ON RATIONALLY CONNECTED 3-FOLDS

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ABSTRACT. We study the existence of Hodge metrics of positive total scalar curvature on projective threefolds of negative Kodaira dimension. We give strong indications about the existence of such metrics on rationally connected threefolds. The proofs rely on the theory of minimal models, the weak factorization theorem and on a specialization technique.

CONTENTS

INTRODUCTION

In this article we address the following question:

Question 0.1. *Let X be a smooth complex n-fold of Kähler type and negative Kodaira dimension. Does X admit Kähler metrics of positive total scalar curvature?*

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If we denote by s_g and $d\mu_g$ the scalar curvature and the volume form of g, respectively, this is the same as asking if there is any Kähler metric g on X such that $\int_X s_g d\mu_g > 0$.

For Kähler metrics, the total scalar curvature has a simpler expression:

$$\int_X s_g d\mu_g = 2\pi n c_1(X) \cup [\omega]^{n-1} \tag{0.1}$$

where $[\omega]$ is the cohomology class of the Kähler form of g. The negativity of the Kodaira dimension is a necessary condition [Yau74] because, arguing by contradiction, if for some $m > 0$, the m^{th} power of the canonical bundle of X is either trivial or has sections one can immediately see that $c_1(X) \cup [\omega]^{n-1}$ is negative, which, by (0.1), would imply that the total scalar curvature of (X, g) was negative.

Question 0.1 has an immediate positive answer in dimension 1. The only smooth complex curve of negative Kodaira dimension is $\mathbb{P}_1$, and the Fubini-Study metric satisfies the required inequality. In complex dimension 2, relying on the classification of complex surfaces, a positive answer was given by S.-T. Yau in [Yau74].

Inspired by Yau's approach, we tackle Question 0.1 in the case of projective threefolds, where a satisfactory theory of minimal models exists. As in [Yau74], we can look for Hodge metrics instead, and ask :

Question 0.2. *Let X be a smooth projective 3-fold, with $\mathrm{Kod}(X) = -\infty$. Is there any ample line bundle H on X such that $K_X \cdot H^2 < 0$?*

A positive answer to this question[1] can be connected to a deep result of S. Mori and Y. Miyaoka [MiyMo86], and regarded as a possible effective characterization of the class of smooth, projective threefolds of negative Kodaira dimension. Also, Question 0.2 can be viewed as extracting some positivity property of the anticanonical bundle. An affirmative answer would yield in dimension three, a weak alternative to the generic semi-positivity theorem of Miyaoka asserting that, for non-uniruled manifolds, the restriction of the cotangent bundle to a general smooth complete intersection curve cut out by elements of $|mH|$ is semi-positive, for any ample divisor H and $m \gg 0$.

In this article, as an attempt to answer the original Question 0.1, we study Question 0.2 in the case of rationally connected threefolds. Recall that cf. [KMM92], a complex projective manifold X of dimension $n \geq 2$ is called *rationally connected* if there is a rational curve passing through any two given points of X. One reason to restrict our attention to this kind of manifolds comes from the observation that for rationally connected manifolds, answering affirmatively to Question 0.1 is equivalent to answering affirmatively to Question 0.2. This follows from their convenient cohomological properties. Namely, if X is such a manifold, then $H^i(X, \mathcal{O}_X) = 0,$ for $i \geq 1$, [KMM92]. But in this case, from the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$$

it follows that $H^2(X, \mathbb{C}) \simeq H^{1,1}(X)$. Thus, we can see that any $(1,1)-$form with real coefficients can be approximated[2] by a $(1,1)-$form with rational coefficients. Hence, up to multiplication by positive integers, any Kähler forms can be approximated by first Chern classes of *ample line bundles*. From this the equivalence of our two questions follows easily. However, this is not the only reason to restrict ourselves to the case of rationally connected threefolds, as we will also make use of their wealth of rational curves [ArKo03].

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[1] The same question has also been raised in a different context by F. Campana, J. P. Demailly, T. Peternell and M. Schneider, [DPSch96], [CP98]

[2] Here we consider $H^2(X, \mathbb{R})$ as a finite dimensional real vector space, endowed with any metric topology

To simplify the exposition, for a projective manifold X, we will say that the property $\mathcal{P}_X$ holds true if there exists a Cartier divisor H on X such that $K_X \cdot H^2 < 0$.

Our quest to provide a positive answer to Question 0.2 for rationally connected threefolds falls short, the difficulty being pointed out by the following:

Conjecture 0.3. *Let X be a projective threefold containing a smooth rational curve $C \subset X$ with $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$, and let $X \dashrightarrow X'$ be its flop. If X' is projective, then*

$$\mathcal{P}_X \text{ holds true} \Longleftrightarrow \mathcal{P}_{X'} \text{ holds true.}$$

Our result is:

Theorem A. *If Conjecture 0.3 holds true, then for every projective, rationally connected manifold X of dimension 3 the property $\mathcal{P}_X$ holds true.*

In Section 3.6 we will formulate a very special case of our Conjecture 0.3 which we actually need in order to prove Theorem A.

Similar to the case of complex surfaces, the idea to prove this theorem is to start with an arbitrary 3-fold X of negative Kodaira dimension, and running the minimal models program to go to a minimal model X_{min}. In doing so, we get a birational map $f : X \dashrightarrow X_{min}$ to a new three dimensional projective variety X_{min}, with at most $\mathbb{Q}-$factorial terminal singularities, which is either a Mori fiber space, or has nef canonical bundle. The hypothesis on the Kodaira dimension exclude [Miy88] the possibility of X_{min} having nef canonical bundle. Hence, X_{min} must be a Mori fiber space, i.e. X_{min} is either a del Pezzo fibration, a conic bundle or a Fano variety. As in [CP98] and [DPSch96], we can see that $\mathcal{P}_{X_{min}}$ holds true. The difficult part of this program is to show that

$$\mathcal{P}_{X_{min}} \Longrightarrow \mathcal{P}_X.$$

In this regard, in Section 2 we will show that it is enough to show that $\mathcal{P}$ is a birational property of *smooth* projective threefolds. We can use now the weak factorization theorem [AKMW02], which says that any birational map between smooth (projective) manifolds can be decomposed into a finite sequence of blow-ups and blow-downs with nonsingular centers of (projective) manifolds. We prove the following:

Proposition 0.4. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold at a point. Then*

$$\mathcal{P}_X \text{ holds true} \Longleftrightarrow \mathcal{P}_Y \text{ holds true.}$$

For the blowing-up along curves the following result is of crucial importance:

Proposition 0.5. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold along a smooth curve C.*

- $\mathcal{P}_X$ holds true $\Longrightarrow \mathcal{P}_Y$ holds true.
- *If $K_X \cdot C < 0$, then*

$$\mathcal{P}_Y \text{ holds true} \Longrightarrow \mathcal{P}_X \text{ holds true.}$$

In the last case to verify, $\mathcal{P}_Y \Longrightarrow \mathcal{P}_X$, where $Y \to X$ is the blowing-up of smooth projective threefolds along smooth curves with $K_X \cdot C \geq 0$, the methods used to prove the previous results do not work anymore. It is only this last hurdle, where we use the rational connectedness assumption.

Our approach is to reduce this last case to the one we have already solved. More precisely, we are going to find a smooth curve $C' \subset X$ with $K_X \cdot C' < 0$, such that $\mathcal{P}_{Y'}$ holds true, where $Y' \to X$ is the blowing-up of X along C', then to use Proposition 0.5.

The condition $K_X \cdot C < 0$ imposed in the previous proposition can be interpreted, by the Riemann-Roch theorem, as saying that the curve C "moves". What we do in our construction is "forcing C to move", by eventually modifying it, while preserving property $\mathcal{P}$. Unfortunately, we were unable to complete entirely this task, but merely to reduce it to the validity of Conjecture 0.3. This is done by a lengthy specialization argument, where we strongly rely on the rational connectedness hypothesis. The construction presented in Section 3.2, on which all the computations are performed is based on the work of Graber, Harris, Starr [GHS03] and Kollár [ArKo03]. We devote to this specialization argument the entire Section 3.

In Section 4 we prove Theorem A. An appendix containing some results used intensively throughout this entire chapter is added for the reader's convenience.

Conventions: We work over the field of complex numbers and we use the standard notations and terminology of Hartshorne's Algebraic Geometry book [Hart77].

1. Minimal models

In this section we introduce the objects which appear in the theory of minimal models and we show that our problem has a positive answer for Mori fiber spaces.

Let X be a variety with dim$X > 1,$ such that K_X is $\mathbb{Q}-$Cartier, i.e. mK_X is Cartier for some positive integer $m.$ If $f : Y \to X$ is a proper birational morphism such that K_Y is a line bundle (e.g. Y is a resolution of X), then mK_Y is linearly equivalent to:

$$f^*(mK_X) + \sum m \cdot a(E_i) \cdot E_i,$$

where the E_i's are the exceptional divisors. Using numerical equivalence, we can divide by m and write:

$$K_Y \equiv_{\mathbb{Q}} f^* K_X + \sum a(E_i) \cdot E_i.$$

Definition 1.1. *We say that X has terminal singularities if for any resolution, and for any $i,\ a(E_i) > 0.$*

Definition 1.2. *We say that a variety X is $\mathbb{Q}-$factorial if for any Weil divisor D there exist a positive integer m such that mD is a Cartier divisor.*

The minimal model program (MMP) studies the structure of varieties via birational morphisms or birational maps of special types to seemingly simpler varieties. The birational morphisms which appear running the MMP are the following:

Definition 1.3 (**divisorial contractions**)**.** *Let X be a projective variety with at most $\mathbb{Q}-$factorial singularities. A birational morphism $f : X \to Y$ is called a divisorial contraction if it contracts a divisor, $-K_X$ is $f-$ample and* $\mathrm{rank}\ NS(X) = \mathrm{rank}\ NS(Y) + 1.$

The main difficulty in the higher dimensional minimal model program is the existence of non-divisorial contractions. When the variety X is $\mathbb{Q}-$factorial with only terminal singularities, one may get contractions, called contractions of flipping type $f : X \to Y,$ where the exceptional locus E has codimension at least 2, but such that $-K_X$ is $f-$ample and $\mathrm{rank}\ NS(X) = \mathrm{rank}\ NS(Y) + 1.$ In this case, K_Y is no longer $\mathbb{Q}-$Cartier. The remedy in dimension 3 is the existence of special birational maps, called flips, which allow to replace X by another $\mathbb{Q}-$factorial variety X^+ with only terminal singularities, but simpler in some sense, [KoMo98]:

Definition 1.4 (**flips**). *Let $f : X \to Y$ be a flipping contraction as above. A variety X^+ together with a map $f^+ : X^+ \to Y$ is called a flip of f if X^+ is $\mathbb{Q}-$factorial varieties with terminal singularities and K_{X^+} is f^+-ample.*

By abuse of terminology, the birational map $X \dashrightarrow X^+$ will also be called a flip.

The minimal model program starts with an arbitrary projective $\mathbb{Q}$-factorial threefold with at most terminal singularities X on which one applies an suitable sequence of divisorial contractions and flips. To describe the outcome of the MMP, we recall the following definition:

Definition 1.5 (**Mori fiber spaces**). *Let X and Y be two irreducible $\mathbb{Q}-$factorial varieties with terminal singularities, dim $X >$ dim Y, and $f : X \to Y$ a morphism. The triplet (X, Y, f) is called a Mori fiber space if $-K_X$ is $f-$ample, and*

$$\text{rank } NS(X) = \text{rank } NS(Y) + 1.$$

Theorem 1.6 (**Mori**). *Let X be a projective variety with only $\mathbb{Q}-$factorial terminal singularities, and dim $X = 3$. Then there exist a birational map $f : X \dashrightarrow X_{min}$, which is a composition of divisorial contractions and flips, such that either $K_{X_{min}}$ is nef or X_{min} has a Mori fiber space structure.*

In their approach to Question 0.2, Campana and Peternell proved some important cases. Because of the simplicity, we include their proofs.

The following easy lemma will be used frequently throughout the proofs, sometimes without referring to it.

Lemma 1.7. *Let X be a projective $\mathbb{Q}-$factorial variety of dimension 3, with (at most) terminal singularities for which there exists a nef line bundle D with $K_X \cdot D^2 < 0$. Then there exists an ample line bundle H on X such that $K_X \cdot H^2 < 0$.*

Proof. Let L be any ample on X. Then for any positive integer $m,\ H_m := mD + L$ is an ample line bundle with:

$$K_X \cdot H_m^2 = K_X \cdot (mD + H)^2 = mK_X \cdot D^2 + 2K_X \cdot D \cdot L + K_X \cdot L^2 < 0$$

for $m \gg 0$. □

Proposition 1.8 (**Mori fiber spaces**). *Let (X, Y, f) be a Mori fiber space, with dim $X = 3$. Then the property $\mathcal{P}_X$ holds true.*

Proof. Since dim $X = 3$ we have 3 cases, according to the dimension of Y :

Case 1 **(dim Y=0) :** In this case X is a $\mathbb{Q}-$Fano variety with $\text{rank}\, NS(X) = 1$. In particular, $-mK_X$ is an ample line bundle, for some integer $m > 0$, and the property $\mathcal{P}_X$ follows immediately.

Case 2 **(dim Y=1) :** Take L_Y be any ample line bundle on Y. Since $-K_X$ is $f-$ample it follows that $K_X \cdot (f^*L)^2 < 0$, and as f^*L_Y is nef, from Lemma 1.7 we can see that $\mathcal{P}_X$ holds true.

Case 3 **(dim Y=2) :** As before we take L_Y be any ample line bundle on Y and H_X be an ample line bundle on X. Then for any positive integer $H_m := mf^*L + H_X$, is an ample line bundle and we have:

$$K_X \cdot H_m^2 = K_X \cdot (mf^*L + H_X)^2 = K_X \cdot H_X^2 + 2mK_X \cdot H_X \cdot f^*L_Y < 0,$$

for $m \gg 0$, again because $-K_X$ is $f-$ample.

