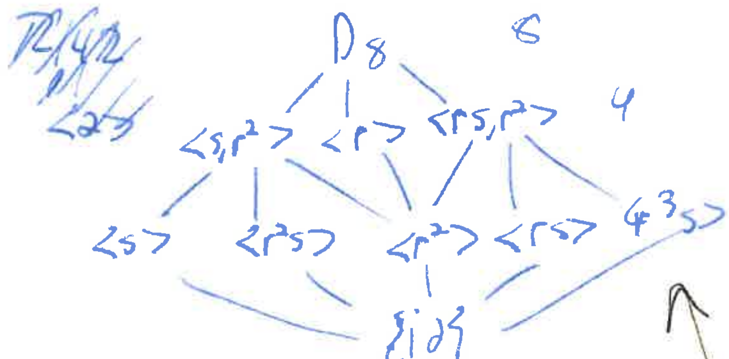
Ex:





Klein Viergruppe





Call reflections, r^2)

leave pic up!

Isomorphism Thms: [we did a lot of groundwork already!]

1st Iso Thm: $\varphi: G \to H$. $\Rightarrow G/\ker(\varphi) \cong \varphi(G)$. Saw before! $\varphi(G) \le H$, $\ker(\varphi) \trianglelefteq G$. [Book makes exercise!]

Pf: ~~Let~~ Take $\bar{\varphi}: G/\ker(\varphi) \to \varphi(G)$ Try the only natural thing!
$$g\ker(\varphi) \mapsto \varphi(g)$$

Well-defined: Change g to gk, $k \in \ker(\varphi)$
$$\Rightarrow \varphi(gk) = \varphi(g)\varphi(k) = \varphi(g) \checkmark.$$

hom: $\bar{\varphi}(g\ker(\varphi)\, h\ker\varphi) = \bar{\varphi}(gh\ker\varphi) = \varphi(gh) = \varphi(g)\varphi(h)$
$= \bar{\varphi}(g\ker G)\,\bar{\varphi}(h\ker\varphi)$

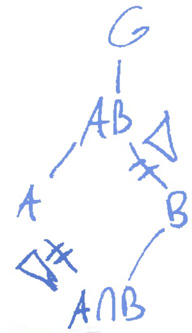
inj: ~~$\varphi(g) = \varphi(h)$~~ $\bar{\varphi}(g\ker\varphi) = \bar{\varphi}(h\ker\varphi)$
or check kernel.
$\Leftrightarrow \varphi(g) = \varphi(h) \Leftrightarrow \varphi(gh^{-1}) = 1 \Leftrightarrow gh^{-1} \in \ker(\varphi)$ $\Leftrightarrow g\ker\varphi = h\ker\varphi$. related: $g\ker\varphi = h\ker(\varphi)$

surj: If $h \in \varphi(G)$, then say $h = \varphi(g) \Rightarrow h = \bar{\varphi}(g\ker\varphi)$ #

Note: This is like the rank-nullity thm:
$$\frac{|G|}{|\ker\varphi|} = |\varphi(G)| \leftrightarrow \dim V = \operatorname{rank}\varphi + \operatorname{null}\varphi.$$

2nd (Diamond) Iso Thm: $A, B \le G$, $A \le N_G(B)$
$$\Rightarrow B \trianglelefteq AB,\ A\cap B \trianglelefteq A,\ AB/B \cong A/A\cap B$$
(we saw that $AB \le G$ already)

Picture:



Pf: $B \trianglelefteq N_G(B)$ is clear, and assumed $A \leq N_G(B) \Rightarrow AB \leq N_G(B)$
$\Rightarrow B \trianglelefteq AB$. Thus, we can consider AB/B.

$\varphi: A \to AB/B$ (N.B: This is restriction of proj. $\pi: AB \to AB/B$ to A)
$a \mapsto aB$.

hom: $\varphi(aa') = aa'(aa')B = aBa'B = \varphi(a)\varphi(a')$

surj: If $abB = aB \in AB/B$, a maps to it.

$\ker(\varphi) = \{a \in A \mid aB = B\} = \{a \in A \mid a \in B\} = A \cap B$.

$\Rightarrow A \cap B \trianglelefteq A$, and by 1st Iso Thm,

$A/A\cap B \cong AB/B$ ∎

3.4 14...

~~Next time~~ Make exercise? Just sketch

3^{rd} Iso Thm:

$H, K \trianglelefteq G$, $H \leq K$.

Can "cancel" ~~$(G/H)/(K/H)$~~ $(G/H)/(K/H) \cong G/K$.

Denoting $\cdot/H$ by $\overline{\cdot}$: $\overline{G}/\overline{K} \cong G/K$.

Pf: $\varphi: G/H \to G/K$ (only choice!) ask them: what would you try?
$gH \mapsto gK$

Well-defined: send $g \mapsto gh$.
$\varphi(gh) = ghK = gK = \varphi(g)$ as $h \in K$.

hom: $\varphi(gg') = gg'K = gKg'K = \varphi(g)\varphi(g')$

surj: $\underset{\text{arb.}}{gK} \leftarrow gH$.

$\ker(\varphi) = \{gH \mid gK = K\} = \{gH \mid g \in K\} = K/H$ (which implies $K/H \trianglelefteq G/H$!)

1st Iso $\Rightarrow (G/H)/(K/H) \cong G/K$ ∎

✓ do.

4^{th} Iso Thm: (Lattice Thm) $N \trianglelefteq G$. Then there is a bijective corr.

~~{subgps of G}~~ $\{N \leq H \leq G\} \longleftrightarrow \{\overline{A} \leq G/N\}$

$H \longmapsto \overline{H} = H \bmod N$. $(= HN)$

$\pi_N: G \to G/N$ $\quad$ $\pi_N^{-1}(A) \longleftarrow A$

(6) $\longrightarrow$ over!

Moreover, ~~if~~ if $N \trianglelefteq A, B$:

$A \le B \Leftrightarrow \bar{A} \le \bar{B}$,

$A \trianglelefteq G \Leftrightarrow \bar{A} \trianglelefteq \bar{G}$.

... A few other properties in the book ... like about indices, generating ~~sets~~ sets...

Pf: Next time.

~~11.8~~

Th^m (4^th isomorphism) Lattice Th^m $N \trianglelefteq G$.

there is a bijective corr.

$$\{N \le H \le G\} \longleftrightarrow \{A \le G/N\}.$$

$$H \longmapsto \bar{H} := H/N \quad (\text{image of } H \text{ in natural proj. } \pi_N : G \to G/N).$$

$$\pi_N^{-1}(A) \longleftarrow A$$

- Moreover 1). if $N \le A, B$,

$$A \le B \Leftrightarrow \bar{A} \le \bar{B}$$

2). $A \trianglelefteq G \Leftrightarrow \bar{A} \trianglelefteq \bar{G}$

... a few other properties in book, like about indices, gen. sets,

Pf: Skip ... like book does ...

lets see a picture!