ON THE SMITH-THOM DEFICIENCY OF HILBERT SQUARES

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Abstract. We give an expression for the Smith-Thom deficiency of the Hilbert square $X^{[2]}$ of a smooth real algebraic variety X in terms of the rank of a suitable Mayer-Vietoris mapping in several situations. As a consequence, we establish a simple necessary and sufficient condition for the maximality of $X^{[2]}$ in the case of projective complete intersections, and show that with a few exceptions no real nonsingular projective complete intersection of even dimension has maximal Hilbert square. We also provide new examples of smooth real algebraic varieties with maximal Hilbert square.

> . . . si beau soit un vaste paysage, les horizons lointains sont toujours un peu vagues . . .
>
> Henri Poincaré, Savants et écrivains.

Contents

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1. Introduction

Recall that on a real algebraic variety X the following inequality holds
$$\dim H_*(X(\mathbb{R});\mathbb{F}_2) \leq \dim H_*(X(\mathbb{C});\mathbb{F}_2).$$
It is traditionally called the Smith inequality and the difference
$$\mathfrak{D}(X) = \dim H_*(X(\mathbb{C});\mathbb{F}_2) - \dim H_*(X(\mathbb{R});\mathbb{F}_2)$$
is called *the Smith-Thom deficiency of* X. A real algebraic variety X is said to be *maximal*, or an *M-variety*, if its Smith-Thom deficiency vanishes.

As it was observed in our previous paper [6], many deformation classes of algebraic surfaces do not contain any real representative X whose Hilbert square $X^{[2]}$ is maximal. Here, we refine results obtained in [6] and generalize them to higher dimensions. The main results are as follows.

Theorem 1.1. *Let X be a real nonsingular projective variety of dimension $n \geq 2$. If the Hilbert square $X^{[2]}$ is maximal, then X is maximal.*

Theorem 1.2. *The Smith-Thom deficiency of the Hilbert square $X^{[2]}$ of a maximal real nonsingular projective complete intersection $X \subset \mathbb{P}^N$ of dimension $n \geq 2$ is given by*
$$\mathfrak{D}(X^{[2]}) = 4\left(\sum_{l=1}^{[\frac{n}{2}]}\sum_{i=0}^{l-1}\beta_i(X(\mathbb{R})) - d(n)\right),$$
where $d(n)$ is equal to $\frac{n(n+2)}{8}$ for n even and $\frac{n^2-1}{8}$ for n odd.

The $\mathbb{F}_2$-Betti numbers $\beta_i(X(\mathbb{R}))$ are non-zero for every maximal real nonsingular complete intersection $X \subset \mathbb{P}^N$ of dimension n whenever $0 \leq i \leq [\frac{n}{2}]$ (see Lemma 2.6). Since $d(n) = \sum_{l=1}^{[\frac{n}{2}]} l$, we get the following criterium of maximality as a consequence of Theorems 1.1 and 1.2.

Corollary 1.3. *Let $X \subset \mathbb{P}^N$ be a real nonsingular projective complete intersection of dimension $n \geq 2$. Then $X^{[2]}$ is maximal if and only if X is maximal and*
$$\beta_i(X(\mathbb{R})) = 1 \quad \textit{for every} \quad 0 \leq i \leq \left[\frac{n}{2}\right] - 1.$$

In particular, Corollary 1.3 yields a direct proof of the following result due to L. Fu [4, Theorem 7.5]:

Corollary 1.4. *The Hilbert square of a real cubic threefold is maximal if and only if the threefold is maximal.*

According to Theorem 11.12 in [4], no real cubic four-fold has maximal Hilbert square. As a consequence of Theorems 1.1 and 1.2 we prove:

Theorem 1.5. *The Hilbert square $X^{[2]}$ of a real nonsingular projective complete intersection $X \subset \mathbb{P}^N$ of positive, even dimension is maximal if and only if X is a linear subspace, a maximal quadric, a maximal intersection of two quadrics, or a maximal cubic surface in $\mathbb{P}^3$.*

As it was communicated to us by L. Fu, for a cubic hypersurface X, the Galkin-Shinder-Voisin diagram (see [1, Section 4]) yields the following relation between the Smith-Thom deficiencies of X, its Hilbert square, and its Fano variety of lines $F(X)$:

$$\mathfrak{D}(X^{[2]}) = (n+1)\,\mathfrak{D}(X) + \mathfrak{D}(F(X)).$$

Therefore, Theorem 1.5 has the following consequence:

Corollary 1.6. *Let X be a real nonsingular cubic hypersurface of dimension $2n$ with $n \geq 2$. If X is maximal* [1]*, then its Fano variety of lines $F(X)$ is not maximal.*

We study next the maximality of the Hilbert square of real algebraic varieties whose complex locus has no odd degree cohomology.

