RELATIVE OPEN GROMOV-WITTEN INVARIANTS

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Abstract. We introduce a theory of relative open Gromov-Witten invariants. This theory counts $J-$holomorphic disks with Lagrangian boundary in symplectic 4-manifolds endowed with an anti-symplectic involution with non-empty fixed locus. The disks are subject to tangency conditions with an invariant smooth divisor, also with non-empty fixed locus.

The complex enumerative geometry benefitted in the 90's from the new perspective of the Gromov-Witten theory and many classical old problems have been elegantly solved. Meanwhile, in the real enumerative geometry many basic questions were wide open. For example, while the number of complex rational plane curves was found in arbitrary degree [KM94], the existence of a real rational plane quartic through 11 generic real points was still an open problem. In 2003, Welschinger [W05] introduced in the mainstream the notion of *real symplectic manifolds* and his numerical invariants in dimension four and six, counting real rational curves with signs. His theory developed in several directions. In one direction, the Welschinger invariants were interpreted in tropical geometry and the existence question of plane real rational curves was given a positive answer in arbitrary degree. A second approach, via the symplectic field theory, was undertaken by Welschinger [W07] who provided precise computations for some of his invariants. The second author [S06] interpreted the Welschinger invariants as open Gromov-Witten invariants, and extended their definition to six-dimensional real Calabi-Yau manifolds. Together with Pandharipande and Walcher, he proved mirror symmetry for the real quintic threefold [PSW08]. Also, the second author found an analog of the WDVV equation in open Gromov-Witten theory [S]. In particular, this equation leads to recursive formulae computing the Welschinger invariants for plane real rational curves.

Despite the recent advances in the study of the real enumerative invariants, many foundational aspects are still to be settled. However, the interpretation of the Welschinger's invariants as open Gromov-Witten invariants suggests a parallelism with the Gromov-Witten theory which nowadays is considerably more developed. We are trying to fill some of the gaps by developing a theory of *relative open Gromov-Witten invariants* for four-dimensional real symplectic manifolds, in analogy with the relative Gromov-Witten theory [IP03, LR01]. The goal is to extract numerical invariants responsible for the counting of pseudo-holomorphic disks subject to tangency conditions with respect to a smooth real symplectic divisor. The tropical analog of such invariants has already been found in the tropical setting [IKS09] together with an appropriate Caporaso-Harris type formula.

To introduce our results, let (X, ω, ϕ) be *real symplectic 4-manifold*. That is X is a closed differentiable $4-$manifold endowed with a non-degenerate 2-form ω with $d\omega = 0$, and an involution ϕ on X satisfying $\phi^*\omega = -\omega$. We assume $\mathbb{R}X :=$

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$Fix(\phi) \neq \emptyset$. Let $V \subset X$ be a smooth symplectic divisor, invariant under the real structure ϕ, such $\mathbb{R}V := V \bigcap \mathbb{R}X \neq \emptyset$. Fix $d \in H_2(X, \mathbb{R}X; \mathbb{Z})$ such that $d = -\phi_* d$. We want to define open Gromov-Witten-type invariants of X with respect to V, counting (with signs) pseudo-holomorphic disks of degree d with boundary in $\mathbb{R}X$. These disks are subject to fixed/moving, boundary/interior tangency conditions (TC) with V. We impose that all of the boundary contact points of the disks with the divisor have odd multiplicities.

We define *the moduli space of relative disks* as a subspace of the moduli space of pseudo-holomorphic disk maps with lagrangian boundary conditions. This is done by imposing the vanishing of the normal jets at all of the contact points of the disks with V. The Gromov-compactification of this moduli space comes naturally equipped with a total evaluation map at all of the marked points, including the fixed tangency points. One would like to define relative numbers as the degree of this evaluation map, under a dimension condition.

There are two major difficulties to overcome in defining such invariants which do not occur in the more familiar Gromov-Witten theory. In general, the spaces involved are not orientable. The second big problem consists in overcoming the presence of the codimension one boundary strata introduced in the Gromov-compactification of the moduli spaces of relative stable maps. This issue occurs when trying to prove the independence of the numbers defined of the choices made.

The orientation issue is dealt with by providing a canonical *relative orientation* for the total evaluation map. Recall that a relative orientation of a map $f : M \to N$ is an isomorphism $TM \simeq f^*TN$. The *relative numbers* are defined as the degree of the evaluation map with respect to its relative orientation.

The independence of the numbers defined above of the choices made usually relies on Stokes' theorem. For moduli spaces of disks, this argument must be treated carefully due to the presence of codimension one strata. In the open Gromov-Witten theory, this issue is overcome by using the flipping procedure [S06] which sends one component of a multi-disk map to its conjugate and leaving the other components unchanged. This procedure shows that the codimension one strata of the moduli space come in pairs of opposite relative orientations. The *new phenomenon* which occurs in our relative setting is that the flipping procedure preserves the canonical relative orientation on some codimension one strata. Since the full cancelling of the codimension one phenomena does not occur, the signed counting *depends* on the choices made. A different approach to cancelling is necessary.

For each boundary surviving the flipping procedure, we identify a suitable cartesian product of moduli spaces of relative disks equipped with relative orientations with respect to an evaluation map (and hence an associated relative number) having the same boundary, but with the reversed relative orientation as one of the boundaries. This suggested a combinatorial approach based on gluing various moduli spaces along appropriate codimension one strata. We define a finite connected graph $\mathfrak{G}_{TC} = (\mathcal{V}, \mathcal{E})$ with a distinguished vertex $\mathbf{v_0} \in \mathcal{V}$. The root $\mathbf{v_0}$ represents the moduli space of relative maps with the tangency conditions (TC). The vertices $\mathbf{v} \in \mathcal{V}$ represent the cartesian products of moduli spaces discussed, and the edges $e \in \mathcal{E}$ represent the codimension one phenomenon cancelled. The graph $\mathfrak{G}_{TC}$ is inductively defined. Its main property is that the valence of each vertex is the number of the surviving codimension one strata of the corresponding moduli space.

Moreover, to each vertex we can associate a *relative number* $n_v \in \mathbb{Z}$, and a weight $w(v) \in \mathbb{Q}$. The main result is

Theorem 0.1. [RS10a] The numbers

$$\mathcal{R}_{TC} = \sum_{v \in \mathcal{V}} w(v) n_v$$

are independent of the choices made. Moreover, $\mathcal{R}_{TC}$ are integers.

The proof relies on gluing results which will appear in our joint work [RS10b]. The result extends the relative invariants introduced by Welschinger [W06].

In our talk, we described how this graph is constructed for some interesting cases. The first one is the case when there are no tangency conditions imposed, and we recover the original Welschinger invariants. In this case, the graph is linear. In the second case, we impose a third order boundary moving tangency. We will use this example to illustrate the difficulties in constructing the graph in general.

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