QUALITATIVE ASPECTS OF COUNTING REAL RATIONAL CURVES ON REAL K3 SURFACES

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ABSTRACT. We study qualitative aspects of the Welschinger-like $\mathbb{Z}$-valued count of real rational curves on primitively polarized real $K3$ surfaces. In particular, we prove that with respect to the degree of the polarization, at logarithmic scale, the rate of growth of the number of such real rational curves is, up to a constant factor, the rate of growth of the number of complex rational curves. We indicate a few instances when the lower bound for the number of real rational curves provided by our count is sharp. In addition, we exhibit various congruences between real and complex counts.

"В числах есть нечто, чего в словах, даже крикнув их, нет."

Иосиф Бродский, Полдень в комнате.

[1]

INTRODUCTION

The discovery by J.-Y. Welschinger [23] of a deformation invariant $\mathbb{Z}$-valued count of real rational curves interpolating real collections of points on a real rational surface has allowed to respond in an affirmative way to the long standing problem of existence of real solutions in this enumerative problem. Moreover, the lower bound on the number of real solutions provided by the Welschinger invariants has happened be so powerful that it allowed [11] to disclose a remarkable new *phenomenon of abundance*: in the logarithmic scale, when the degree of curves is growing, the number of real solutions happens to be of about the same growth rate as the number of complex ones. Later on, similar abundance phenomena were observed in a few other, even more classical, enumerative problems, like enumerating linear subspaces on projective hypersurfaces (see [7] and references therein). All this originates further natural questions, which are essential for applications: *what are the asymptotic and arithmetical properties of the lower bounds provided*

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[1]"In the numbers there is something, which in the words, even having shouted them, is not." Joseph Brodsky, "Noon in a Room".

by such an invariant $\mathbb{Z}$-valued count; how non trivial and sharp are these lower bounds?

A response to these questions requires a comparison with the behavior of the numbers of solutions over the complex field, which in the above mentioned problems are given by some Gromov-Witten and Schubert numbers, respectively. Up to our knowledge, the corresponding aspects of the complex enumerative algebraic geometry are rarely treated in the literature: for related information we refer the interested reader to [8], [9], and [13]. The sharpness of Welschinger lower bounds is also little studied; here, we can cite only [24].

In our previous paper [15], we considered the problem of counting real rational curves on primitively polarized real $K3$ surfaces, introduced an appropriate invariant $\mathbb{Z}$-valued count and expressed the answer in a closed form, which can be viewed as a real version of the Yau-Zaslow formula (see Sect. 1 below). Our aim here is to show that thus obtained invariant lower bounds have similar peculiar asymptotic and arithmetic properties as those that were observed in the previously studied real enumerative problems, and to indicate some instances where the lower bounds are sharp.

The note is organized as follows. First of all we recall the precise statement of the real version of the Yau-Zaslow formula. Then, we start the qualitative analysis by relating our formula with the Dedekind eta-function and use one of Jacobi identities to establish some positivity property of the Welchinger invariants. In the next subsection we apply Hardy-Ramanujan-Uspensky results [10, 21] on the asymptotic behavior of the number of partitions to determine the asymptotic behavior of the Welschinger invariants in the logarithmic scale and to exhibit an abundance phenomenon. The third subsection is devoted to the comparison modulo $2, 3, 4$, and 8 of our Welschinger-type invariants, with the corresponding reduced Gromov-Witten invariants in the complex case, computed by the Yau-Zaslow formula. Finally, in the last subsection we apply Kulikov's type I and II degenerations [17] to establish the sharpness of our lower bounds for certain real deformation types of real $K3$ surfaces. A closing section contains a couple of concluding remarks. Some numerical data collected to illustrate the results obtained is shown in the table in the appendix.

1. Real version of the Yau-Zaslow formula

Let X be a generic real $K3$ surface admitting a complete real g-dimensional linear system of curves of genus g. If $g \geq 2$, assume, in addition, that X is of Picard rank 1 and the curves in the linear system belong to a primitive divisor class. Let c_g denote the number of complex rational curves in this linear system, and $w_g = n_+ - n_-$ the number of real rational curves in the same linear system counted with Welschinger sign; that is with the sign $+$, if the number of real solitary points is even, and with the sign $-$, otherwise (recall that by Chen's theorem [3, Theorem 1.1] the genericity assumption ensures that all the rational curves in our linear system are nodal).

The numbers c_g depend only on g and not on a specific choice of the surface X, and obey the Yau-Zaslow formula [25]

$$\sum_{g\geq 0} c_g q^g = \prod_{s\geq 1} \frac{1}{(1-q^s)^{e_{\mathbb{C}}}}, \tag{1.1}$$

where $e_{\mathbb{C}} = 24$ is the Euler characteristic of X. As we proved in [15], w_g depends only on the Euler characteristic $e_{\mathbb{R}}$ of the real part $X_{\mathbb{R}}$ of X, and for $e_{\mathbb{R}}$ fixed the generating function for w_g is as follows:

$$\sum_{g\geq 0} w_g q^g = \prod_{r\geq 1} \frac{1}{(1+q^r)^{e_{\mathbb{R}}}} \prod_{s\geq 1} \frac{1}{(1-q^{2s})^{\frac{e_{\mathbb{C}}-e_{\mathbb{R}}}{2}}}. \tag{1.2}$$

Note that $e_{\mathbb{R}}$ is always even and its values vary between -18 and 20 (for references and more details on the topology of real $K3$ surfaces, see [5]).