Theorem 1.7. *Let X be a real nonsingular projective variety of dimension ≥ 2* [2] *satisfying $H_{\mathrm{odd}}(X) = 0$. If the Hilbert square $X^{[2]}$ is maximal, then $\beta_k(X(\mathbb{R})) = \beta_{2k}(X(\mathbb{C}))$ for every $k \geq 0$. In particular, $X(\mathbb{R})$ is connected.*

The conditions found in Theorem 1.7 are not sufficient to ensure the maximality of the Hilbert square in general (*cf.*, Theorem 1.5). However, we identify a case when they are sufficient.

Theorem 1.8. *Let X be a maximal real nonsingular projective variety such that $H_{\mathrm{odd}}(X) = 0$. If for every $k \leq \dim X$, the group $H_k(X(\mathbb{R}))$ is generated by fundamental classes of real loci of real smooth algebraic submanifolds, then the Hilbert square $X^{[2]}$ is maximal.*

Corollary 1.9. *The Hilbert square is maximal for projective spaces, nonsingular toric varieties, nonsingular quadrics in $\mathbb{P}^{2n+1}$, and products of the former ones (always equipped with standard real structure).*

More advanced examples of real projective manifolds with maximal Hilbert square will be presented elsewhere [7], in a different context.

The results obtained in Theorems 1.1, 1.7 and 1.8, including their proofs, literally extend from real algebraic setting to compact complex manifolds equipped with an anti-holomorphic involution.

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[1]By common belief, maximal real cubic hypersurfaces should exist in all dimensions. However, up to our knowledge, published proofs cover the statement only up to dimension 4.

[2]The only exception in dimensions < 2 is that of $X = \mathbb{P}^1$ with a real structure without real points.

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Notations and conventions:

1) By a complex variety equipped with a real structure, we mean a pair (Y, c) consisting of a complex variety Y and an anti-holomorphic involution $\mathrm{c} : Y \to Y$. When the anti-holomorphic conjugation is understood from the context, we will simply say that Y is defined over the reals.
2) Let Y be an algebraic variety defined over $\mathbb{R}$, and G denote the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The group G is a cyclic group of order 2 and acts on the locus of complex points $Y(\mathbb{C})$. The non-trivial element of G acts as an anti-holomorphic involution, which we will denote by c, and the fixed point set of the action coincides with the set of real points of Y. The pair (Y, c) is a variety equipped with a real structure. To mediate between the notations traditionally used for varieties equipped with real structures and for algebraic varieties defined over $\mathbb{R}$, we will use from now on Y to denote the set of complex points, and $Y(\mathbb{R})$ the set of real points.
3) Unless explicitly stated, all the homology and cohomology groups have coefficients in the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. We use $\beta_i(\,\cdot\,)$ and $b_i(\,\cdot\,)$ to denote the Betti numbers when the coefficients are in $\mathbb{F}_2$ or in $\mathbb{Q}$, respectively. For convenience, we allow the index i to be an arbitrary integer, by setting $\beta_i = 0$ for $i < 0$. We will use the notations $\beta_*(\,\cdot\,)$ and $b_*(\,\cdot\,)$ for the corresponding total Betti numbers, while $\beta_{\mathrm{odd}}(\,\cdot\,)$ and $\beta_{\mathrm{even}}(\,\cdot\,)$ denote the sums $\sum_{i\geq 0} \beta_{2i+1}(\,\cdot\,)$ and $\sum_{i\geq 0} \beta_{2i}(\,\cdot\,)$, respectively.

2. Preliminaries

2.1. **Smith theory.** Most results cited in this section are due to P.A. Smith. Proofs can be found, e.g., in [2, Chapter 3] and [3, Chapter 1].

Throughout the section we consider a topological space X with a cellular involution $c : X \to X$, *i.e.,* X is a CW-complex, c transforms cells into cells and acts identically on each invariant cell [3]. Let $F = \mathrm{Fix}\, c$, $\bar{X} = X/c$, and denote by $\mathrm{in} : F \hookrightarrow X$ and $\mathrm{pr} : X \to \bar{X}$ be the natural inclusion and projection, respectively.