□

Corollary 1.9. *Let X be a $\mathbb{Q}-$factorial, projective variety of dimension three with at most terminal singularities. If $\mathrm{Kod}(X) = -\infty$ then $\mathcal{P}_{X_{min}}$ holds true.*

Proof. From Theorem 1.6 we know that X_{min} is either a Mori fiber space, for which $\mathcal{P}_{X_{min}}$ holds true, or $K_{X_{min}}$ is nef. To exclude the second possibility, we note that from Miyaoka's abundance theorem [MiPe97, page 88], in dimension three this would imply that $\mathrm{Kod}(X_{min}) \geq 0$. However, this is impossible since the Kodaira dimension is a birational invariant. □

2. VARIOUS REDUCTIONS

Our first reduction takes care of the singularities. Let X be a projective $\mathbb{Q}-$factorial variety X of dimension three, with terminal singularities. A nice feature of these singularities is that, in dimension three, they are isolated points. By Hironaka's resolution of singularities we can always find resolution $p : X' \to X$ which is an isomorphism outside the singular points of X. We begin by proving the following:

Proposition 2.1. *$\mathcal{P}_{X'}$ holds true if and only if $\mathcal{P}_X$ holds true.*

Proof. Suppose first that $\mathcal{P}_X$ holds true. Hence there exists an ample line bundle L on X such that $K_X \cdot L^2 < 0$, and let $D' = p^*L$. Then D is a nef line bundle on X' and

$$K_{X'} \cdot {D'}^2 = (p^*K_X + \sum_i a_iE_i) \cdot p^*L \cdot p^*L = K_X \cdot L^2 < 0.$$

Using Lemma 1.7 it follows that $\mathcal{P}_{X'}$ holds also true.

Conversely, suppose now $\mathcal{P}_{X'}$ holds true, and let H' be a an ample line on X' such that $K_{X'} \cdot {H'}^2 < 0$. Without loss of generality we can assume H' very ample and represented by an irreducible divisor, still denoted by H'. Let $D := p(H')$ its pushforward in X. Following [KoMo98, Lemma 3.39] we have

$$p^*D \equiv_{\mathbb{Q}} H' + \sum_i c_iE_i,$$

where the E_i 's are the exceptional divisors of the resolutions and $c_i \geq 0$.

Now, if C is any curve in X, let C' be its strict transform in X' and so $p_*C' = C$. Then, $D \cdot C = D \cdot p_*C' = p^*D \cdot C' = H' \cdot C' + \sum c_i(E_i \cdot C') > 0,\ C'$ not being contained in any of the exceptional divisors. Thus D is a strictly nef divisor. The singularities of X being terminal, $K_{X'} \equiv_{\mathbb{Q}} p^*K_X + \sum a_iE_i$, with $a_i > 0$. Again, since the singularities of X are a finite number of isolated points, $p^*L \cdot E_i = 0$ for any Cartier divisor L on X. We immediately obtain:

$$\begin{aligned} K_X \cdot D^2 &= (p^*K_X) \cdot (p^*D)^2 = (p^*K_X) \cdot (H')^2 \\ &= K_{X'} \cdot {H'}^2 - \sum a_iE_i \cdot {H'}^2 < 0. \end{aligned}$$

The proposition follows now from Lemma 1.7. □

Proposition 2.1 allows us to interpret the results we proved in the previous section in in the following way. From the minimal models program we obtain a birational map $f : X \dashrightarrow Y$ from a smooth projective threefold X to a singular threefold Y for which $\mathcal{P}_Y$ holds true. We can replace now Y by a *smooth* projective threefold X' for which $\mathcal{P}_{X'}$ holds true. Thus the problem we study reduces to the following:

Question 2.2. *Is $\mathcal{P}$ a birational property of the class of projective threefolds of negative Kodaira dimension?*

This is already a major simplification, because we can use now the weak factorization theorem [AKMW02] of Abramovich, Karu, Matsuki and Włodarczyk:

Theorem 2.3 (**Abramovich, Karu, Matsuki, Włodarczyk).** *A birational map between projective nonsingular varieties over an algebraically closed field K of characteristic zero is a composite of blowings up and blowings down with smooth centers of smooth projective varieties.*

Therefore, what is left to prove is that the property $\mathcal{P}$ is preserved under blowing-ups and blowing-downs at points and smooth curves, respectively.

2.1. **Blowing-up at points.**

Proposition 2.4. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold at a point. Then $\mathcal{P}_X$ holds true if and only if $\mathcal{P}_Y$ holds true.*

Proof. Let E be the exceptional divisor of p. Then by [Hart77, Ex. II.8.5], $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}[E]$ and $K_Y = p^*K_X + 2E$.

Suppose first that $\mathcal{P}_X$ holds true and let H_X be an ample line bundle on X such that $K_X \cdot H_X^2 < 0$. Then $D_Y \stackrel{\text{def}}{=} p^*H_X$ is a nef line bundle on Y such that $K_Y \cdot D_Y^2 = (p^*K_X + 2E) \cdot p^*H_X \cdot p^*H_X = p^*K_X \cdot p^*H_X \cdot p^*H_X = K_X \cdot H_X^2 < 0$. Using again Lemma 1.7 it follows that $\mathcal{P}_Y$ holds true.

Conversely, suppose that $\mathcal{P}_Y$ holds true, and let H_Y be an ample line bundle on Y such that $K_Y \cdot H_Y^2 < 0$. Then $H_Y = p^*D_X - aE$, for some line bundle $D_X \in \mathrm{Pic}(X)$, and some positive integer a. As in the proof of Proposition 2.1, we can show that D_X is nef and $K_X \cdot D_X^2 < 0$. Let C be any curve in X and let C' be its strict transform in Y. Then $p_*C' = C$, and $D_X \cdot C = D_X \cdot p_*C' = p^*D_X \cdot C' = H_X \cdot C' + aE \cdot C' > 0$, because H_X is ample and C' is not contained in E. Therefore D_X is a nef line bundle and

$$K_X \cdot D_X^2 = p^*K_X \cdot p^*D_X \cdot p^*D_X = p^*K_X \cdot H_Y \cdot H_Y = K_Y \cdot H_Y^2 - 2E \cdot H_Y^2 < 0.$$

Applying again Lemma 1.7 we can conclude the proof of the proposition. □

2.2. **Blowing-up along curves.**

In the case of 1-dimensional blowing-up centers, it is easy to prove in one direction:

Proposition 2.5. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold along a smooth curve C. If $\mathcal{P}_X$ holds true then $\mathcal{P}_Y$ holds true.*

Proof. Let H_X be an ample line bundle on X satisfying $K_X \cdot H_X^2 < 0$ and let $D_Y \stackrel{\text{def}}{=} p^*H_X$. Then D_Y is a nef line bundle on Y and we have

$$K_Y \cdot D_Y^2 = (p^*K_X + E) \cdot p^*H_X \cdot p^*H_X = p^*K_X \cdot p^*H_X \cdot p^*H_X = K_X \cdot H_X^2 < 0.$$

The conclusion follows again from Lemma 1.7. □

For the converse of Proposition 2.5 the following proposition is the key step in our line of argument. Its proof is rather long, but elementary, based on Proposition C.3.

Proposition 2.6. *Let $p : Y \to X$ be the blowing-up of a smooth, projective 3-fold along a smooth curve C such that $K_X \cdot C < 0$. If $\mathcal{P}_Y$ holds true then $\mathcal{P}_X$ holds true.*

Proof. Let H_Y be an ample line bundle on X such that $\mathcal{P}_Y$ holds true. Without loss of generality, we can assume that H_X is very ample. Since $p : Y \to X$ is the blowing-up of X along $C \subset X$, the exceptional divisor $E = \mathbb{P}_C(N_{C/X}^\vee)$ will be a ruled surface over C.

Let $d = \deg_C(N_{C/X})$, and let g be the genus of C. Let f be a fiber of $p_{|_E} : E \to C$. We denote by a the intersection number $(H_Y \cdot f)$ in the Chow ring $A(Y)$. We can write:

$$H_Y = p^* L_X - aE \tag{2.1}$$

for some line bundle $L_X \in \mathrm{Pic}(X)$. As in the proof of Proposition 2.4, we can check that $L_X \cdot C' > 0$ for any irreducible curve $C' \subset X$, different than C. Let $\tilde{C}$ be the strict transform of C'. Then:

$$L_X \cdot C' = (H_Y + aE) \cdot C' = H_Y \cdot \tilde{C} + aE \cdot \tilde{C} > 0,$$

because $E \cdot \tilde{C} \geq 0$, the curve $\tilde{C}$ is an irreducible curve, obviously not contained in E. Thus, in order to show that L_X is nef we only have to check that $L_X \cdot C \geq 0$. A straightforward application of the projection formula and of Proposition B.2 gives:

$$\begin{aligned} K_Y \cdot H_Y^2 &= (p^* K_X + E) \cdot (p^* L_X - aE) \cdot (p^* L_X - aE) \\ &= p^* K_X \cdot p^* L_Y \cdot p^* E - 2ap^* K_X \cdot p^* L_X \cdot E + a^2 p^* K_X \cdot E^2 \\ &\quad + E \cdot p^* L_X \cdot p^* L_X - 2aE^2 \cdot p^* L_X + a^2 E^3 \\ &= K_X \cdot L_X^2 - 2aE^2 \cdot p^* L_X + a^2 p^* K_X \cdot E^2 + a^2 E^3 \\ &= K_X \cdot L_X^2 + 2a(L_X \cdot C) - a^2 (K_X \cdot C) - a^2 d \\ &= K_X \cdot L_X^2 + 2a(L_X \cdot C) - a^2(2g-2). \end{aligned}$$

Observation 2.7. Since $K_Y \cdot H_Y^2 < 0$, to conclude the proof of the Proposition 2.6 it will suffice to prove that

$$2a(L_X \cdot C) - a^2(2g-2) \geq 0, \tag{2.2}$$

because we would obtain:

- $K_X \cdot L_X^2 < 0$,
- $L_X \cdot C \geq 0$, *if* $g \geq 1$.

If $g = 0$, we still have to check that $L_X \cdot C \geq 0$.

For a better understanding of (2.2) the following considerations are necessary.

On $E = \mathbb{P}_C(N^\vee_{C/X})$, let C_0 be the section of minimal self-intersection $C_0^2 = -e$. We use $\{C_0, f\}$ as a basis for $\mathrm{Num}_{\mathbb{Z}}(E)$. With respect to this basis:

$$\begin{aligned} H_{X|_E} &\equiv aC_0 + bf, \text{ for some } b \in \mathbb{Z}\ ; \\ E_{|_E} &\equiv xC_0 + yf, \end{aligned}$$

where x and y can be determined as follows:

$$\begin{aligned} -1 = E \cdot f = E_{|_E} \underset{E}{\cdot} f = (xC_0 + yf) \underset{E}{\cdot} f = x; \\ -d = E^3 = E_{|_E} \underset{E}{\cdot} E_{|_E} = (-C_0 + yf)^2 = -e - 2y, \end{aligned}$$

so $y = \frac{d-e}{2}$. Here we denoted by " $\underset{E}{\cdot}$ " the intersection product on the exceptional smooth divisor E.

Remark 2.8. Note that the two invariants, d and e, of $E = \mathbb{P}_C(N^\vee_{C/X})$, have the same parity.

Lemma 2.9. *In the above notations, we have:*

$$2a(L_X \cdot C) - a^2(2g-2) = 2ab - a^2 e - a^2(K_X \cdot C). \tag{2.3}$$

Proof. Computing $H_Y \cdot E \cdot p^*L_X$ in two ways, we obtain:

$$H_Y \cdot E \cdot p^*L_X = (L_X \cdot C)L_X \cdot f = a(L_X \cdot C);$$
$$H_Y \cdot E \cdot p^*L_X = H_Y \cdot E \cdot (H_Y + aE) = H_Y^2 \cdot E + aH_Y \cdot E^2.$$

Thus

$$\begin{aligned} 2a(L_X \cdot C) - a^2(2g-2) &= 2(H_Y^2 \cdot E + aH_Y \cdot E^2) - a^2(2g-2) \\ &= 2(H_Y^2 \cdot E + aH_Y \cdot E^2) - a^2 d - a^2(K_X \cdot C). \end{aligned}$$

Furthermore:

$$\begin{aligned} 2(H_Y^2 \cdot E + aH_Y \cdot E^2) &= 2[(H_{X|_E} \underset{E}{\cdot} H_{X|_E}) + a(H_{X|_E} \underset{E}{\cdot} E_{|E})] \\ &= 2(aC_0 + bf)^2 + a(aC_0 + bf) \underset{E}{\cdot} [-2C_0 + (d-e)f] \\ &= 2a^2C_0^2 + 4ab - 2a^2C_0^2 - 2ab + a^2(d-e) \\ &= 2ab + a^2(d-e). \end{aligned}$$

Therefore $2a(L_X \cdot C) - a^2(2g-2) = 2ab - a^2 e - a^2(K_X \cdot C)$. □

We can finish now the proof of Proposition 2.6:

- If $g \geq 0$, by Proposition C.3, we have two subcases:
  - **i)** ***Case*** $\mathbf{e \geq 0}$: Since H_Y is ample, $H_{Y|_E}$ is ample, and so, by Proposition C.3, $a > 0$ and $b > ae$. Remembering that $K_X \cdot C < 0$, from (2.3) we can see that:

    $$\begin{aligned} 2a(L_X \cdot C) - a^2(2g-2) &= 2ab - a^2 e - a^2(K_X \cdot C) \\ &> a^2 e - a^2(K_X \cdot C) > 0. \end{aligned}$$

    By the crucial Observation 2.7 and by Proposition 1.7 we are done.
  - **ii)** ***Case*** $\mathbf{e < 0}$: Similarly, since H_Y is ample, $H_{Y|_E}$ is ample, too. Thus, by Proposition C.3, $a > 0$ and $b > \frac{1}{2}ae$. Then:

    $$\begin{aligned} 2a(L_X \cdot C) - a^2(2g-2) =& 2ab - a^2 e - a^2(K_X \cdot C) \\ >& - a^2(K_X \cdot C) > 0, \end{aligned}$$

    and we are again done.
- If $g = 0$ then $e \geq 0$, and in this case it suffices to show that $2ab + a^2(d-e) \geq 0$. Since $H_{Y|_E}$ is ample, $a > 0$ and $b > ae$. We have:

  $$2ab + a^2(d-e) > a^2 e + a^2 d = a^2(e - 2 - K_X \cdot C).$$

  So, if $K_X \cdot C \leq -2$ it follows immediately that $L_X \cdot C > 0$, and with the help of Observation 2.7 and Lemma 1.7 we are done again. If $K_X \cdot C = -1$, then $d = -1$ and since d and e have the same parity, $e \geq 1$ and we obtain again $L_X \cdot C > 0$, and we can conclude as above.