2. Analysis of the real version

2.1. **Positivity.**

Theorem 2.1. *For each fixed $e_{\mathbb{R}} < 0$, all w_g are positive and form a strictly increasing sequence $|e_{\mathbb{R}}| = w_1 < w_2 < w_3 < \dots$. For $e_{\mathbb{R}} = 0$, all w_g with odd g are zero, while those with even g are positive and form a strictly increasing sequence $12 = w_2 < w_4 < \dots$. For each fixed $e_{\mathbb{R}} > 0$, all the terms of the sequence $(-1)^g w_g$ are positive and form a strictly increasing sequence $e_{\mathbb{R}} = -w_1 < w_2 < -w_3 < \dots$.*

Proof. To prove these statements in the case $e_{\mathbb{R}} \leq 0$ it is sufficient to notice that the second product power series in (1.2) has nonzero coefficients only in even degrees and each of these coefficients is positive, while the first product power series has all coefficients positive as soon as $e_{\mathbb{R}} < 0$. The strict monotonicity claim follows immediately by noticing that multiplying positive power series with increasing sequence of coefficients yields a positive series with increasing sequence of coefficients.

Assume now that $e_{\mathbb{R}} > 0$. First, we rewrite our formula (1.2) in terms of the Dedekind eta-function

$$\eta(z) = e^{\pi i z/12} \prod_{n\geq 1} (1 - e^{2\pi i n z}) = q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n),$$

and of the modular discriminant

$$\Delta(z) = \eta^{24}(z) = q \prod_{n\geq 1} (1-q^n)^{24},$$

where z is in the upper half-plane, and $q = e^{2\pi i z}$. In this notation, formula (1.2) can be written as

$$\sum_{g\geq 0} w_g q^g = \frac{q}{\sqrt{\Delta(2z)}} \left(\frac{\eta^2(z)}{\eta(2z)}\right)^{\frac{e_{\mathbb{R}}}{2}} \tag{2.1}$$

For the above eta-quotient there is the following remarkable Gauss identity (see, for example, [16, Corollary 1.3]):

$$\frac{\eta^2(z)}{\eta(2z)} = 1 + 2\sum_{n\geq 1}(-1)^n q^{n^2}. \tag{2.2}$$

To finish the proof it is sufficient now to notice that the property to have nonpositive coefficients in odd degrees and nonnegative coefficients in even degrees is preserved under multiplication of power series (with such a property) and that the power series

$$\frac{q}{\sqrt{\Delta(2z)}} = \prod_{n\geq 1}(1-q^{2n})^{-12} \tag{2.3}$$

has positive coefficients in each even degree and zero coefficients in each odd degree. Another possible approach is to replace q by $-q$, which makes all the power series involved to have positive coefficients, and then to apply the same arguments as above in the case $e_{\mathbb{R}} < 0$. This argument yields the strict monotonicity claim. □

2.2. **Asymptopia.**

Lemma 2.2. *Let $\sum a_n q^n$ and $\sum b_n q^n$ be two power series with positive coefficients, and $\sum p_n q^n = (\sum a_n q^n)(\sum b_n q^n)$ the product power series. If, at a logarithmic scale, the coefficients a_n and b_n have the asymptotic behavior $\log a_n \sim (an)^\alpha$ and $\log b_n \sim (bn)^\alpha$, for some real constant $0 < \alpha < 1$, then $\log p_n \sim (cn)^\alpha$ where $c = (a^{\frac{\alpha}{1-\alpha}} + b^{\frac{\alpha}{1-\alpha}})^{\frac{1-\alpha}{\alpha}}$.*

Proof. The result follows from

$$\max_{0\leq k\leq n} \log a_k b_{n-k} \leq \log p_n \leq \log n + \max_{0\leq k\leq n} \log a_k b_{n-k}$$

and

$$\max_{0\leq k\leq n} \log a_k b_{n-k} \sim \left((a^{\frac{\alpha}{1-\alpha}} + b^{\frac{\alpha}{1-\alpha}})^{\frac{1-\alpha}{\alpha}} n \right)^\alpha .$$

To derive the latter relation it is sufficient to bound, from above and from below, the sequences a_n, b_n by sequences of the type $\exp(k_\pm + (1\pm\epsilon)(an)^\alpha)$ and $\exp(k_\pm + (1\pm\epsilon)(bn)^\alpha)$, respectively, and then let ϵ go to 0. The bounds needed here can be found by using the fact that $((a^{\frac{\alpha}{1-\alpha}} + b^{\frac{\alpha}{1-\alpha}})^{\frac{1-\alpha}{\alpha}} n)^\alpha$ is exactly the maximal value of the function $(at)^\alpha + (b(n-t))^\alpha$ on the interval $[0, n]$. □