[3]This rather traditional condition that X is a CW-complex (or a simplicial complex, as in [2]) can be relaxed at the cost of using Čech cohomology and assuming X to be finite dimensional locally compact Hausdorff topological space. In this paper, Smith theory is applied to smooth manifolds and smooth involutions, so the CW-complex assumption is largely enough.

Consider the *Smith chain complexes* defined by

$$\mathrm{Sm}_*(X) = \mathrm{Ker}[(1 + c_*) \,:\, S_*(X) \to S_*(X)],$$
$$\mathrm{Sm}_*(X, F) = \mathrm{Ker}[(1 + c_*) \,:\, S_*(X, F) \to S_*(X, F)].$$

and their *Smith homology* $H_r(\mathrm{Sm}_*(X))$ and $H_r(\mathrm{Sm}_*(X, F))$, respectively. There exists a canonical isomorphism $\mathrm{Sm}_*(X, F) = \mathrm{Im}[(1 + c_*) \,:\, S_*(X) \to S_*(X)]$. The *Smith sequences* are the long homology and cohomology exact sequences associated with the short exact sequence of complexes

$$0 \to \mathrm{Sm}_*(X) \xrightarrow{\text{inclusion}} S_*(X) \xrightarrow{1+c_*} \mathrm{Sm}_*(X, F) \to 0. \tag{2.1}$$

The latter one canonically splits, $\mathrm{Sm}_*(X) = S_*(F) \oplus \mathrm{Im}(1 + c_*)$, and the transfer homomorphism $\mathrm{tr}^* : S_*(\bar{X}, F) \to \mathrm{Sm}_*(X, F)$ is an isomorphism [2, Chapter 3] (see also *op. cit.* for the cohomology version). In view of these identifications the long exact sequences associated to (2.1) yield:

Theorem 2.1. *There are two natural, in respect to equivariant maps, exact sequences, called* (homology and cohomology) Smith sequences of (X, c):

$$\cdots \to H_{p+1}(\bar{X}, F) \xrightarrow{\Delta} H_p(\bar{X}, F) \oplus H_p(F) \xrightarrow{\mathrm{tr}^* + \mathrm{in}_*} H_p(X) \xrightarrow{\mathrm{pr}_*} H_p(\bar{X}, F) \to ,$$
$$\to H^p(\bar{X}, F) \xrightarrow{\mathrm{pr}^*} H^p(X) \xrightarrow{\mathrm{tr}_* \oplus \mathrm{in}^*} H^p(\bar{X}, F) \oplus H^p(F) \xrightarrow{\Delta} H^{p+1}(\bar{X}, F) \to \cdots .$$

The homology and cohomology connecting homomorphisms Δ *are given by*

$$x \mapsto x \cap \omega \oplus \partial x \quad \textit{and} \quad x \oplus f \mapsto x \cup \omega + \delta f,$$

respectively, where $\omega \in H^1(\bar{X} \setminus F)$ *is the characteristic class of the double covering* $X \setminus F \to \bar{X} \setminus F$*. The images of* $\mathrm{tr}^* + \mathrm{in}_*$ *and* pr^* *consist of invariant classes:* $\mathrm{Im}\,\mathrm{tr}^* \subset \mathrm{Ker}(1 + c_*)$ *and* $\mathrm{Im}\,\mathrm{pr}^* \subset \mathrm{Ker}(1 + c^*)$.

The following immediate consequences of Theorem 2.1, which we state in the homology setting, have an obvious counterpart for cohomology.

Corollary 2.2. *Let* (X, c) *be a CW complex equipped with a cellular involution. Then*

$$\dim\ H_*(F) + 2\sum_p \dim \mathrm{Coker}(\mathrm{tr}^p + \mathrm{in}_p) = \dim\ H_*(X). \tag{2.2}$$

As a consequence, we have

$$\dim\ H_*(F) \le \dim\ H_*(X) \quad \text{(Smith inequality)}. \tag{2.3}$$

Definition 2.3. *Let* (X, c) *be a topological space equipped with a cellular involution. The integer*

$$\mathfrak{D}(X, \mathrm{c}) = 2\sum_p \dim \mathrm{Coker}(\mathrm{tr}^p + \mathrm{in}_p)$$

is called the Smith-Thom deficiency of (X, c)*. If* $\mathfrak{D}(X, \mathrm{c}) = 0$*, the topological space* X *is called* maximal*, or an* M*-space, and* c *is called an* M-involution*.*

When the involution is understood from the context, it will be omitted from the notation of the Smith-Thom deficiency.