With this Proposition 2.6 is completely proved. □

Remark 2.10. The proof of Proposition 2.6 also works when $K_X \cdot C = 0$ and $g > 0$. However, when C is a rational curve $d = \deg N_{C/X} = -2$, and the above arguments show that a possible exception occurs only when $e = 0$, and $0 < b < a$. In this case, $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$, and what fails is only the nefness of L_X.
When $K_X \cdot C > 0$, nothing can be said with the above approach.

This remark is the one that inspires Conjecture 0.3.

3. SPECIALIZATION ARGUMENT

From what we proved so far, to show that $\mathcal{P}$ is a birational property in the class of smooth, projective threefolds it would be enough to answer affirmatively to the following question:

Question 3.1. *Let $p : X_C \to X$ be the blowing up of smooth projective threefold X along a smooth curve $C \subset X$ with $K_X \cdot C \geq 0$. Suppose that $\mathcal{P}_{X_C}$ holds true. Does $\mathcal{P}_X$ also hold true?*

Proposition 2.6 is inspirational, suggesting that a positive answer is possible if we can replace the blowing-up $p : X_C \to X$ of X along the curve C by the blowing-up $p' : X_{C'} \to X$ of X along a smooth curve $C' \subset X$, but such that $K_X \cdot C' < 0$, as long as we are able to show that $\mathcal{P}_{X_{C'}}$ also holds true. We will show that such an approach works in the case of $rationally\ connected$ projective threefolds, if we assume the validity of Conjecture 0.3.

A more precise description of our strategy to answer Question 3.1, and the outline of the structure of this section is the following. In the next subsection we introduce the results from the theory of rationally connected manifolds we need. Then using the outcome of Theorem 3.5, we can construct a smooth family over the unit disk $\mathcal{X} \to \Delta$, whose general fiber X_{C_t} is the blowing-up of X along a smooth curve C_t with $K_X \cdot C_t < 0$. The central fiber of this family will be a normal crossing divisor whose irreducible components are smooth rationally connected threefolds. In subsection 3.3 we show that any line bundle on the central fiber of $\mathcal{X} \to \Delta$ extends to $\mathcal{X}$. Moreover, if the line bundle on the central fiber is chosen to be ample, by eventually shrinking Δ, its extension restricted to X_{C_t} will also be ample. We will next show how a positive answer to Conjecture 0.3 allows us to construct an ample line bundles on the whole central fiber of $\mathcal{X} \to \Delta$. Finally, after setting-up the intersection theory of the central fiber, we will show that on the central fiber of $\mathcal{X} \to \Delta$ satisfies property $\mathcal{P}$ holds true, which will imply that $\mathcal{P}_{X_{C_t}}$ holds true, too.

3.1. **Rationally connected manifolds.**

In this section we collect the necessary information from the theory of rationally connected manifolds. For the definitions and the main results presented we refer the interested reader to [KMM92], [Ko96] and especially to [ArKo03].

Let X denote a complex projective manifold with dim$X \geq 2$.

Definition 3.2. *A nonsingular, complex, projective variety X will be called* rationally connected *if any pair of points in X can be connected by a rational curve.*

The main properties and characterizations of rationally connected manifolds are summarized in the following:

Theorem 3.3.
1) *Rationally connectedness is a birational property and is invariant under smooth deformations.*
2) *Rationally connected manifolds are simply connected and satisfy*

$$H^0(X, \Omega_X^{\otimes m}) = 0 \ for \ m > 0 \ and \ H^i(X, \mathcal{O}_X) = 0 \ for \ i > 0.$$

3) *X is rationally connected if and only if for any point $x \in X$ there exists a smooth rational curve $L \subset X$ passing through x, with arbitrarily prescribed tangent direction and such that its normal bundle $N_{L|X}$ is ample.*

Definition 3.4. *A* comb *with* n teeth *is a projective curve with $n+1$ irreducible components $C, L_1, \ldots, L_n$ such that:*

- *The curves $L_1, \cdots, L_n$ are mutually disjoint, smooth rational curves.*
- *Each $L_i,\ i \neq 0$ meets C transversely in a single smooth point of C.*

The curve C is called the handle *of the comb, and $L_1, \ldots, L_n$ are called the* teeth.

The key result we use is the following theorem of Graber, Harris and Starr [GHS03], which we present in the shape given by J. Kollár [ArKo03]:

Theorem 3.5. *Let X be a smooth, complex, projective variety of dimension at least 3. Let $C \subset X$ be a smooth irreducible curve. Let $L \subset X$ be a rational curve with ample normal bundle intersecting C and let $\mathcal{L}$ be a family of rational curves on X parametrized by a neighborhood of $[L]$ in $\mathrm{Hilb}(X)$. Then there are curves $L_1, \ldots, L_n \in \mathcal{C}$ such that $C_0 = C \cup L_1 \cup \cdots \cup L_n$ is a comb and satisfies the following conditions:*

1) The sheaf $N_{C_0/X}$ is generated by the global sections.
2) $H^1(C_0, N_{C_0/X}) = 0$.

For a better understanding of this theorem the following corollary [ArKo03] is very useful. Since we consider that its proof gives some useful information about our construction, we include for convenience Kollár's proof.

Corollary 3.6. $\mathrm{Hilb}(X)$ *has a unique irreducible component containing $[C_0]$. This component is smooth at $[C_0]$ and a non-empty subset of it parametrizes smooth, irreducible curves in X.*

Proof. Since the curve C_0 is locally complete intersection, its normal sheaf $N_{C_0/X}$ is locally free. We have an exact sequence

$$0 \to N_{C/X} \to N_{C_0/X}{}_{|C} \to Q \to 0$$

where Q is a torsion sheaf supported at the points $P_i = C \cap L_i,$ for $i = 1, \ldots n$. Since $N_{C_0/X}$ is globally generated, we can find a global section $s \in H^0(C_0, N_{C_0/X})$ such that, for each i, the restriction of s to a neighborhood of P_i is not in the image of $N_{C/X}$. This means that s corresponds to a first-order deformation of C_0 that smoothes the nodes P_i of C_0. From the vanishing of $H^1(C_0, N_{C_0/X})$ we see that there are no obstructions finding a global deformation of C_0 that smoothes its nodes P_i.

To be more explicit, we choose local holomorphic coordinates, so that near one of its nodes $P,\ C_0$ is given by:

$$z_1 z_2 = z_3 = \cdots = z_n = 0.$$

Consider now a general $1-$parameter deformation corresponding to a section of $N_{C_0/X}$ which does not belong to the subspace of $N_{C_0/X,P}$ generated by $z_3, \cdots, z_n$. This deformation will be given by the equations:

$$z_1 z_2 + t f(t, \mathbf{z}) = z_3 + t f_3(t, \mathbf{z}) = \cdots = z_n + t f_n(t, \mathbf{z}),$$

and $f(t, \mathbf{z}) \neq 0$, by assumption. We can change new coordinates ${z_1}' := z_1,\ {z_2}' := z_2$ and ${z_i}' := z_i + t f_i(t, \mathbf{z})$ for $i = 3, \ldots n$, to get new, simpler equations:

$$ {z_1}'{z_2}' + t(a + F(t, \mathbf{z})) = {z_3}' = \cdots = {z_n}' = 0, \tag{3.1}$$

where $a \neq 0$ and $F(0,0) = 0$. The singular points are given by the equations:

$${z_1}' + t\frac{\partial F}{\partial {z_2}'} = {z_2}' + t\frac{\partial F}{\partial {z_1}'} = \cdots = {z_n}' = 0.$$

Substituting back these equations into ${z_1}'{z_2}' + t(a + F(t, \mathbf{z}) = 0$ we get a new equation for the supposed singular point:

$$ta = -tF(t, \mathbf{z}) - t^2 \frac{\partial F}{\partial {z_1}'} \frac{\partial F}{\partial {z_2}'}$$

The latter has no solution for $t \neq 0$ and $t, {z_1}', {z_2}', \dots {z_n}'$ small since $a \neq 0$ and $F(0, 0) = 0$. □

Remark 3.7. Using the implicit function theorem we can change one more time the coordinates in (3.1) such that near the node P_i, C_0 is given by:

$$z_1 z_2 + t = z_3 = \cdots = z_n = 0. \tag{3.2}$$

This change of coordinates is given by $z_i := {z_i}'$ for $i = 1, \dots, n$, and $t := t(a + F(t, \mathbf{z}))$.

3.2. **Construction of the specialization.** We start with our blowing-up $p : X_C \to X$ of a projective, rationally connected threefold X along a smooth curve $C \subset X$, with E the exceptional divisor.

Since X is rationally connected, we can always attach [ArKo03] to the curve $C \subset X$ a finite number of disjoint, smooth rational curves $L_1, \dots L_n \subset X$, with ample normal bundle, each meeting C transversely at exactly one point $P_i = C \cap L_i$, $i = 1, \dots, n$. Using Theorem 3.5 and Corollary 3.6, the comb $C_0 = C \cup L_1 \cup \cdots \cup L_n$ is smoothable for $n \gg 0$. As in the proof of Corollary 3.6, this means that we can find a small deformation of C_0 parametrized by a one dimensional disk $\Delta \subset \mathrm{Hilb}(X)$ centered in $[C_0]$. That is there exists a smooth submanifold $\mathcal{C} \subset X \times \Delta$, such that its projection $\pi : \mathcal{C} \to \Delta$ is flat, and

$$\pi^{-1}(t) = \begin{cases} C_0, & \text{if } t = 0 \\ C_t, & \text{if } t \neq 0, \end{cases}$$

where C_t is a smooth irreducible curve. From Corollary 3.6 and Remark 3.7, in local coordinates chosen w.r.t. a neighborhood of the node P_i, π is the projection

$$(z_1, z_2, z_3, t) \mapsto t$$

and $\mathcal{C}$ is given by $z_1 z_2 + t = z_3 = 0$. In these local coordinates, $C \subset \mathcal{C}$ is given by $z_1 = z_3 = t = 0$, and L_i by $z_2 = z_3 = t = 0$.

Let $\varpi : \mathcal{X}_{\mathcal{C}} \to X \times \Delta$ be the blow-up of $X \times \Delta$ along $\mathcal{C}$, and let

$$\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$$

be the projection onto Δ.

Lemma 3.8. **(Structure of $\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$)**

i) *$\mathcal{X}_{\mathcal{C}}$ is a smooth variety, and $\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$ is a flat, proper family of projective varieties.*

ii) *For $t \neq 0$, $X_{\mathcal{C},t} = \Pi^{-1}(t)$ is the blowing-up of X along C_t, while $X_{\mathcal{C},0} = \Pi^{-1}(0)$ is the blowing-up of X along the ideal sheaf of $C_0 \subset X$.*

Proof. i) This are standard facts about blowing-up, see sections II. 7 and II. 8 of [Hart77].

ii) For the proof we can either quote the universal property of the blowing-up, Corollary II.7.15 of [Hart77] or use local equations as in the proof of Corollary 3.6. We adopt the latter. The results we want to prove here are of local nature. In a neighborhood of a node of $C_0 \subset U \subset X \times \Delta$, $\mathcal{C}$ is given by the equations:

$$z_1 z_2 + t = z_3 = 0.$$

$\mathcal{X}_{\mathcal{C}_{|U}} \subset U \times \mathbb{P}_1$ will therefore given by the equations:

$$(z_1 z_2 + t)v = z_3 u, \tag{3.3}$$

where $[u : v]$ are the homogeneous coordinates on $\mathbb{P}_1$, and the conclusion follows now immediately. $\square$

Let $\Pi_0 : X_{\mathcal{C},0} \to X$ denote the blowing-up map of X along the ideal sheaf of C_0.

Lemma 3.9. (Structure of the central fiber $X_{\mathcal{C},0}$)

i) $X_{\mathcal{C},0}$ *has exactly* n *distinct ordinary double points as singularities.*
ii) *The exceptional divisor of* Π_0, *denoted by* E^* *is a union of smooth Weil divisors* $E^*_C,\ E^*_1, \ldots, E^*_n$.

Proof. i) From the arguments used in the above Lemma we can see that the singular points of $X_{\mathcal{C},0}$ can only occur over the singular points of C_0. For every node of C_0, in the above coordinates, there is exactly one singular point of $X_{\mathcal{C},0}$, which appears in the chart where $v \neq 0$ and is given by the local equation

$$z_1 z_2 = z_3 u', \tag{3.4}$$

where $u' = \frac{u}{v}$.

ii) Since the center of the blowing-up has exactly $n+1$ components, it follows the exceptional divisor of $X_{\mathcal{C},0}$ has $n+1$ components too, one over each of the components of C_0. Using (3.4) the other claims easily follow. $\square$

Let $Q_i \in \mathcal{X}_{\mathcal{C}},\ 1 = 1, \ldots, n$ denote the singular points of $X_{\mathcal{C},0}$.