Theorem 2.3. *The following asymptotic relations hold:*

i) *If $e_{\mathbb{R}} < 0$, then*

$$\log w_n \sim \pi \sqrt{\frac{4(e_{\mathbb{C}} - 3e_{\mathbb{R}})}{e_{\mathbb{C}}} \cdot n} \sim \sqrt{\frac{e_{\mathbb{C}} - 3e_{\mathbb{R}}}{4e_{\mathbb{C}}}} \log c_n.$$

ii) *If $e_{\mathbb{R}} > 0$, then*

$$\log |w_n| \sim 2\pi\sqrt{n} \sim \frac{1}{2}\log c_n.$$

iii) *If* $e_{\mathbb{R}} = 0$*, then*

$$\log w_{2n} \sim \frac{1}{2} \log c_{2n}.$$

Proof. Hardy-Ramanujan and Uspensky results [10, 21] on the the asymptotic behavior of the coefficients in the power series expansions

$$\prod_{n\geq 1}(1+q^n) = \prod_{n\geq 1} \frac{1}{1-q^{2n-1}} = \sum_{n\geq 1} Q(n)q^n \quad \text{and} \quad \prod_{n\geq 1} \frac{1}{1-q^n} = \sum_{n\geq 1} P(n)q^n$$

give us the following equivalence relations:

$$Q(n) \sim \frac{e^{\pi\sqrt{\frac{n}{3}}}}{4 \cdot 3^{\frac{1}{4}} \cdot n^{\frac{3}{4}}} \tag{2.4}$$

and

$$P(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}. \tag{2.5}$$

Thus, in the logarithmic scale, $\log P(n) \sim \log Q(2n)$ and $\log Q(n) \sim \pi\sqrt{n/3}$. Then Lemma 2.2 implies that $\log c_n \sim \pi\sqrt{e_{\mathbb{C}} \cdot \frac{2n}{3}} = 4\pi\sqrt{n}$.

If $e_{\mathbb{R}} < 0$, then the coefficients u_r in the power expansion $\sum u_r q^r$ of the first product in the formula (1.2) are positive, and, according to Lemma 2.2 and formula (2.4), they grow in the log-scale as $\pi\sqrt{\frac{-e_{\mathbb{R}} r}{3}}$. The coefficients v_r in the power expansion $\sum v_r q^r$ of the second product are vanishing in odd degree and they are positive in even degree. Lemma 2.2 and formula (2.5) imply that the coefficients v_{2s} grow in the log-scale as $\pi\sqrt{\frac{e_{\mathbb{C}}-e_{\mathbb{R}}}{2} \cdot \frac{2s}{3}}$. It follows that

$$\log w_{2n} \sim \pi\sqrt{\frac{e_{\mathbb{C}} - 3e_{\mathbb{R}}}{2} \cdot \frac{2n}{3}}.$$

Wherefrom, by monotonicity of the sequence w_n,

$$\log w_n \sim \pi\sqrt{\frac{e_{\mathbb{C}} - 3e_{\mathbb{R}}}{2} \cdot \frac{n}{3}} \sim \sqrt{\frac{e_{\mathbb{C}} - 3e_{\mathbb{R}}}{4e_{\mathbb{C}}}} \log c_n.$$

If $e_{\mathbb{R}} \geq 0$, then the proof is similar. This time we start from formula (2.1). Notice that according to formula (2.3) and Lemma 2.2, the coefficients of the first factor in the odd degrees are zero, while the coefficients in degree $2n$ grow in the log-scale as $\pi\sqrt{12 \cdot \frac{2n}{3}} \sim \frac{1}{2} \log c_{2n}$. Thus, there remains to notice that, due to formula (2.2), the coefficients of the second factor have polynomial growth, and, then, if $e_{\mathbb{R}} > 0$, to apply the monotonicity. □

Corollary 2.4. *The number* $r_g(X)$ *of real rational curves in the divisor class of the primitive polarization of* X *satisfies the following bounds:*

$$\phi(g) = |w_g| \leq r_g(X) \leq c_g = \psi(g),$$

where

$$\log \psi(g) = 4\pi\sqrt{g} + o(\sqrt{g}),$$

and, for a fixed $e_{\mathbb{R}} \neq 0$*,*

$$\log \phi(g) = 4\pi\rho\sqrt{g} + o(\sqrt{g})$$

with $\rho = \frac{1}{2}$ *if* $e_{\mathbb{R}} > 0$ *and* $\rho = \sqrt{\frac{e_{\mathbb{C}} - 3e_{\mathbb{R}}}{4e_{\mathbb{C}}}}$ *if* $e_{\mathbb{R}} < 0$*, while for* $e_{\mathbb{R}} = 0$ *it holds*

$$\log \phi(2g) = 2\pi\sqrt{2g} + o(\sqrt{g}).$$

Similar abundance of real solutions phenomena are observed in several other real enumerative problems, see [11], [12], [2], [6], [7]. There, like for $e_{\mathbb{R}} \geq 0$ in the above Corollary, a magic factor $1/2$ occurs in quite a few cases.

2.3. **Congruences.** The modularity of the generating series for the real and complex counting described in (1.1) and (1.2) allows us to exhibit noteworthy congruences that go much further than $w_g \equiv c_g \pmod 2$, straightforward from definitions.