Remark 3.10. It can be seen that $X_{\mathcal{C},0}$ is a Gorenstein non $\mathbb{Q}-$factorial variety. Hence push-forward arguments we used in the previous section no longer apply.

In order to perform the computation to follow, we need a better understanding of the components E^*_i's of E^*.

Proposition 3.11. (The components $E^*_i,\ i = 1, \ldots n$)

i) $E^*_i = \mathbb{P}_{L_i}(N^{\vee}_{C_0/X_{|L_i}})$.
ii) *The conormal bundle of* E^*_i *is given by the extension:*

$$0 \longrightarrow \mathcal{O}_{E^*_i} \longrightarrow N^{\vee}_{E^*_i/\mathcal{X}_{\mathcal{C}}} \longrightarrow \mathcal{J}_{Q_i} \otimes \mathcal{O}_{E^*_i}(1) \otimes \mathcal{O}_{E^*_i}(f) \longrightarrow 0,$$

where, by $\mathcal{J}_{Q_i}$ *we denoted the ideal sheaf of* Q_i *on* $E^*_i,\ \mathcal{O}_{E^*_i}(1)$ *is the dual of the tautological bundle of the ruled surface* E^*_i, *and* f *is its fiber.*

Proof. i) This is a well-known fact. We include a short proof for convenience. The general theory of blowing-ups tells us that, since $L_i \subset \mathcal{C},\ E^*_i = \mathbb{P}_{L_i}(N^{\vee}_{\mathcal{C}/X \times \Delta_{|L_i}})$. To compute

$N^{\vee}_{\mathcal{C}/X\times\Delta_{|L_i}}$ we use the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{L_i} & = & \mathcal{O}_{L_i} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{\mathcal{C}/X\times\Delta_{|L_i}} & \longrightarrow & N^{\vee}_{L_i/X}\oplus\mathcal{O}_{L_i} & \longrightarrow & \mathcal{O}_{L_i}(P_i) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C_0/X_{|L_i}} & \longrightarrow & N^{\vee}_{L_i/X} & \longrightarrow & \mathcal{O}_{P_i}(P_i) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.5}$$

The first row is given by the exact sequence of conormal bundles of the inclusions $L_i \subset \mathcal{C} \subset X \times \Delta$. We have the obvious isomorphism $N^{\vee}_{L_i/X\times\Delta} \simeq N^{\vee}_{L_i/X} \oplus \mathcal{O}_{L_i}$.

On the smooth surface $\mathcal{C}$, since the L_i's are mutually disjoint *rational curves* and meet C transversally at exactly one point, we have:

$$0 = L_i \cdot C_t = L_i \cdot C_0 = L_i \cdot (C + L_1 + \cdots L_n) = 1 + L_i^2.$$

Therefore, the L_i's are actually $(-1)-$curves and $N^{\vee}_{L_{i|\mathcal{C}}} \simeq \mathcal{O}_{L_i}(P_i)$.

The second row is the exact sequence of Andreatta-Wisniewski [AnWi98, page 265]. From the snake lemma, we can see now that

$$N^{\vee}_{\mathcal{C}/X\times\Delta_{|L_i}} \simeq N^{\vee}_{C_0/X_{|L_i}}.$$

ii) Let $\bar{F} : \bar{\mathcal{X}} \to \mathcal{X}_{\mathcal{C}}$ be the blowing-up of $\mathcal{X}_{\mathcal{C}}$ at the points Q_i, for $i = 1, \ldots, n$, and $\bar{\Pi} : \bar{\mathcal{X}} \to \Delta$ the projection onto Δ. We denote by $\bar{X}$ the strict transform of $X_{\mathcal{C},0}$, and by $Z_i,\ i = 1, \ldots, n$, the exceptional divisors of $\bar{F}$. $\bar{\Pi}$ has the following fibers:

$$\bar{\Pi}^{-1}(t) = \begin{cases} X_{C_t}, & \text{if } t = 0 \\ \bar{X} + 2Z_1 + \ldots 2Z_n, & \text{if } t \neq 0. \end{cases}$$

Since $\mathcal{X}_{\mathcal{C}}$ is smooth, we have $Z_i \simeq \mathbb{P}_3$, and $N_{Z_i/\bar{\mathcal{X}}} \simeq \mathcal{O}_{\mathbb{P}_3}(-1)$. The multiplicity of the Z_i's in the central fiber are caused by the ordinary double point singularities of $X_{\mathcal{C},0}$. The reduced component $\bar{X}$ is a big resolution of $X_{\mathcal{C},0}$. The induced map $\bar{X} \to X_{\mathcal{C},0}$ has n exceptional divisors, $T_i = \bar{X} \cap Z_i,\ i = 1, \ldots, n$, each of them isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$, with $N_{T_i/\bar{X}} \simeq \mathcal{O}(-1,-1)$. Moreover, $N_{T_i/Z_i} \simeq \mathcal{O}(1,1)$, for all $i = 1, \ldots, n$.

To compute the conormal bundle $N^{\vee}_{E_i^*/\mathcal{X}_{\mathcal{C}}}$, we need a good understanding of the main component $\bar{X}$ of $\bar{\Pi}^{-1}(0)$. Let $\bar{p} : \bar{X} \to X$ be the natural morphism onto X. This has the following alternative description :

- Consider $p_L : X_L \to X$, the blowing up of X, along the disjoint union of curves $L_1, \ldots, L_n$. Let $E_1, \ldots, E_n$ denote the exceptional divisors and $\bar{C}$ denote the strict transform of C and $\{x_i\} = \bar{C} \cap E_i$. The E_i's are rational ruled surfaces over $L_i,\ E_i = \mathbb{P}_{L_i}(N^{\vee}_{L_i/X})$, with $N_{E_i/X_L} \simeq \mathcal{O}_{E_i}(-1)$. Consider $f_i \in E_i$, the fiber of E_i through x_i, for all $i = 1, \ldots, n$.

- Let $p_{\bar{C}} : X_{L,\bar{C}} \to X_L$ be the blowing-up of X_L along $\bar{C}$. We denote by $E_{\bar{C}}$ the exceptional divisor, and by $\bar{E}_i$ the strict transforms of E_i, for all $i = 1, \dots, n$. Each of the $\bar{E}_i{}'$s is the blowing-up of E_i at x_i. Let ℓ_i denote the exceptional divisor of these blowing-up. $E_{\bar{C}}$ and $\bar{E}_i$ meet transversally along ℓ_i, and ℓ_i sits in $E_{\bar{C}}$ as fiber. Moreover, we have $N_{\bar{E}_i/X_{L,\bar{C}}} = p^*_{\bar{C}} N_{E_i/X_L}$ (see [Ful]). In each $\bar{E}_i$, we denote by $\bar{f}_i$ the strict transform of f_i.

  We can immediately see that $N_{\bar{E}_i/X_{L,\bar{C}}{}_{|\bar{f}_i}} \simeq \mathcal{O}_{\mathbb{P}_1}(-1)$, and the exact sequence:
$$0 \to N_{\bar{f}_i/\bar{E}_i} \to N_{\bar{f}_i/X_{L,\bar{C}}} \to N_{\bar{E}_i/X_{L,\bar{C}}{}_{|\bar{f}_i}} \to 0$$
  yields
$$N_{\bar{f}_i/X_{L,\bar{C}}} \simeq \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1).$$
- We blow-up now $X_{L,\bar{C}}$ along $\bar{f}_i$, for all $i = 1, \dots, n$. The resulting $3-$fold is isomorphic to $\bar{X}$, where the exceptional divisors of the last blowing-up coincide with $T_i,\ i = 1, \dots, n$. Let $p_{\bar{F}} : \bar{X} \to X_{L,\bar{C}}$ be the blowing-up map. The map $\bar{p}$ is the composition:
$$\bar{p} = p_L \circ p_{\bar{C}} \circ p_{\bar{F}}.$$
  Denote by $R_i = p^*_{\bar{F}} \bar{E}_i - T_i$ the strict transforms of $\bar{E}_i$. Since $\bar{f}_i \subset \bar{E}_i,\ R_i$ is isomorphic to $\bar{E}_i$, the blowing-up of of E_i at x_i. Let $\bar{E}$ be the strict transform of $E_{\bar{C}}$, and $\bar{\ell}_i$ be the strict transform of ℓ_i, for all $i = 1, \dots, n$. $\bar{E}$ is isomorphic to $E_{\bar{C}}$ blown-up at the intersection points of $E_{\bar{C}}$ with the curves $\bar{f}_i$, and intersects R_i transversally along $\bar{\ell}_i$ for all $i = 1, \dots, n$.

First we need to determine $N_{R_i/\bar{X}}$. To do this, we have to analyze more closely the position of the exceptional divisors of the map $\bar{p}$.

- R_i and T_i meet transversally along h_i, one of the rulings of T_i which coincides with $\bar{f}_i$, under the identification of R_i with $\bar{E}_i$;
- $\bar{E}$ and T_i meet transversally along k_i, the other ruling of T_i, for all $i = 1, \dots, n$;
- $\bar{E} \cap R_i \cap T_i = \{\text{point}\}$, for all $i = 1, \dots, n$;
- $R_i \cap R_j = \emptyset$, for $i \neq j$.

Let $p_i : R_i \to E_i$ be the blowing up of E_i at x_i, where h_i is the strict transform of the fiber through x_i, and $\bar{\ell}_i$ denotes the exceptional divisor. Using i) we can see that E_i^* is actually the elementary transform of E_i centered at x_i. Consequently, we denote by $q_i : R_i \to E_i^*$, the blowing-down of $\bar{f}_i$, for every $i = 1, \dots, n$.

Lemma 3.12. $N_{R_i/\bar{X}} \simeq \mathcal{O}_{R_i}(-h_i) \otimes p_i^* \mathcal{O}_{E_i}(-1)$.

Proof of Lemma 3.12. Since R_i is the blowing-up of E_i, we can write $N_{R_i/\bar{X}}$ as
$$\mathcal{O}_{R_i}(ah_i) \otimes p_i^* \mathcal{O}_{E_i}(b) \otimes p_i^* \mathcal{O}_{E_i}(cf),$$
where f is the generic fiber of the ruled surface E_i. Let $d_i = \deg N_{L_i/X}$. Let also denote by $\bar{f}$ the strict transform in $\bar{\mathcal{X}}$ of the generic fiber of E_i. From the fact that $R_i = p^*_{\bar{F}} \bar{E}_i - T_i$ and the projection formula, we compute:
$$\begin{aligned} R_i \cdot \bar{f} &= (p^*_{\bar{F}} \bar{E}_i - T_i) \cdot \bar{f} = E_i \cdot f = -1; \\ R_i \cdot \bar{\ell}_i &= (p^*_{\bar{F}} \bar{E}_i - T_i) \cdot \bar{\ell}_i = p^*_{\bar{C}} E_i \cdot \ell_i - T_i \cdot \bar{\ell}_i = -1; \\ R_i^3 &= (p^*_{\bar{F}} \bar{E}_i - T_i)^3 = \bar{E}_i{}^3 - 3 p^*_{\bar{F}} \bar{E}_i \cdot p^*_{\bar{F}} \bar{E}_i \cdot T_i + 3 p^*_{\bar{F}} \bar{E}_i \cdot T_i^2 - T_i^3 \\ &= E_i^3 + 3(\bar{E}_i \cdot \bar{f}_i) \cdot (T_i \cdot k_i) + 2 = -d_i + 3 - 2 = -d_i + 1. \end{aligned}$$

On the other hand, computing on the surface R_i, we have:

- $R_i \cdot \bar{f} = (ah_i + p_i^*\mathcal{O}_{E_i}(b) + cp_i^* f) \cdot p_i^* f = b$, and so $b = -1$.
- $R_i \cdot \bar{\ell}_i = (ah_i + p_i^*\mathcal{O}_{E_i}(b) + cp_i^* f) \cdot \bar{\ell}_i = a(p_i^* f - \bar{\ell}_i) \cdot \bar{\ell}_i = a$, and so $a = -1$;
- $R_i^3 = (-h_i - p_i^*\mathcal{O}_{E_i}(1) + cp_i^* f)^2 = -1 - d_i + 2 - 2c = -d_i + 1 - 2c$, and so $c = 0$.

□

We compute now $N^\vee_{R_i/\bar{\mathcal{X}}}$ from the conormal sequence of the inclusions $R_i \subset \bar{X} \subset \bar{\mathcal{X}}$:

$$0 \to N^\vee_{\bar{X}/\bar{\mathcal{X}}_{|R_i}} \to N^\vee_{R_i/\bar{\mathcal{X}}} \to N^\vee_{R_i/\bar{X}} \to 0. \tag{3.6}$$

In $\bar{\mathcal{X}}$, we have $\bar{X} + 2Z_1 + \cdots 2Z_n \sim 0$, (linearly equivalence) and so

$$N^\vee_{\bar{X}/\bar{\mathcal{X}}} \simeq \mathcal{O}_{\bar{X}}(2T_1 + \cdots 2T_n).$$

Tensoring by $\mathcal{O}_{R_i}$, we get $N^\vee_{\bar{X}/\bar{\mathcal{X}}_{|R_i}} \simeq \mathcal{O}_{R_i}(2h_i)$. Hence we obtained:

$$0 \to \mathcal{O}_{R_i}(2h_i) \to N^\vee_{R_i/\bar{\mathcal{X}}} \to \mathcal{O}_{R_i}(h_i) \otimes p_i^*\mathcal{O}_{E_i}(1) \to 0. \tag{3.7}$$

On the other hand, since $\mathcal{X}_{\mathcal{C}}$ and E_i^* are smooth, R_i is the strict transform of E_i^* in $\bar{\mathcal{X}}$. Moreover, the restriction of blowing-up map $\bar{F}$ to R_i coincides with the blowing-up q_i with h_i as exceptional divisor. Therefore, by [Ful, page 437],

$$N_{R_i/\bar{\mathcal{X}}} \simeq q_i^* N_{E_i^*/\mathcal{X}_{\mathcal{C}}} \otimes \mathcal{O}_{R_i}(-h_i).$$

From (3.7) we obtain:

$$0 \to \mathcal{O}_{R_i}(h_i) \to q_i^* N^\vee_{E_i^*/\mathcal{X}_{\mathcal{C}}} \to p_i^*\mathcal{O}_{E_i}(1) \to 0. \tag{3.8}$$

Lemma 3.13. *On the surface R_i, we have:*

$$p_i^*\mathcal{O}_{E_i}(1) = q_i^*\mathcal{O}_{E_i^*}(1) \otimes q_i^*\mathcal{O}_{E_i^*}(f) \otimes \mathcal{O}_{R_i}(-h_i),$$

where here f denotes the generic fiber of E_i^*.

Proof of Lemma. Computing the canonical line bundle of R_i in two ways, we get:

$$p_i^*\mathcal{O}_{E_i}(K_{E_i}) \otimes \mathcal{O}_{R_i}(\bar{\ell}_i) = q_i^*\mathcal{O}_{\bar{E}_i}(K_{\bar{E}_i}) \otimes \mathcal{O}_{R_i}(h_i). \tag{3.9}$$

Using the canonical bundle formula for ruled surfaces, and the fact that E_i^* is the elementary transform of E_i centered at x_i, from (3.9) we have:

$$\begin{aligned} &p_i^*\mathcal{O}_{E_i}(-2) \otimes p_i^*\mathcal{O}_{E_i}(-d_i f) \otimes \mathcal{O}_{R_i}(\bar{\ell}_i) = \\ &q_i^*\mathcal{O}_{E_i^*}(-2) \otimes q_i^*\mathcal{O}_{E_i^*}((-d_i - 1)f) \otimes \mathcal{O}_{R_i}(h_i). \end{aligned} \tag{3.10}$$

But, $\mathcal{O}_{R_i}(\bar{\ell}_i) = q_i^*\mathcal{O}_{E_i}(f) \otimes \mathcal{O}_{R_i}(-h_i)$, and $p_i^*\mathcal{O}_{E_i}(f) = q_i^*\mathcal{O}_{E_i^*}(f)$. Simplifying (3.10) we get:

$$p_i^*\mathcal{O}_{E_i}(-2) = q_i^*\mathcal{O}_{E_i^*}(-2) \otimes q_i^*\mathcal{O}_{E_i^*}(-2f) \otimes \mathcal{O}_{R_i}(2h_i),$$

and the proof of the lemma follows. □

To finish the proof of the proposition, notice that we obtained the following exact sequence:

$$0 \to \mathcal{O}_{R_i}(h_i) \to q_i^* N^\vee_{E_i^*/\mathcal{X}_{\mathcal{C}}} \to q_i^*\mathcal{O}_{E_i^*}(1) \otimes q_i^*\mathcal{O}_{E_i^*}(f) \otimes \mathcal{O}_{R_i}(-h_i) \to 0.$$

By pushing forward on E_i^*, since $R^1 q_{i*}\mathcal{O}_{R_i}(h_i) = 0$ and $q_{i*}\mathcal{O}_{R_i}(h_i) = \mathcal{O}_{E_i^*}$, the projection formula yields:

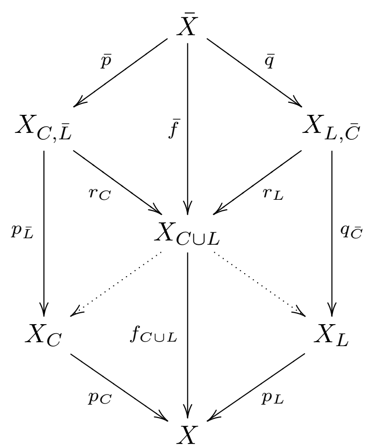
$$0 \longrightarrow \mathcal{O}_{E_i^*} \longrightarrow N^\vee_{E_i^*/\mathcal{X}_{\mathcal{C}}} \longrightarrow \mathcal{J}_{Q_i} \otimes \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f) \longrightarrow 0,$$

where, $\mathcal{J}_{Q_i}$ is the ideal sheaf of the point Q_i, and we are done. □

Corollary 3.14. *The Chern classes of* $N^{\vee}_{E_i^*/\mathcal{X}_{\mathcal{C}}}$ *are:*

- $\det(N^{\vee}_{E_i^*/\mathcal{X}_{\mathcal{C}}}) = \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f)$;
- $c_2(N^{\vee}_{E_i^*/\mathcal{X}_{\mathcal{C}}}) = 1$.

Remark 3.15. The description of $\bar{X}$ in the proof of the above proposition is of local nature and it comes from the well-known diagram below [EiHa00, pages 178-179]. This commutative diagram exhibits the relation between the two small resolutions and the natural big resolutions of a three-dimensional ordinary double points, as those appearing as singularities of $X_{\mathcal{C},0}$.



(3.11)

To simplify the notation and explain the diagram (3.11), let X be an arbitrary threefold, and $C,\ L \subset X$ be two smooth curves intersecting transversally at exactly one point $\{x\} = C \cap L$.

- $f_{C\cup L} : X_{C\cup L} \to X$ is the blowing-up of X along the ideal sheaf of $C \cup L$;
- $\bar{f} : \bar{X} \to X_{C\cup L}$ is the big resolution of $\bar{X}$ obtained by blowing-up the singular point.
- $p_C : X_C \to X$ is the blowing-up of X along C. Let f_C be the fiber of the exceptional divisor over x.
- $q_L : X_L \to X$ is the blowing-up of X along L. Let f_L be the fiber of the exceptional divisor over x.
- $p_{\bar{L}} : X_{C,\bar{L}} \to X$ is the blowing-up of X_C along $\bar{L}$, the proper transform of L in X_C. Let $\bar{f}_C$ denote the proper transform of f_C.
- $q_{\bar{C}} : X_{L,\bar{C}} \to X$ is the blowing-up of X_L along $\bar{C}$, the proper transform of C in X_L; Let $\bar{f}_L$ denote the proper transform of f_L.
- $\tilde{p} : \tilde{X} \to X_{C,\bar{L}}$ is the blowing-up of $X_{C,\bar{L}}$ along $\bar{f}_C$.
- $\tilde{q} : \tilde{X} \to X_{L,\bar{C}}$ is the blowing-up of $X_{L,\bar{C}}$ along $\bar{f}_L$.
- $r_L : X_{L,\bar{C}} \to \bar{X}$ and $r_C : X_{C,\bar{L}} \to \bar{X}$ are the two *small resolutions* of the singular point of $\bar{X}$.

We will modify the family $\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$ to produce a flat, proper map

$$\Phi : \mathcal{X} \to \Delta$$

with normal crossing central fiber, and with X_{C_t} as the general fiber.

The map Φ is obtained as the composition

$$\mathcal{X} \xrightarrow{F} \mathcal{X}_{\mathcal{C}} \xrightarrow{\Pi} \Delta,$$

where $F : \mathcal{X} \to \mathcal{X}_{\mathcal{C}}$ is the blowing-up $X_{\mathcal{C}}$ along E_i^*, for $i = 1, \ldots, n$.

It is easy to see that the generic fiber of Φ is X_{C_t}, the blowing-up of X along the smooth curve C_t. The central fiber of Φ is a normal crossing thereefold

$$X_0 = X_p \cup X_1 \cup \cdots \cup X_n,$$

with exactly $n+1$ irreducible, smooth components.

To describe the main component, denoted by X_p, as before, let $X_L \to X$ denote the blowing-up of X along the curves $L_i,\ i = 1, \ldots, n$. Then X_p is obtained by blowing-up X_L along $\bar{C}$, the strict transforms of the C. It actually coincides with the $3-$fold $X_{L,\bar{C}}$, described in the above proposition. We denote by

$$f_p : X_p \to X,$$

the projection onto X.

The other components,

$$X_i = \mathbb{P}_{E_i^*}(N^{\vee}_{E_i^*|\mathcal{X}_{\mathcal{C}}}) \xrightarrow{f_i} E_i^*,$$

are the exceptional divisors of the of the blowing-up of $\mathcal{X}_{\mathcal{C}}$ along E_i^*, for $i = 1, \ldots, n$.

Moreover, the intersections $S_i := X_p \cap X_i$ are smooth surfaces, the blowing-up of E_i^* at Q_i. Again, the S_i actually coincides with the surfaces $\bar{E}_i$ described in the above proposition.

It is easy to notice that all of the components of the central fiber and their intersections are in fact rationally connected manifolds.

3.3. **Extensions of line bundles.** In this section we will show that any line bundle on the central fiber of Φ extends to a line bundle on an open neighborhood of the central fiber. The results are probably standard and well-known, e.g. [Per77, Theorem Q, page 50]. However, since we couldn't find any reference for the proof, we include it for completeness. But first, we must describe the line bundles on X_0. To do this we follow [Fr83].

Since the components X_p and X_i of X_0 intersect transversally along S_i, and $X_i \cap X_j = \emptyset$, for $i \neq j$, a line bundle H_{X_0} on X_0 consists of a $(n+1) - uple$ of line bundles $(H_p, H_1, \ldots H_n)$ on X_p and $X_i,$ for $i = 1, \ldots, n$, respectively, such that

$$H_{p|_{S_i}} \simeq H_{i|_{S_i}},$$

for any $i = 1, \ldots, n$.

As an example, we consider [Fr83] the case of dualizing sheaf of ω_{X_0}, an example which will be needed later. We have:

$$\begin{aligned}\omega_p &= \omega_{X_0|_{X_p}} = \mathcal{O}_{X_p}(K_{X_p} + S_1 + \cdots S_n),\\ \omega_i &= \omega_{X_0|_{X_i}} = \mathcal{O}_{X_i}(K_{X_i} + S_i).\end{aligned}$$

By adjunction, we have the canonical isomorphisms:

$$\omega_{p|_{S_i}} = \mathcal{O}_{X_p}(K_{X_p} + S_i)_{|_{S_i}} = \mathcal{O}_{R_i}(K_{S_i}) = \mathcal{O}_{X_i}(K_{X_i} + S_i)_{|_{S_i}} = \omega_{i|_{S_i}},$$

showing that ω_p agrees with ω_i on S_i, and so

$$\omega_{X_0} = (\mathcal{O}_{X_p}(K_{X_p} + R_1 + \cdots R_n), \mathcal{O}_{X_1}(K_{X_i} + R_1), \ldots \mathcal{O}_{X_n}(K_{X_n} + R_n)). \quad (3.12)$$

Lemma 3.16. $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = 0.$

Proof. Recall that the central fiber X_0 has $(n+1)$ smooth components

$$X_0 = X_p \cup X_1 \cup \cdots \cup X_n,$$

where $S_i = X_p \cap X_s$, is the blowing up of a rational ruled surface, and $X_i \cap X_j = \emptyset$. for $i \neq j$. Their structure sheaves are related by the Mayer-Vietoris sequence:

$$0 \to \mathcal{O}_{X_0} \to \mathcal{O}_{X_p} \oplus \mathcal{O}_{X_1} \oplus \cdots \oplus \mathcal{O}_{X_n} \to \mathcal{O}_S \to 0, \tag{3.13}$$

where S is the reduced, reducible surface $S_1 + \cdots + S_n$.

The relevant part of the cohomology sequence is:

$$\begin{aligned}\cdots \to H^0(X_p, \mathcal{O}_{X_p}) \oplus H^0(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^0(X_n, \mathcal{O}_{X_n}) \xrightarrow{s} H^0(S, \mathcal{O}_S) \\ \to H^1(X_0, \mathcal{O}_{X_0}) \to H^1(X_p, \mathcal{O}_{X_p}) \oplus H^1(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^1(X_n, \mathcal{O}_{X_n}) \\ \to H^1(S, \mathcal{O}_S) \to H^2(X_0, \mathcal{O}_{X_0}) \to H^2(X_p, \mathcal{O}_{X_p}) \oplus H^2(X_1, \mathcal{O}_{X_1}) \oplus \cdots \\ \oplus H^2(X_n, \mathcal{O}_{X_n}) \to \cdots .\end{aligned} \tag{3.14}$$

Under the identification

$$H^k(S, \mathcal{O}_S) \simeq H^k(S_1, \mathcal{O}_{S_1}) \oplus \cdots \oplus H^k(S_n, \mathcal{O}_{S_n}), \ \forall k \geq 0,$$

the map:

$$s : H^0(X_p, \mathcal{O}_{X_p}) \oplus H^0(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^0(X_n, \mathcal{O}_{X_n}) \to H^0(S, \mathcal{O}_S)$$

is given by $(s_p, s_1, \ldots, s_n) \mapsto (s_p - s_1, \ldots, s_p - s_n)$, and is obviously surjective.