Theorem 2.5. *The following congruences hold:*

i) *We have* $w_g \equiv c_g \pmod 2$ *for any* $g \geq 1$*, and* $w_g \equiv c_g \equiv 0 \pmod 2$ *for every* g *with* $g \not\equiv 0 \pmod 8$*.*
ii) *If* $e_{\mathbb{R}} \equiv 0 \pmod 4$ *then* $w_g \equiv c_g \pmod 4$ *for any* $g \geq 1$*, and if in addition* $g \not\equiv 0 \pmod 4$ *then* $w_g \equiv c_g \equiv 0 \pmod 4$*.*
iii) *If* $e_{\mathbb{R}} \equiv 0 \pmod 8$ *then* $w_g \equiv c_g \pmod 8$ *for any* $g \geq 1$*, and if in addition* $g \not\equiv 0 \pmod 2$ *then* $w_g \equiv c_g \equiv 0 \pmod 8$*.*
iv) *If* $e_{\mathbb{R}} \equiv 0 \pmod 3$ *and* $g \not\equiv 0 \pmod 3$ *then* $w_g \equiv c_g \equiv 0 \pmod 3$*.*

Proof. As a consequence of (2.2) and (2.3) we get

$$\sum_{g\geq 0} w_g q^g \equiv \prod_{n\geq 1} \frac{1}{(1-q^{2n})^{12}} \equiv \prod_{n\geq 1} \frac{1}{(1-q^{8n})^3} \pmod 2. \tag{2.6}$$

In particular, it implies that w_g is even for every g with $g \not\equiv 0 \pmod 8$.

Likewise,

$$\sum_{g\geq 0} c_g q^g = \prod_{n\geq 1} \frac{1}{(1-q^{n})^{24}} \equiv \prod_{n\geq 1} \frac{1}{(1-q^{8n})^3} \equiv \sum_{g\geq 0} w_g q^g \pmod 2. \tag{2.7}$$

Making an additional assumption that $e_{\mathbb{R}} \equiv 0 \pmod 4$, we get in a similar way

$$\begin{aligned}\sum_{g\geq 0} w_g q^g &\equiv \left(1 + e_{\mathbb{R}}\left(\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)\right) \prod_{n\geq 1} \frac{1}{(1-q^{2n})^{12}} \pmod 4\\ &\equiv \prod_{n\geq 1} \frac{1}{(1-q^{2n})^{12}} \pmod 4\\ &\equiv \sum_{g\geq 0} c_g q^g \pmod 4.\end{aligned}$$

We conclude that w_g and c_g are congruent modulo 4 for any g, and that they are divisible by 4 for each $g \geq 1$ with $g \not\equiv 0 \pmod 4$, as it follows from

$$(1-q^{2n})^{12} \equiv (1 + 2q^{4n} + q^{8n})^3 \pmod 4.$$

Assuming now that $e_{\mathbb{R}} \equiv 0 \pmod 8$, the same proof as in the previous case yields

$$\sum_{g\geq 0} w_g q^g \equiv \prod_{n\geq 1} \frac{1}{(1-q^{2n})^{12}} \equiv \sum_{g\geq 0} c_g q^g \pmod 8,$$

and so we conclude that $w_g \equiv c_g \pmod 8$ for any $g \geq 1$. Moreover, since

$$(1-q^{2n})^{12} \equiv (1+4q^{2n}+6q^{4n}+4q^{6n}+q^{8n})^3 \pmod 8,$$

we notice that $w_g \equiv c_g \equiv 0 \pmod 8$ for any $g \equiv 1 \pmod 2$.

If $e_{\mathbb{R}} \equiv 0 \pmod 3$, we get in a similar way

$$\sum_{g\geq 0} w_g q^g \equiv \left(1+\sum_{n=1}^{\infty}(-1)^{n+1}q^{3n^2}\right)^{\frac{e_{\mathbb{R}}}{6}} \prod_{n\geq 1} \frac{1}{(1-q^{6n})^4} \pmod 3$$

and

$$\sum_{g\geq 0} c_g q^g \equiv \prod_{n\geq 1} \frac{1}{(1-q^{3n})^8} \pmod 3.$$

We conclude that $w_g \equiv c_g \equiv 0 \pmod 3$ for each $g \geq 1$ with $g \not\equiv 0 \pmod 3$. $\square$

2.4. **Sharpness.** In [15, Sect 5.2] we observed that the lower bound for the count of real rational curves on primitively polarized $K3$ surfaces given by the absolute value of w_g is sharp for any g as soon as the surface has no real points. The reasoning is simple: there is no real nodal rational curve in the corresponding linear system when g is odd, since such a curve would have at least one real point among its singular points, which is impossible; while when g is even, the singular points of such a curve split into pairs of conjugate ones, and therefore the curve counts with positive Welschinger sign. Here we prove that the lower bound is optimal in a few more cases.