Since, as discussed in Theorem 3.3, the structural sheaf of a rationally connected manifold has no higher cohomology, it follows that:

$$H^k(X_p, \mathcal{O}_{X_p}) = H^k(X_i, \mathcal{O}_{X_i}) = H^k(S_i, \mathcal{O}_{S_i}) = 0,$$

for $k = 1, 2$ and $i = 1, \ldots, n$ and the lemma follows now from a simple inspection of (3.14). □

Corollary 3.17. $R^1\Phi_*\mathcal{O}_{\mathcal{X}} = R^2\Phi_*\mathcal{O}_{\mathcal{X}} = 0.$

Proof. Since Φ is proper and flat, and we have just proved that $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = 0$, from the semi-continuity theorem it follows that

$$R^1\Phi_*\mathcal{O}_{\mathcal{X}} = R^2\Phi_*\mathcal{O}_{\mathcal{X}} = 0.$$

Of course, we shrink Δ whenever necessary. □

Theorem 3.18. *Any line bundle $\mathcal{L}_{X_0} \in \mathrm{Pic}(X_0)$ extends to a line bundle over $\mathcal{X}$.*

Proof. Applying Φ_* over the exponential sequence

$$0 \to \mathbb{Z}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}^*_{\mathcal{X}} \to 0$$

we obtain

$$0 = R^1\Phi_*\mathcal{O}_{\mathcal{X}} \to R^1\Phi_*\mathcal{O}^*_{\mathcal{X}} \to R^2\Phi_*\mathbb{Z}_{\mathcal{X}} \to R^2\Phi_*\mathcal{O}_{\mathcal{X}} = 0.$$

Hence $R^1\Phi_*\mathcal{O}^*_{\mathcal{X}} \simeq R^2\Phi_*\mathbb{Z}_{\mathcal{X}}$, and passing to global sections we get

$$\mathrm{Pic}(\mathcal{X}) \simeq H^2(\mathcal{X}, \mathbb{Z}).$$

However, X_0 is a deformation retract of $\mathcal{X}$, and therefore

$$H^2(X_0, \mathbb{Z}) \simeq H^2(\mathcal{X}, \mathbb{Z}).$$

But, from Lemma 3.16 and the exponential sequence we can see that

$$\mathrm{Pic}(X_0) \simeq H^2(X_0, \mathbb{Z}),$$

and the proposition follows. □

The following proposition is well-known. It is the key that allows us to solve the ampleness issue discussed in Remark 2.10. For its proof we refer the interested reader to [KoMo98, Proposition 1.41].

Proposition 3.19. *Let $f : X \to Y$ be a proper morphism, and D a Cartier divisor on X. Let $y \in Y$ be a point and X_y the fiber of f over y. If $\mathcal{O}_{X_y}(D)$ is ample, then D is ample over some open set $U \ni y$ of Y.*

3.4. **Intersection Theory I.** In this section we set up the intersection theory of the central fiber X_0 of $\Phi : \mathcal{X} \to \Delta$. To do this, we will adopt a cohomological approach, as discussed in Appendix A. In our situation we need the following:

Proposition 3.20. *If $L_0^i = (L_p^i, L_1^i, \dots L_n^i),\ i = 1, 2, 3$ are three line bundles on X_0, then*

$$L_0^1 \cdot L_0^2 \cdot L_0^3 = L_p^1 \cdot L_p^2 \cdot L_p^3 + \sum_{k=1}^{n} L_k^1 \cdot L_k^2 \cdot L_k^3.$$

Proof. Let $L_0 = (L_p, L_1, \dots L_n)$ be an arbitrary line bundle on X_0. Of course, $L_p \in \operatorname{Pic}(X_p),\ L_i \in \operatorname{Pic}(X_i)$ for $i = 1, \dots, n$, with the property that:

$$L_{S_i} \overset{\text{def}}{=} L_{p|S_i} \simeq L_{i|S_i}.$$

Tensoring the Mayer-Vietoris sequence (3.14) by $L_0^\vee$ we get:

$$0 \to L_0^\vee \to L_p^\vee \oplus L_1^\vee \oplus \cdots \oplus L_n^\vee \to L_{S_1}^\vee \oplus \cdots \oplus L_{S_n}^\vee \to 0 \tag{3.15}$$

From (3.15) we immediately obtain:

$$\chi(-L_0) = \chi(-L_p) + \sum_{k=1}^{n} \chi(-L_k) - \sum_{k=1}^{n} \chi(-L_{S_k}). \tag{3.16}$$

Now using (3.16) in (A.1) we see that

$$L_0^1 \cdot L_0^2 \cdot L_0^3 = L_p^1 \cdot L_p^2 \cdot L_p^3 + \sum_{k=1}^{n} L_k^1 \cdot L_k^2 \cdot L_k^3 - \sum_{k=1}^{n} L_{S_k}^1 \cdot L_{S_k}^2 \cdot L_{S_k}^3.$$

But, since S_k is a smooth surface $L_{S_k}^1 \cdot L_{S_k}^2 \cdot L_{S_k}^3 = 0$, and the conclusion follows. □

Corollary 3.21. *Let $\mathcal{L}^i,\ i = 1, 2, 3$ be three line bundles on $\mathcal{X}$, and denote by L_t^i their restriction to the fiber $\Phi^{-1}(t)$. If $L_0^i = (L_p^i, L_1^i, \dots L_n^i),\ i = 1, 2, 3,$ then:*

$$L_t^1 \cdot L_t^2 \cdot L_t^3 = L_p^1 \cdot L_p^2 \cdot L_p^3 + \sum_{k=1}^{n} L_k^1 \cdot L_k^2 \cdot L_k^3,\ \forall t \in \Delta.$$

Proof. Let $\mathcal{L}$ be an arbitrary line bundle on $\mathcal{X}$. We will denote by L_t its restriction to $\Phi^{-1}(t)$ an arbitrary fiber of Φ. Since the Euler characteristic is constant in flat families, we have:

$$\chi(L_0) = \chi(L_t),\ \forall t \in \Delta. \tag{3.17}$$

From (A.1), (3.17) and the above proposition, the result follows. □

Remark 3.22. The proposition and its corollary are nothing but the three dimensional version of [Per77, Proposition 2.4.1]. There, U. Persson sets up the intersection theory on the central fiber of degenerations of surfaces for extendable line bundles. However, our cohomological approach is different than his topological approach.

3.5. **Construction of the line bundle.** Recall now the initial package. We have a smooth threefold X containing a smooth curve C, with $K_X \cdot C \geq 0$, and let $p : X_C \to X$ be the blowing-up of X along C. On X_C we have an ample line bundle H_{X_C} such that $K_{X_C} \cdot H^2_{X_C} < 0$.

We want to find an ample line bundle H_0 on X_0 such that $K_{X_0} \cdot H^2_{X_0} < 0$. What we are going to do is to start with the line bundle on the main component X_p, and by studying its restriction to S_i we will extend it across X_0.

Remember that $f_p : X_p \to X$ is gotten by first blowing-up X along the curves $L_1, \ldots, L_n$, followed by blowing-up $\bar{C}$, the strict transform of C. We denoted the exceptional divisors by $S_1, \ldots, S_n$, and $E_{\bar{C}}$, respectively. The S_i's can be either viewed as blowing-ups of E_i, the exceptional divisors of X along the L_i's, at one point or as blowing-ups of E_i^* at one point. The components X_i of X_0 are pairwise disjoint and meet X_p transversally along S_i. For the moment, we pick an arbitrary line bundle H_p on X_p.

To simplify the exposition, we introduce the following notations:

- $d_i = \deg_{L_i}(N_{L_i/X}) > 2$, since L_i is a very free rational curve;
- $p_i : S_i \to E_i$ is the blowing-up of E_i at $P_i = \bar{C} \cap E_i$ with e_i its exceptional divisor.
- k_i be the strict transform in S_i of the fiber of the ruled surface E_i that passes through P_i;
- $q_i : S_i \to E_i^*$ is the contraction of k_i;
- $h_i = \mathcal{O}_{E_i}(1),\ h_i^* = \mathcal{O}_{E_i^*}(1)$;
- f denotes the generic fiber of either E_i or E_i^*. We will still abusively denote by f its strict transform in S_i.
- $c_1 = \det(N^{\vee}_{E_i^*/\mathcal{X}_\mathcal{C}}) = h_i^* + f$;
- $\xi_i = \mathcal{O}_{X_i}(1)$.

What we do next is to find the restriction of H_p to S_i, for each $i = 1, \ldots, n$. For convenience we will denote the restriction of H_p to S_i by H_{S_i}.

Since $\mathrm{Pic}(S_i) \simeq q_i^* \,\mathrm{Pic}(E_i^*) \oplus \mathbb{Z}[k_i]$, we can write:

$$H_{S_i} = q_i^* M_i - x_i k_i, \tag{3.18}$$

for some line bundle $M_i \in \mathrm{Pic}(E_i^*)$, and some integer x_i.

To construct the line bundle on X_i, recall that

$$X_i = \mathbb{P}(N^{\vee}_{E_i^*|\mathcal{X}_C}) \xrightarrow{f_i} E_i^*$$

is the exceptional divisor of the blowing-up of $\mathcal{X}$ along E_i^*, and X_i intersects transversally X_p. This transversality can be used, as in [Per77, page 41], to see that

$$N_{X_i/\mathcal{X}_{|S_i}} = N_{S_i/X_p}.$$

In our notations and using Lemma 3.13 this translates into:

$$\xi_{i|S_i} = p_i^* h_i = q_i^* h_i^* + q_i^* f - k_i. \tag{3.19}$$

We can write now:

$$H_{S_i} = q_i^*(M_i - x_i h_i^* - x_i f) + x_i \xi_{i|S_i}. \tag{3.20}$$

Thus, if we let $N_i = M_i - x_i h_i^* - x_i f = M_i - x_i c_1$ we can see that we can write H_{S_i} as

$$H_{S_i} = q_i^* N_i + x_i \xi_{i|S_i}. \tag{3.21}$$

With this is now clear that, we have a unique choice for H_i, namely:

$$H_i = f_i^* N_i + x_i \xi_i. \tag{3.22}$$

As can be easily seen from the above consideration, the choice of the line bundle $H_p \in \mathrm{Pic}(X_p)$ is not necessary to extend it across X_0. Thus, we have just proved:

Proposition 3.23. *Let H_p be an arbitrary line bundle on X_p. Then H_p extends uniquely to a line bundle H_0 on X_0.*

For furher considerations, we need to show the ampleness of the line bundles H_i, provided the ampleness of H_p. Before we proceed we need the following:

Lemma 3.24. $\mathcal{O}_{X_i}(S_i) = \xi_i$.

Proof. In additive notations, we can write $\mathcal{O}_{X_i}(S_i) = a\xi_i + bf_i^*h_i^* + cf_i^*\mathcal{O}_{E_i^*}(f)$, with $a,\ b,\ c \in \mathbb{Z}$. Restricting to S_i, and using (3.19) we obtain:

$$\begin{aligned}\mathcal{O}_{S_i}(S_i) &= a\xi_{i|S_i} + bq_i^*h_i^* + cq_i^*\mathcal{O}_{E_i^*}(f)\\ &= (a+b)q_i^*h_i^* + (a+c)q_i^*\mathcal{O}_{E_i^*}(f) - ak_i.\end{aligned} \tag{3.23}$$

On the other hand, the restriction of $\mathcal{O}_{X_i}(S_i)$ to S_i is the normal bundle N_{S_i/X_i}. But, from the transversality of the intersection of X_i with X_p we have [Per77, page 41], $N_{S_i/X_i} \simeq N^\vee_{S_i/X_p} = p_i^*h_i$, and so:

$$\mathcal{O}_{S_i}(S_i) = p_i^*h_i = q_i^*h_i^* + q_i^*\mathcal{O}_{E_i^*}(f) - k_i$$

Comparing with (3.23) in $\mathrm{Pic}(S_i)$, we can immediately see that $a = 1$ and $b = c = 0$. □

For the computations involved in proving the ampleness of the H_i's, we will work with the more traditional basis of $\mathrm{Pic}(E_i^*)$ described in Appendix C. $\mathrm{Pic}(E_i^*)$ is generated by $\{\mathcal{O}_{E_i^*}(C_0),\ \mathcal{O}_{E_i^*}(f)\}$, where $\mathcal{O}_{E_i^*}(C_0)$ is the line bundle associated to the section of E_i^* of minimal self-intersection, and f is the class of a fiber. Since E_i^* is rational, $C_0^2 = -e \le 0$. It is easy to see that $h_i^* = C_0 - \frac{d_i+1-e}{2}f$. Using this basis, any line bundle $L \in \mathrm{Pic}(X_i)$ can be written as $L = a\xi_i + bf_i^*\mathcal{O}_{E_i^*}(C_0) + cf_i^*\mathcal{O}_{E_i^*}(f)$, where $a,\ b,\ c \in \mathbb{Z}$. We have:

Proposition 3.25. *Any line bundle $L \in \mathrm{Pic}(X_i)$ which admits sections, can be written as*

$$L = \lambda(\xi_i - f_i^*c_1) + \beta f_i^*\mathcal{O}_{E_i^*}(C_0) + (\gamma + \beta e)f_i^*\mathcal{O}_{E_i^*}(f),$$

with $\lambda,\ \beta,\ \gamma \ge 0$.

Proof. Let $L = a\xi_i + bf_i^*\mathcal{O}_{E_i^*}(C_0) + cf_i^*\mathcal{O}_{E_i^*}(f)$ be an arbitrary line bundle on X_i. For simplicity, we will abusively denote $\mathcal{O}_{E_i^*}(C_0)$ by C_0 and $\mathcal{O}_{E_i^*}(f)$ by f. If L has sections, its restriction to an arbitrary fiber f of f_i has sections. But $L_{|f} \simeq \mathcal{O}_{\mathbb{P}_1}(a)$, and so $a \ge 0$.

Next we restrict our line bundle to S_i. Using again (3.19) have:

$$\begin{aligned}L_{|S_i} &= a\xi_{i|S_i} + bf_i^*C_{0|S_i} + cf_i^*f_{|S_i}\\ &= a(q_i^*h_i^* + q_i^*f - k_i) + bq_i^*C_0 + cq_i^*f\\ &= (a+b)q_i^*C_0 + (c + a - a\frac{d_i+1-e}{2})q_i^*f - ak_i.\end{aligned} \tag{3.24}$$

Now, if L has sections, $L_{|S_i}$ restricted to f, the strict transform in S_i of a generic fiber of E_i^*, still has sections. But this can happen only if $a + b \ge 0$. We restrict now to the strict transform $\bar{C}_0$ of C_0. Since E_i^* is the elementary transform of E_i, $\bar{C}_0$ will intersect k_i transversally at one point. If we assume that L has sections, it will follow as before that $c - a\frac{d_i+1-e}{2} - (a+b)e \ge 0$. By a change of notation, if we let $\lambda = a \ge 0,\ \beta = a + b \ge 0$ and $\gamma = c - a\frac{d_i+1-e}{2} - (a+b)e \ge 0$, we see that we can write any line bundle $L \in \mathrm{Pic}(X_i)$ with sections as:

$$L = \lambda(\xi - f_i^*h_i^*) + \beta f_i^*C_0 + (\gamma + \beta e)f_i^*f,$$

where $\lambda,\ \beta,\ \gamma \ge 0$. □

Remark 3.26. One could study if the necessary conditions we found are also sufficient for line bundles on X_i to admit sections. However, the result we proved is enough for our purpose.