Theorem 2.6. *For any g, the lower bound on the number of real solutions by the absolute value of w_g is sharp for surfaces whose real locus is a torus or a pair of tori.*

Proof. First, we treat the case of a pair of tori and g odd. Let $\pi : Y \to \mathbb{P}^1 \times \mathbb{P}^1$ be the double covering ramified along a real nonsingular curve of bi-degree $(4,4)$ without real points (*cf.,* [22, 20]), where $\mathbb{P}^1 \times \mathbb{P}^1$ is considered with the standard product real structure (i.e., the hyperboloid). Such a double covering carries two real structures that differ by the automorphism of the covering, and we pick the one for which the real locus is formed by two tori. We denote by F_1 and F_2 the generators of $\mathbb{P}^1 \times \mathbb{P}^1$. The linear system $|F_1 + nF_2|$ embeds $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^{2n+1}$, while its pull-back $|\pi^*(F_1+nF_2)|$ provides a representation of Y as a hyperelliptic $K3$ surface in $\mathbb{P}^{2n+1}$. In such a representation the pull-back of hyperplane sections form a complete $2n+1$ dimensional linear system of curves of genus $g = 2n+1$. Finally, we take as X an embedding into $\mathbb{P}^{2n+1}$ of a real $K3$ surface obtained by a small generic real perturbation of Y (its existence follows from the period space description, see for example [20]). Notice now that $Y_{\mathbb{R}}$ consists of a pair of disjoint tori which are non-contractible in $\mathbb{P}^{2n+1}_{\mathbb{R}}$ since the real locus of the starting ruled surface is non-contractible in $\mathbb{P}^{2n+1}_{\mathbb{R}}$. This implies that $X_{\mathbb{R}}$ consists of a pair of disjoint, non-contractible tori as well. Hence, every real hyperplane section of $X_{\mathbb{R}}$ has at least 2 components, and thus can not be rational. This proves the sharpness claim, since according to formula (1.2) we have $w_g = 0$ for odd values of g, as soon as $e_{\mathbb{R}}$ is zero.

To treat the case of a pair of tori and g even, we consider a degeneration of a $K3$ to a double covering of the blown up projective plane. Namely, we start from $\mathbb{P}^2(1)$, the projective plane blown up at a real point, and consider a double covering $\varpi : Y \to \mathbb{P}^2(1)$ ramified along the proper transform of a one-nodal sextic with a real solitary point at the center of the blow-up and two ovals surrounding this solitary point. The standard real structure on $\mathbb{P}^2(1)$ lifts to two real structures on Y, and we pick the one for which the real locus consists of two tori. We embed now $\mathbb{P}^2(1)$ into $\mathbb{P}^{2n}$ by the linear system $|E + nF|$, where E is the exceptional divisor and F stands for the straight lines through the center of the blow-up. As above, we take $X \subset \mathbb{P}^{2n}$ to be a generic small real perturbation of Y. The $K3$ surface Y carries a natural real elliptic fibration given by the pull-back of the pencil of lines through the center of the blow-up. Since the starting sextic admits no real tangents passing through the node, all the singular fibers of this elliptic fibration are imaginary. This implies that every real rational hyperplane section of X, which as is known (*cf.,* [1, Proposition 4.1]) is a perturbation of the section, $\varpi^{-1}(E)$, and a collection of, possibly multiple, singular fibers, has no real singular points. Therefore, all the inputs into w_g in such a construction are positive, wherefrom the sharpness for this other particular case: $X_{\mathbb{R}}$ is a pair of tori and g is even, $g = 2n$. Notice that in this case $w_g > 0$ and it grows fast, as discussed in Theorems 2.1 and 2.3, respectively.

In the case of one torus and g even, we start again from $\mathbb{P}^2(1)$, the projective plane blown up at a real point, and consider a double covering $\varpi : Y \to \mathbb{P}^2(1)$ ramified in a proper transform of a one-nodal sextic with a real solitary point at the center of the blow-up and, this time, no other real points. As a real structure on Y, we select that lift of the standard real structure on $\mathbb{P}^2(1)$ for which the real locus of Y is a torus. We embed $\mathbb{P}^2(1)$ into $\mathbb{P}^{2n}$ by the linear system $|E + nF|$, where E is the exceptional divisor and F stands for the linear system of lines through the center of the blow-up. As above, we can assume that all the singular fibers of the associated elliptic fibration are imaginary. For that it is sufficient to make our sextic generic, since a generic nodal sextic has no double tangents passing through the node. Hence, we can argue as in the previous case, that is to use the pull-back $|\varpi^*(E + nF)|$ to represent the double covering under consideration by a hyperelliptic $K3$ surface in $\mathbb{P}^{2n}$, and then to take as X a generic small real perturbation of Y. Once more all the inputs in such a construction are positive. Wherefrom the sharpness for this other particular case: $X_{\mathbb{R}}$ is a torus and g is even, $g = 2n$. Notice again that in this case again $w_g > 0$ and it grows fast.