Lemma 3.27. *If H_p is ample, then the line bundles H_i constructed above are ample for any $i = 1, \dots n$.*

Proof. **Step 1:** We show that for any $i = 1, \dots n$, the line bundle $N_i = M_i - x_i(h_i^* + f) \in \mathrm{Pic}(E_i^*)$ is ample.

Writing the line bundle $M_i = ah_i^* + bf \in \mathrm{Pic}(E_i^*)$ $a,\ b \in \mathbb{Z}$ in our preferred basis $\{C_0,\ f\}$, we see that

$$M_i - x_i(h_i^* + f) = (a - x_i)C_0 + [(b - x_i) - (a - x_i)\frac{d_i + 1 - e}{2}]f.$$

To check the ampleness of this this line bundle, we use the conditions of Proposition C.3. First, if in (3.18) we take the intersection with the fiber of E_i^* passing through the center of the blowing-up, we immediately see that $a - x_i > 0$. The other condition to check is:

$$(b - x_i) - (a - x_i)\frac{d_i + 1 - e}{2} > (a - x_i)e \Longleftrightarrow (b - x_i) > (a - x_i)\frac{d_i + 1 + e}{2} \quad (3.25)$$

Now, using the Nakai-Moishezon criterion is easy to check that since $H_{S_i} = q_i^* M_i - x_i k_i$ is ample, then $M_i \in \mathrm{Pic}(E_i^*)$ is ample, too. Proposition C.3 yields that:

$$b > a\frac{d_i + 1 + e}{2}, \quad (3.26)$$

which at its turn implies (3.25).

Step 2: We show that H_i is strictly nef. Let $C \subset X_i$ be an arbitrary irreducible curves. Depending on the position of the curve C with respect to the surface $S_i \subset X_i$ we distinguish three cases.

(1) $C \subseteq S_i$. In this case the intersection pairing can be computed on S_i, where

$$H_i \cdot C = H_{S_i} \underset{S_i}{\cdot} C > 0,$$

since H_{S_i} is ample.

(2) $C = f$ is a fiber of $X_i \to E_i^*$. Then

$$H_i \cdot C = (f_i^* N_i + x_i \xi_i) \cdot f = x_i(\xi_i \cdot f) = x_i > 0.$$

(3) When $C \nsubseteq S_i$, using Lemma 3.23, we have:

$$H_i \cdot C = (f_i^* N_i + x_i \xi_i) \cdot C = f_i^* N_i \cdot C + x_i S_i \cdot C \geq f_i^* N_i \cdot C > 0,$$

as N_i is ample and C is not contained in a fiber of f_i.

Step 3: We show that H_i is ample. Using the Nakai-Moishezon, all we need to prove are the following:

(1) $H_i^2 \cdot S > 0$ for any irreducible surface $S \subseteq X_i$. Using Lemma 3.25, we can write the line bundle associated to S as $\lambda(\xi - f_i^* h_i^*) + \beta f_i^* C_0 + (\gamma + \beta e) f_i^* f$, where $\lambda,\ \beta,\ \gamma \geq 0$, are not simultaneously zero. We have:

$$\begin{aligned} H_i^2 \cdot S &= (f_i^* N_i + x_i \xi_i)^2 \cdot (\lambda(\xi - f_i^* h_i^*) + \beta f_i^* C_0 + (\gamma + \beta e) f_i^* f) \\ &= \lambda (f_i^* N_i + x_i \xi_i)^2 \cdot (\xi - f_i^* h_i^*) + \beta (f_i^* N_i + x_i \xi_i)^2 \cdot f_i^* C_0 \\ &\quad + (\gamma + \beta e)(f_i^* N_i + x_i \xi_i)^2 \cdot f_i^* f. \end{aligned} \quad (3.27)$$

We compute now separately the terms involved.

$$\begin{aligned}(f_i^*N_i + x_i\xi_i)^2 \cdot f_i^*f =&(f_i^*N_i \cdot f_i^*N_i + 2x_i\xi_i \cdot f_i^*N_i + x_i^2\xi_i^2) \cdot f_i^*f \\ &= 2x_i\xi_i \cdot f_i^*N_i \cdot f_i^*f + x_i^2\xi_i^2 \cdot f_i^*f \\ &= 2x_i(N_i \cdot f) + x_i^2(f \cdot c_1) \\ &= 2x_i[(M_i - x_ic_1) \cdot f] + x_i^2[f \cdot (h_i^* + f)] \\ &= 2ax_i - x_i^2 > 0, \end{aligned} \tag{3.28}$$

since $a > x_i$.

$$\begin{aligned}(f_i^*N_i + x_i\xi_i)^2 \cdot f_i^*C_0 &= (f_i^*N_i \cdot f_i^*N_i + 2x_i\xi_i \cdot f_i^*N_i + x_i^2\xi_i^2) \cdot f_i^*C_0 \\ &= 2x_i\xi_i \cdot f_i^*N_i \cdot f_i^*C_0 + x_i^2\xi_i^2 \cdot f_i^*C_0 \\ &= 2x_i(N_i \cdot C_0) + x_i^2(C_0 \cdot c_1) \\ &= x_i(N_i \cdot C_0) + x_i(M_i \cdot C_0), \end{aligned} \tag{3.29}$$

as N_i and M_i ar both ample line bundles on E_i^*. Finally,

$$\begin{aligned}(f_i^*N_i + x_i\xi_i)^2 \cdot (\xi - f_i^*h_i^*) &= (f_i^*N_i + x_i\xi_i)^2 \cdot \xi - (f_i^*N_i + x_i\xi_i)^2 f_i^*h_i^* \\ &= H_{S_i}^2 - 2x_i(N_i \cdot h_i^*) - x_i^2(h_i^*)^2 \\ &= M_i^2 - 2x_i(M_i \cdot h_i^*) + x_i^2(h_i^*)^2 + x_i^2. \end{aligned} \tag{3.30}$$

To show that this last term is positive we use (3.26) to get:

$$\begin{aligned}&M_i^2 - 2x_i(M_i \cdot h_i^*) + x_i^2(h_i^*)^2 + x_i^2 \\ &= (ah_i^* + bf)^2 - 2x_i[(ah_i^* + bf) \cdot h_i^*] + x_i^2(h_i^*)^2 + x_i^2 \\ &= 2b(a - x_i) - a^2(d_i + 1) + 2ax_i(d_i + 1) - d_ix_i^2 \\ &> a(a - x_i)(d_i + 1) - a^2(d_i + 1) + 2ax_i(d_i + 1) - d_ix_i^2 \\ &= ax_i(d_i + 1) - d_ix_i^2 > 0, \end{aligned} \tag{3.31}$$

since $a > x_i$.

From (3.27),(3.28),(3.29), (3.30), (3.31), and since $\lambda,\ \beta$ and γ cannot simultaneously vanish, this step is concluded.

(2) $H_i^3 > 0$. This follows again by a direct computation and Step 1.

$$\begin{aligned}H_i^3 &= (f_i^*N_i + x_i\xi_i)^3 \\ &= [(f_i^*N_i + x_i\xi_i)^2 \cdot f_i^*N_i] + x_i[\xi_i \cdot (f_i^*N_i + x_i\xi_i)^2] \\ &= 2x_i[\xi_i \cdot (f_i^*N_i)^2] + x_i^2(\xi_i^2 \cdot f_i^*N_i) + x_iH_{S_i}^2 \\ &= 2x_i(N_i \cdot N_i) + x_i^2(N_i \cdot c_1) + x_iH_{S_i}^2 \\ &= x_i(N_i \cdot N_i) + x_i(N_i \cdot M_i) + x_iH_{S_i}^2 > 0, \end{aligned} \tag{3.32}$$

since $x_i > 0$, and the line bundles $M_i,\ N_i \in \mathrm{Pic}(E_i^*)$ are ample.

□

3.6. **Intersection Theory II.** In this subsection we will make a choice for the line bundle H_p used throughout the previous section. Assuming the validity of Conjecture 0.3, noticing again that X_p is the flop of $X_{C,\bar{L}}$, for which obviously $\mathcal{P}_{X_{C,\bar{L}}}$ holds true, it will follow that $\mathcal{P}_{X_p}$ also holds true. That is X_p is endowed with ample line bundle H_p with $K_{X_p} \cdot H_{X_p}^2 < 0$. We conclude our arguments of this entire section, by proving the the following:

Proposition 3.28. *If* $H_{X_0} = (H_p, H_1, \dots, H_n)$ *is the line bundle constructed above,*

$$K_{X_0} \cdot H_{X_0}^2 < 0.$$

Proof. Using Proposition A.3 and Corollary 3.20 we compute now $K_{X_0} \cdot H_{X_0}^2$:

$$\begin{aligned} K_{X_0} \cdot H_{X_0}^2 =&(K_{X_p} + S_1 + \dots + S_n) \cdot H_p^2 + \sum_{i=1}^{n} (K_{X_i} + S_i) \cdot H_i^2 \\ =&K_{X_p} \cdot H_p^2 + \sum_{i=1}^{n} (K_{X_i} \cdot H_i^2 + 2H_{S_i}^2). \end{aligned} \tag{3.33}$$

We compute separately $K_{X_i} \cdot H_i^2$. From Proposition C.1 we have that $K_{X_i} = f_i^* L_i - 2\xi_i$, where

$$\begin{aligned} L_i =&K_{E_i^*} + c_1 = -2h_i^* - (d_i + 3)f + h_i^* + f \\ =&-h_i^* - (d_i + 2)f = -c_1 - (d_i + 1)f. \end{aligned} \tag{3.34}$$

A straightforward computation using Proposition C.1 again yields:

$$\begin{aligned} K_{X_i} \cdot H_i^2 =&(f_i^* L_i - 2\xi_i) \cdot (f_i^* N_i + x_i \xi_i)^2 \\ =&(f_i^* L_i - 2\xi_i) \cdot (f_i^* N_i \cdot f_i^* N_i + 2x_i f_i^* N_i \cdot \xi_i + x_i^2 \xi_i^2) \\ =&2x_i f_i^* L_i \cdot f_i^* N_i \cdot \xi_i + x_i^2 f_i^* L_i \cdot \xi_i^2 \\ &- 2f_i^* N_i \cdot f_i^* N_i \cdot \xi_i - 4x_i f_i^* N_i \cdot \xi_i^2 - 2x_i^2 \xi_i^3 \\ =&2x_i (L_i \cdot N_i) + x_i^2 (L_i \cdot c_1) - 2(N_i \cdot N_i) \\ &- 4x_i (N_i \cdot c_1) - 2x_i^2 (c_1^2 - c_2). \end{aligned} \tag{3.35}$$

Making use of (3.34), we complete this computation as follows:

$$\begin{aligned} (L_i \cdot c_1) &= [(-c_1 - (d_i + 1)f) \cdot c_1] = -c_1^2 - (d_i + 1)(c_1 \cdot f) \\ &= d_i - 1 - d_i - 1 = -2; \\ (L_i \cdot N_i) &= [L_i \cdot (M_i - x_i c_1)] = (L_i \cdot M_i) - x_i (L_i \cdot c_1) = (L_i \cdot M_i) + 2x_i \\ (N_i \cdot N_i) &= [(M_i - x_i c_1) \cdot (M_i - x_i c_1)] = (M_i \cdot M_i) - 2x_i (M_i \cdot c_1) + x_i^2 c_1^2; \\ &= H_{S_i}^2 + x_i^2 - 2x_i (M_i \cdot c_1) + x_i^2 c_1^2; \\ (N_i \cdot c_1) &= [(M_i - x_i c_1) \cdot c_1] = (M_i \cdot c_1) - x_i c_1^2. \end{aligned} \tag{3.36}$$

From (3.35), we get:

$$\begin{aligned} K_{X_i} \cdot H_i^2 + 2H_{S_i}^2 =& 2H_{S_i}^2 + 2x_i (L_i \cdot M_i) + 2x_i^2 - 2x_i^2 - 2H_{S_i}^2 \\ &- 2x_i^2 + 4x_i (M_i \cdot c_1) - 2x_i^2 c_1^2 - 4x_i (M_i \cdot c_1) \\ &+ 4x_i^2 c_1^2 - 2x_i^2 c_1^2 + 2x_i^2 \\ =&2x_i (L_i \cdot M_i). \end{aligned} \tag{3.37}$$

It is not hard to see, from [Hart77] that on the ruled surface E_i^*, the line bundle $-L_i = h_i^* + (d_i + 2)f$ has a section which is also a section of E_i^*, (actually we can say much more, namely that $-L_i$ is ample) and since M_i is ample it follows immediately that

$$K_{X_i} \cdot H_i^2 + 2H_{S_i}^2 < 0.$$

Finally, from (3.33) we obtain:

$$K_{X_0} \cdot H_{X_0}^2 = K_{X_p} \cdot H_p^2 + \sum_{i=1}^{n} (K_{X_i} \cdot H_i^2 + 2H_{S_i}^2) < K_{X_p} \cdot H_p^2 < 0. \tag{3.38}$$

□

Corollary 3.29. *$\mathcal{P}_{X_0}$ holds true.*

Proof. The line bundle $H_{X_0} = (H_p, H_1, \ldots, H_n)$ has H_p ample from our assumption and, by Lemma 3.27, the H_i 's are also ample. Thus H_{X_0} is ample. With this, Proposition 3.28 actually says that $\mathcal{P}_{X_0}$ holds true. □

Remark 3.30. It would seem that the choice of the line bundle H_p is not important, as by attaching more curves to C we could arbitrarily decrease $K_{X_0} \cdot H_{X_0}^2$. However, this is not correct, as H_p depends on the number of curves we attach, which makes the intersection number $K_{X_0} \cdot H_{X_0}^2$ hard to control.