To construct sharp examples for the remaining case, a $K3$ surface in $\mathbb{P}^{2n+1}$ with a torus as the real locus, we start from a suitable type II degeneration of such a $K3$ surface [17]. Namely, we start, as in the case of a pair of tori, from the same real rational ruled surface $\mathbb{P}^1 \times \mathbb{P}^1 = \Sigma_0 \subset \mathbb{P}^{2n+1}$ and consider an elliptic normal curve E without real points that is cut on this ruled surface by a real rational ruled surface Σ_2 having empty real locus and intersecting Σ_0 transversally along E with $E^2 = -2$ on this Σ_2. To construct an explicit example, *cf.,* [4, Lemma 1, page 643], one can start from the elliptic curve $\mathbb{C}^2/(\mathbb{Z} + i\mathbb{Z})$ equipped with the real structure $z \mapsto \bar{z} + \frac{1}{2}$, embed it into $\mathbb{P}^{2n+1}$ by means of the linear system $n(A + B) + (\frac{1}{2} + A) + (\frac{1}{2} + B) \sim (n + 1)(A + B)$ taking $A = 0, B = \frac{i}{2}$, and choose as a scroll $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ containing E, the scroll corresponding to the hyperelliptic involution defined by the divisor $A + C$, $C = \frac{1}{2}$. Under such choices the scroll Σ_0 becomes real and has a torus as its real part, while the scroll corresponding to the divisor $A + B$ gives us a real Σ_2 with an empty real part, as required. After that, there remains to pick as X a small generic real perturbation of $\Sigma_0 \cup \Sigma_2$. The existence of a smooth such surface X is guaranteed by [4, Theorem 1, page 644]. The generic such surface X is primitively polarized [4, Theorem 2, page 646], and, as always due to the surjectivity of the period map, it can be deformed further to a surface generic in the sense of Chen's theorem [3], which guarantees that the rational curves in the primitive polarization are all nodal.

Under such a choice, X does not have any real rational hyperplane section. Indeed, if such a section exists, then by the compactness of Kontsevich's space of stable curves, there would exist a real projective connected nodal curve Z of arithmetic genus 0 and a real regular map $f : Z \to \Sigma_0 \cup \Sigma_2$ that realizes a hyperplane section of $\Sigma_0 \cup \Sigma_2$. On the other hand, the hyperplane sections of Σ_0 form the linear system $|F_1 + nF_2|$, where F_1, F_2 are generators of $\mathbb{P}^1 \times \mathbb{P}^1 = \Sigma_0$, while the hyperplane sections of Σ_2 form the linear system $|F_1' + (n + 1)F_2'|$, where F_1' stands for the (-2)-section and F_2' for the generator of Σ_2. The standard generators F_1, F_2 and the (-2)-section are complex conjugation invariant. Hence, every real rational map $f : Z \to \Sigma_0 \cup \Sigma_2$ as above, which represents a hyperplane section of $\Sigma_0 \cup \Sigma_2$, should have in its source, Z, an irreducible real component of type $F_1 + aF_2$ and an irreducible real component of type $F_1' + aF_2'$. Since Z is rational and connected, each two real components should be connected by a chain of real components. This is impossible for real components belonging one to Σ_0 and another to Σ_2, since any two such components intersect only at pairs of complex conjugated points. Such a contradiction ends the proof. □

3. Concluding remarks

3.1. **On asymptotics.** Hardy and Ramanujan have obtained not only an asymptotic approximation for coefficients $P(n), Q(n)$ (see formulas (2.4), (2.5)), but also a full asymptotic expansion, which later was even improved by Rademacher up to a convergent series expression. These can be applied to get similar expansions for w_g. We have restricted ourselves here to asymptotic approximations, since for our aim, to demonstrate the abundance phenomenon, it is not necessary to go further. Moreover, it would only obscure the presentation by a much heavier and lengthy analysis.

3.2. **On congruences modulo** 2**.** Table 3.4 below that gives the values of w_g and c_g for $g \leq 20$ shows that these values are odd if $g = 8$ and 16. Thus, it may give the impression that they should be odd for all $g \equiv 0 \pmod 8$ (*cf.,* Section 2.3). In fact, the situation is much different.

Let $\{i_n\}_{n\geq 0}$ be the parity sequence given by $i_n \equiv w_{8n} \equiv c_{8n} \pmod 2$, $n \geq 0$.

Proposition 3.1. *The sequence* $\{i_n\}_{n\geq 0}$ *contains infinitely many* 0*'s and infinitely many* 1*'s.*

Proof. As noticed in (2.6), the following identity holds

$$\sum_{n\geq 0} i_n q^{8n} \equiv \sum_{g\geq 0} w_g q^g \equiv \prod_{n\geq 1} \frac{1}{(1-q^{8n})^3} \pmod 2.$$

Moreover, we have

$$\prod_{n\geq 1} \frac{1}{(1-q^{8n})^3} \equiv \prod_{n\geq 1} \frac{(1-q^{16n})^2}{(1-q^{8n})} \prod_{n\geq 1} \frac{1}{(1-q^{16n})^3} \pmod 2.$$

Since

$$\prod_{n\geq 1} \frac{1}{(1-q^{16n})^3} \equiv \prod_{n\geq 1} \frac{1}{(1-q^{n})^{48}} \equiv \left(\sum_{n\geq 0} i_n q^{8n}\right)^2 \pmod 2,$$

we obtain

$$\sum_{n\geq 0} i_n q^{8n} \equiv \prod_{n\geq 1} \frac{(1-q^{16n})^2}{(1-q^{8n})} \left(\sum_{n\geq 0} i_n q^{16n}\right) \pmod 2. \tag{3.1}$$

Furthermore, from the Jacobi identity [16, Corollary 1.4]

$$\prod_{n\geq 1} \frac{(1-q^{16n})^2}{(1-q^{8n})} \equiv \prod_{n\geq 1}(1-q^{8n})^3 \equiv \sum_{n=0}^{\infty} q^{4n(n+1)} \pmod 2$$

we notice that the power series development of the term $\prod_{n\geq 1} \frac{(1-q^{16n})^2}{(1-q^{8n})}$ in (3.1) contains infinitely many odd and infinitely many even coefficients and the gaps between odd coefficients are growing quadratically.

Suppose now that the sequence $\{i_n\}_{n\geq 0}$ contains finitely many 1's. That would imply (in arithmetics over $\mathbb{Z}/2$) the equality between a non-zero polynomial and a product of an infinite series with a polynomial, which is impossible. Furthermore, if we suppose now that the sequence $\{i_n\}_{n\geq 0}$ contains finitely many 0's, to get a contradiction, it is sufficient to write $\sum_{n\geq 0} i_n q^{8n}$ as a sum of a polynomial with $\sum_{n\geq C} q^{8n}$, and to observe that the coefficients of $\sum_{n\geq 0} i_n q^{8n}$ in powers $4(4k+1)(4k+2)$ and $4(4k+2)(4k+3)$ have opposite parities for all k sufficiently large with respect to C, which is impossible. $\square$

3.3. **Other congruences.** In Section 2.3, we discussed congruences between the real and complex invariants, modulo primes and their powers that divide 24, i.e., congruences modulo $2, 3, 4$, and 8. In fact, when the powers of 2 and 3 do not divide 24, interesting congruences are still expected to occur. We discuss below some congruences modulo 9 and 16.

Proposition 3.2. *The following congruences hold:*

i) *If* $e_{\mathbb{R}} \equiv 0 \pmod 9$ *then* $w_g \equiv c_g \equiv 0 \pmod 9$ *for all* $g \equiv 4 \pmod 6$.
ii) *If* $e_{\mathbb{R}} \equiv 0 \pmod{16}$ *then* $w_g \equiv c_g \equiv 0 \pmod{16}$ *for all odd* $g > 1$.

Proof. We assume first that $e_{\mathbb{R}}$ is divisible by 9 (it happens with $e_{\mathbb{R}} = -18, 0$, and 18), and discuss the congruence of the two invariants modulo 9.

As Guo-Niu Han kindly pointed to us, the Jacobi identity [16, Corollary 1.4] shows that

$$\prod_{n\geq 1}(1-q^n)^3 = A(q^3) + 3qB(q^3),$$

where $A, B \in \mathbb{Z}[[q]]$, $A(0) = 1$, which implies that for any integer k,

$$\prod_{n\geq 1}(1-q^n)^{3k} = A_k(q^3) + 3qB_k(q^3) + 9q^2C_k(q^3), \tag{3.2}$$

for some power series $A_k, B_k, C_k \in \mathbb{Z}[[q]]$, $A_k(0) = 1$. In particular, when $k = -8$, it makes evident the congruence $c_g \equiv 0 \pmod 9$ for all $g \equiv 2 \pmod 3$.

To prove the claim for the real invariant, observe first that

$$\left(1 + 2\sum_{n\geq 1}(-1)^n q^{n^2}\right)^9 \equiv E(q^3) \pmod 9,$$

where $E \in \mathbb{Z}[[q]]$, $E(0) = 1$. This implies that

$$\left(1 + 2\sum_{n\geq 1}(-1)^n q^{n^2}\right)^{9\ell} \equiv E_l(q^3) \pmod 9,$$

for any $\ell \in \mathbb{Z}$, where $E_\ell \in \mathbb{Z}[[q]]$, $E_\ell(0) = 1$. Thus, using (2.2), (2.3), and (3.2), we find that

$$\sum_{g\geq 0} w_g q^g \equiv E_{e_{\mathbb{R}}/18}(q^3)\left(A_{-4}(q^6) + 3q^2B_{-4}(q^6) + 9q^4C_{-4}(q^6)\right) \pmod 9.$$

The multiplication of $A_{-4}(q^6) + 3q^2B_{-4}(q^6)$ by $E_{e_{\mathbb{R}}/18}(q^3)$ does not produce any term of exponent $6k+4$. Therefore, $w_g \equiv 0 \pmod 9$, for all g such that $g \equiv 4 \pmod 6$, and claim i) is proved.

Congruences modulo 16 for the complex invariants c_g can be detected by using Klein's modular function $j(z)$ and the classical formula

$$j(q) = \frac{1}{q} + \sum_{n=0}^{\infty} a(n)q^n = \frac{(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n)^3}{q\Pi_{n=1}^{\infty}(1-q^n)^{24}},$$

where $\sigma_3(n) = \sum_{d|n} d^3$. As $(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n)^3 \equiv 1 \pmod{16}$, we have

$$\sum_{g=0} c_g q^g \equiv qj(q) \pmod{16}. \tag{3.3}$$

On the other hand, according to Lehner [18], one has $a(2k) \equiv 0 \pmod{2^{11}}$, for all $k > 0$. Hence, c_g is divisible by 16 for all $g > 1$ with $g \equiv 1 \pmod 2$.