Remark 3.31. From the above considerations, it also follows that our Conjecture 0.3 is much stronger than what we need. The only thing we assumed here is the following special case:

Conjecture 3.32. $\mathcal{P}_{X_C} \Longrightarrow \mathcal{P}_{X_p}$

4. Proof of the Main Theorem

Proposition 4.1. *Let X be a rationally connected, projective threefold, and let X_C be the blowing-up of X along a smooth curve $C \subset X$, and assume the validity of Conjecture 0.3. Then $\mathcal{P}_{X_C}$ holds true if and only if $\mathcal{P}_X$ holds true.*

Proof. The implication $\mathcal{P}_X \Longrightarrow \mathcal{P}_{X_C}$ is the conclusion of Proposition 2.5.

Conversely, we have two cases.

• If $K_X \cdot C < 0$ the result follows from Proposition 2.6.

• If $K_X \cdot C \geq 0$ applying the results of the previous section we obtain a flat, proper family $\Phi : \mathcal{X} \to \Delta$ over the unit disk, such that:

1) $\mathcal{X}$ is smooth;
2) For any $t \in \Delta,\ t \neq 0,\ \Phi^{-1}(t) = X_{C_t}$, the blowing-up of X along a smooth curve with $K_X \cdot C_t < 0$;
3) The central fiber $X_0 = \Phi^{-1}(0)$ is a normal crossing divisor in $\mathcal{X}$ with smooth rationally connected components. Moreover, $\mathcal{P}_{X_0}$ holds true.

By eventually shrinking Δ, we know from Theorem 3.18 and Proposition 3.19 that H_{X_0} can be extended to a line bundle on $\mathcal{X}$, whose restriction $H_{X_{C_t}}$ to the fiber X_{C_t} is ample. Moreover, since $K_{X_0} \cdot H_{X_0}^2 < 0$, from Corollary 3.21 it follows that $K_{X_t} \cdot H_{X_{C_t}}^2 < 0$. Thus, $\mathcal{P}_{X_{C_t}}$ holds true.

Since for all $i = 1, \ldots, n,\ K_X \cdot L_i < 0$, by attaching sufficiently many such curves, $K_X \cdot C_0 < 0$ and by the conservation law we will have $K_X \cdot C_t < 0,$ for any $t \in \Delta$. Thus the blowing-up $X_{C_t} \to X$ is along the curve C_t for which $K_X \cdot C_t < 0$. Applying again Proposition 2.6 it follows that $\mathcal{P}_{X_{C_t}} \Longrightarrow \mathcal{P}_X$ and we are done. □

From the weak factorization theorem and Propositions 2.4 and 4.1, if we assume the validity of Conjecture 0.3, or of the weaker Conjecture 3.32, we easily get:

Theorem 4.2. *$\mathcal{P}$ is a birational property of rationally connected, projective threefolds.*

We can turn now to the main result of this paper:

Proof of Theorem A. If X is a rationally connected threefold, its Kodaira dimension is negative, and X is birational to a Mori fiber space, which we denote by X_{min}. Then, by Proposition 1.8, $\mathcal{P}_{X_{min}}$ holds true. Moreover, if X' is an appropriate desingularization of X_{min}, by Proposition 2.1 $\mathcal{P}_{X'}$ also holds true. Since X and X' ar smooth and birational, the conclusion of the Theorem A follows now from Theorem 4.2. □

APPENDIX A. INTERSECTION THEORY

Throughout this thesis we used the intersection theory algebraic varieties, from a cohomological viewpoint. This is introduced via the the following theorem due to Snapper [Kle66].

Theorem A.1. *Let X be a proper scheme over a field k, $\mathcal{F}$ a coherent sheaf on X, and let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be r Cartier divisors on X. Then the Euler-Poincaré function*

$$(m_1, \ldots, m_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes m_r})$$

takes the same values on $\mathbb{Z}^r$ as a polynomial with rational coefficients of degree at most the dimension of the support of $\mathcal{F}$.

Definition A.2. *Let $L_1, \ldots, L_r$ be Cartier divisors on a proper scheme X, $\mathcal{F}$ a coherent sheaf on X. Suppose $r \geq$ dim $Supp\ Z$. The intersection number of $L_1, \ldots, L_r$ with $\mathcal{F}$, denoted by*

$$(L_1 \cdots L_r \cdot \mathcal{F})$$

is the coefficient of $m_1, \cdots, m_r$ in the polynomial

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes m_r})$$

Notations: Let X be a proper scheme over a field $\mathbf{k}$. If $Z \subset X$ a closed subscheme of dimension s and $L_1, \ldots, L_r$ Cartier divisors on X, for any $r \geq s$ we will denote by $(L_1 \cdots L_r \cdot Z)$ the intersection number $(L_1 \cdots L_r \cdot \mathcal{O}_Z)$. If $L_1 = \cdots = L_d$ then we also use the notation $(L^d \cdot Z)$. When $Z = X$ we use $(L_1 \cdots L_d)$ if no confusion is likely.

The basic properties of intersection numbers [Kle66] are summarized in the following:

Proposition A.3. *The intersection numbers $(L_1 \cdots L_r \cdot \mathcal{F})$ are uniquely determined by the following properties:*

1) *The intersection number is an integer.*
2) *$(L_1 \cdots L_r \cdot Z)$ is symmetric and multilinear in the L_i.*
3) *$(L_1 \cdots L_r \cdot Z) = 0$ if dim $Z < r$.*
4) *For any exact sequence of coherent sheaves*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

we have

$$(L_1 \cdots L_r \cdot \mathcal{F}) = (L_1 \cdots L_r \cdot \mathcal{F}') + (L_1 \cdots L_r \cdot \mathcal{F}'').$$

5) *If j denotes the the inclusion $j : Z \to X$, we have*

$$(j^*L_1 \cdots j^*L_r \cdot Z)_Z = (L_1 \cdots L_r \cdot Z)_X.$$

6) *If D is an effective Cartier divisor and $L_r = \mathcal{O}_X(D)$, we have*

$$(L_1 \cdots L_r) = (L_1 \cdots L_{r-1} \cdot \mathcal{O}_D) = (L_1 \cdots L_{r-1} \cdot D).$$

7) *If $f : X' \to X$ is a morphism of finite degree, and $r \geq$ dim $X =$ dim X', then*

$$(L_1 \cdots L_r) = \deg(f)(f^*L_1 \cdots f^*L_r).$$

Corollary A.4. *We have:*

1) *If $L_1 \equiv 0$ then $(L_1 \cdots L_d \cdot Z) = 0$.*
2) *If $X \subset \mathbb{P}_N$ and $\mathcal{O}_X(L) = \mathcal{O}_X(1)$ then $(L^d \cdot Z) = \deg\ Z$.*
3) *If $f : X' \to X$ is a birational morphism, and $r \geq$ dim $X =$ dim X', then*

$$(L_1 \cdots L_r \cdot X) = (f^*L_1 \cdots f^*L_r \cdot X').$$

For our computations we use the following identity (see [Deb01, page 8]), which follows easily from Proposition A.3.4:

$$(L_1 \cdots L_r) = \sum_{I \subset \{1,\dots,r\}} \varepsilon_I \chi \left(X, -\sum_{i \in I} L_i \right) \tag{A.1}$$

where $\varepsilon_I = (-1)^{\mathrm{Card}(I)}$.

Appendix B. The Blowing Up

Of particular importance for the computations we performed was the multiplication table in the Chow ring of the blowing-up of threefolds. The results [GH78] we used are:

Proposition B.1. *Let X be the a smooth projective threefold, and let $p : X' \to X$ be the blow-up at a point. Let $E \simeq \mathbb{P}_2$ be the exceptional divisor, and f the class of a line of E in the Chow ring $A^*(X')$. Then*

$$A^*(X') = p^*A^*(X) \oplus \mathbb{Z} \cdot E \oplus \mathbb{Z} \cdot f$$

as an additive group. Moreover $p_*E = p_*f = 0$, and $p_*p^*A^*(X) = A^*(X)$. The multiplicative structure of $A^*(X')$ is determined by:

$$E^2 = -f,\ E^3 = -E \cdot f = 1,\ E \cdot p^*Z = f \cdot p^*Z = 0$$

for any cycle $Z \in A^*(X)$.

Proposition B.2. *Let X be the a smooth projective threefold, and let $p : X' \to X$ be the blow-up along a smooth curve $C \subset X$. Let $E \simeq \mathbb{P}(N^\vee_{C/X})$ be the exceptional divisor, and f the class of a fiber of E in the Chow ring $A^*(X')$. Then*

$$A^*(X') = p^*A^*(X) \oplus \mathbb{Z} \cdot E \oplus \mathbb{Z} \cdot f$$

as an additive group. Moreover $p_*E = p_*f = 0$, and $p_*p^*A^*(X) = A^*(X)$. The multiplicative structure of $A^*(X')$ is determined by:

$$\begin{aligned}
&E^2 = -p^*C + \deg_C(N_{C/X}) \cdot f,\ E^3 = -\deg_C(N_{C/X}),\ E \cdot f = -1,\\
&E \cdot p^*D = (C \cdot D)f,\ f \cdot p^*D = 0,\ \forall D \in A^1(X),\\
&E \cdot p^*Z = f \cdot p^*Z = 0,\ \forall Z \in A^2(X).
\end{aligned}$$

In addition, the usual relation holds:

$$\deg_C(N_{C/X}) = 2g(C) - 2 - K_X \cdot C,$$

where $g(C)$ is the genus of the curve C, and $N_{C/X}$ is the normal bundle of C in X.

APPENDIX C. RULED MANIFOLDS

The computations in Proposition 3.28 are based on the following:

Proposition C.1. *Let $X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} S$ be the projectivization of a rank 2 vector bundle $\mathcal{E}$ over a smooth surface S. Let $\xi = \mathcal{O}_X(1),\ c_1 = \det \mathcal{E}$, and $c_2 = c_2(\mathcal{E})$. We have:*

i) $f^*L \cdot f^*K \cdot f^*M = 0,\ \forall L, K, M \in \operatorname{Pic}(S)$;
ii) $f^*L \cdot f^*K \cdot \xi = (L \cdot K),\ \forall L, K \in \operatorname{Pic}(S)$;
iii) $f^*L \cdot \xi^2 = (L \cdot c_1),\ \forall L \in \operatorname{Pic}(S)$;
iv) $\xi^3 = c_1^2 - c_2$;
v) $K_X = f^*L - 2\xi$, *where* $L = K_S + c_1$.

Proof. $i)$ and $ii)$ are obvious, while for $iii)$ and $iv)$ we use the identity [GH85]:

$$\xi^2 - f^*c_1 \cdot \xi + f^*c_2 = 0.$$

Thus:

$$f^*L \cdot \xi^2 = f^*L \cdot f^*c_1 \cdot \xi - f^*L \cdot f^*c_2 = (L \cdot c_1);$$
$$\xi^3 = f^*c_1 \cdot \xi^2 - f^*c_2 \cdot \xi = c_1^2 - c_2.$$

$v)$ is also well-known. □

Similarly, for projectivizations of rank three vector bundles we needed the following:

Proposition C.2. *Let $X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} C$ be the projectivization of a rank 3 vector bundle $\mathcal{E}$ over a smooth curve C. Let $\xi = \mathcal{O}_X(1),\ c_1 = \det \mathcal{E}$, and $c_2 = c_2(\mathcal{E})$. We have:*

i) $f^*L \cdot f^*K \cdot M = 0,\ \forall L, K \in \operatorname{Pic}(C), M \in \operatorname{Pic}(X)$;
ii) $f^*L \cdot \xi^2 = \deg L,\ \forall L \in \operatorname{Pic}(C)$;
iii) $\xi^3 = \deg \mathcal{E}$;
iv) $K_X = f^*L - 3\xi$, *where* $L = K_S + c_1$.

Proof. As in the proof of the previous proposition, $i)$ and $ii)$ are obvious and $iv)$ is well known. For two we use again the identity [GH85]:

$$\xi^2 - f^*c_1 \cdot \xi = 0.$$

Multiplying by ξ, from $ii)$ we immediately get $\xi^3 = \deg \mathcal{E}$. □

Also, of particular importance are the classical ampleness conditions on ruled surfaces. For the following results, the interested reader is referred to [Hart77].

Recall that if $\pi : X \to C$ is a ruled surface, one can write $X \simeq \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a normalized, locally free, rank two sheaf on C. The integer $e = -\deg(\mathcal{E})$ is an invariant of X, and there is a section $\pi_0 : C \to X$ with image C_0, such that $\mathcal{L}(C_0) \simeq \mathcal{O}_X(1)$. Any element of $\operatorname{Pic}(X)$ can be written as $aC_0 + \pi^*\ell$ with $a \in \mathbb{Z}$ and $\ell \in \operatorname{Pic}(C)$, and any element of the $\operatorname{Num}(X)$ can be written $aC_0 + bf$ with $a, b \in \mathbb{Z}$ and where f is the class of a fiber of X. We denote by g the genus of the curve C.

Proposition C.3. *Let $\pi : X \to C$ be a ruled surface, and a divisor $D \equiv aC_0 + bf$.*

(a) *If $g = 0$, then $e \geq 0$, and D is ample if and only if $a > 0$ and $b > ae$.*
(b) *If $g > 0$, according to the the sign of e we have:*
  i) *If $e \geq 0,\ D$ is ample if and only if $a > 0,\ b > ae$.*
  ii) *If $e < 0,\ D$ is ample if and only if $a > 0,\ b > \frac{1}{2}ae$.*

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