If $e_{\mathbb{R}} \equiv 0 \pmod{16}$ (it happens with $e_{\mathbb{R}} = 0, 16$, and -16), then the same arguments as in the proof of congruences modulo 4 and 8 show that

$$w_g \equiv \prod_{n \geq 1} \frac{1}{(1-q^{2n})^{12}} \pmod{16}.$$

This implies $w_g \equiv 0 \pmod{16}$ for any odd $g > 1$, and claim ii) is proved. □

Remark 3.3. *The congruence (3.3) holds modulo* 9 *as well, and* $a(3k) \equiv 0 \pmod{3^5}$ *for all* $k > 0$ *according to* [18]*. Hence* $c_g \equiv 0 \pmod 9$ *if* $g > 1$ *and* $g \equiv 1 \pmod 3$.

3.4. **On sharpness.** A couple of other instances of real $K3$ surfaces where the lower bound for the number of real rational curves given by $|w_g|$ is optimal were already pointed in our previous paper [15]. One such example was the case of Harnack surfaces of degree 4 in $\mathbb{P}^3$.

On the other hand, for real nonsingular $K3$ surfaces of degree 4 in $\mathbb{P}^3$ having the real locus consisting of 6 spheres and a sphere with 5 handles the lower bound given by $|w_g|$, which is equal to 48 in this special case, is not sharp.

Proposition 3.4. *The number of real rational hyperplane sections of a generic real* $K3$ *surface degree* 4 *in* $\mathbb{P}^3$ *is at least* 272 *if the real locus of the surface consists of* 6 *spheres and a sphere with* 5 *handles.*

Proof. The Euler characteristic $e_{\mathbb{R}}$ of the real part of such a surface is 4, and thus for such surfaces $w_g = -48$ (here $g = 3$). But, in fact, there are always at least 272 real rational hyperplane sections on such a surface. Indeed, each real line and each real plane intersect the component of genus 5, as such a component is not homotopy trivial in $\mathbb{RP}^3$ (see [14, Theorem II]). By Bézout theorem, it implies that each of the 6 spheres is convex (that is bound convex balls) and each 2 of them are contained in a convex set disjoint from other spheres. By genus argument, a real hyperplane through 3 of the 6 spheres does not intersect the 3 others, and thus these 3 spheres, as any 3 disjoint convex spheres in a real affine 3-space such that each 2 of them are contained in a convex set disjoint from the third, have 8 common supporting planes. Each of the supporting planes gives us a real rational curve with 3 solitary points, hence of Welschinger weight -1. Since the total number of supporting planes obtained in such a way is equal to $8 \times \binom{6}{3} = 160$ and $w_g = -48$, the total number of real rational curves is at least $160 + (160 - 48) = 272$. □

The same argument can be applied to other deformation classes of real nonsingular $K3$ surfaces of degree 4 in $\mathbb{P}^3$ whose real locus contains a non contractible component and $m \geq 3$ components contractible in $\mathbb{RP}^3$ (for a full deformation classification of real nonsingular $K3$ surfaces of degree 4 in $\mathbb{P}^3$ one can look at the survey [5] or at the original Nikulin's paper [19]; some of them have $e_{\mathbb{R}} = 0$ and are different from tori). More precisely, as above by Bézout theorem each of the contractible components is an affine convex sphere and each 2 of them are contained in a convex set disjoint of the third, while, as it is easy to show, any 3 affine convex spheres with such a property are contained in a common affine space. Thus, as above each 3 contractible components provide 8 tritangent planes and give an input -8 into w_g. In particular, for these kind of $K3$ surfaces, if $e_{\mathbb{R}} = 0$ we get $8 \times \binom{m}{3}$ as an improved lower bound.

Appendix

The table below is based on formulas (1.2) and (1.1). It provides the number of real rational curves counted with the Welschinger sign on primitively polarized $K3$ surfaces of degrees up to twenty, in the cases when $e_{\mathbb{R}} = 0, -18$, and 20. The last column gives, for comparison, the corresponding number of complex curves.

	Real Case			Complex Case
g	$e_{\mathbb{R}} = 0$	$e_{\mathbb{R}} = -18$	$e_{\mathbb{R}} = 20$	
0	1	1	1	1
1	0	18	-20	24
2	12	192	192	324
3	0	1536	-1200	3200
4	90	10152	5630	25650
5	0	58284	-21744	176256
6	520	299776	73600	1073720
7	0	1410048	-226688	5930496
8	2535	6155079	648195	30178575
9	0	25207736	-1742320	143184000
10	10908	97675200	4446912	639249300
11	0	360471552	-10863840	2705114880
12	42614	1273876088	25553402	10914317934
13	0	4329852624	-58129280	4 2189811200
14	153960	14207361792	128365440	156883829400
15	0	45144664064	-276044032	563116739584
16	521235	139288329729	579574795	1956790259235
17	0	418257062220	-1190636016	6599620022400
18	1669720	1224808431104	2397710720	21651325216200
19	0	3503958594048	-4740978480	69228721526400
20	5098938	9808358121720	9217285614	216108718571250

TABLE A. Numbers of real rational curves vs. complex curves on $K3$ surfaces

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