Notice from Corollary 2.2 that X is maximal if and only if $\dim\ H_*(F) = \dim\ H_*(X)$, and from Theorem 2.1 we find the following characterization of maximality.

Corollary 2.4. *Let (X, c) be a topological space equipped with a cellular involution. Then X is maximal if and only if the sequence*

$$0 \to H_{k+1}(\bar{X}, F) \xrightarrow{\Delta} H_k(\bar{X}, F) \oplus H_k(F) \to H_k(X) \to 0$$

is exact for every $k \geq 0$.

Lemma 2.5. *If a d-dimensional space (X, c) is maximal and $r \leq d$, then*

$$\beta_r(\bar{X}, F) = \sum_{k=r}^{d} (\beta_k(X) - \beta_k(F)). \tag{2.4}$$

If, in addition, $d = 2n$, X is a smooth closed manifold, c a smooth involution, and each component of F is n-dimensional, then we have

$$\beta_r(\bar{X}, F) = \sum_{k=r}^{2n} \beta_k(X), \quad \text{for every } r \geq n+1, \tag{2.5}$$

and

$$\beta_*(\bar{X}, F) = \frac{n}{2} \beta_*(X). \tag{2.6}$$

Proof. According to Corollary 2.4, under maximality assumption, for every $k \geq 0$ we have $\beta_{k+1}(\bar{X}, F) + \beta_k(X) = \beta_k(\bar{X}, F) + \beta_k(F)$. We obtain (2.4) by summing up these equalities over all k with $r \leq k \leq d$ and noticing that $\beta_{d+1}(\bar{X}, F) = 0$. In the manifold case, (2.5) is a direct consequence of (2.4).

To prove (2.6), by adding the equalities (2.4) over all r with $1 \leq r \leq d$, we obtain

$$\beta_*(\bar{X}, F) = \sum_{r=0}^{2n} r(\beta_r(X) - \beta_r(F)). \tag{2.7}$$

Due to Poincaré duality applied to both X and F, we have

$$\sum_{r=0}^{2n} r\beta_r(X) = \sum_{r=0}^{2n} (2n - r)\beta_r(X) = 2n\beta_*(X) - \sum_{r=0}^{2n} r\beta_r(X)$$

and

$$\sum_{r=0}^{2n} r\beta_r(F) = \sum_{r=0}^{n} r\beta_r(F) = \sum_{r=0}^{n} (n - r)\beta_r(F) = n\beta_*(F) - \sum_{r=0}^{n} r\beta_r(F),$$

respectively. Hence,

$$\sum_{r=0}^{2n} r\beta_r(X) = n\beta_*(X), \quad \sum_{r=0}^{2n} r\beta_r(F) = \frac{n}{2}\beta_*(F), \tag{2.8}$$

and (2.6) follows from (2.7), (2.8), and the maximality assumption. $\square$

2.2. **A non-vanishing result.** An ingredient needed for the proofs of results announced in §1 is an observation of the first author [5] regarding the non-vanishing of the cohomology of real hypersurface. We include a short proof here for the convenience of the reader.

Lemma 2.6. *Let $X \subset \mathbb{P}^N$ be a maximal complete intersection of dimension n and $\upsilon \in H^1(X(\mathbb{R}))$ the class of hyperplane sections $H(\mathbb{R}) \subset X(\mathbb{R})$. Then $\upsilon^r \neq 0$ for every $r \in \mathbb{N}, r \leq [\frac{n}{2}]$. In particular, $\beta_r(X(\mathbb{R})) \geq 1$ for every $r \in \mathbb{N}, r \leq [\frac{n}{2}]$.*

Proof (*cf.*, [5]). Let d be the degree of $X \subset \mathbb{P}^N$. If d is odd, then $\upsilon^r \neq 0$ for every $0 \leq r \leq n$, just because $(\upsilon^r \cup \upsilon^{n-r}) \cap [X(\mathbb{R})] = d \mod 2$.

If d is even, the result is a straightforward consequence of the following properties:

- If k is odd, then $H^k(X;\mathbb{Z})$=0.
- If k is even and $n < k \leq 2n$, then $H^k(X;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and generated by $h^{\frac{k}{2}}$ where $h \in H^2(X;\mathbb{Z})$ is the class of hyperplane sections $H \subset X$.
- If k is even and $0 \leq k < n$, then $H^k(X;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and $h^{\frac{k}{2}}$ is a d-multiple of a generator.
- if n is even, then $h^{\frac{n}{2}}$ is a primitive element of $H^n(X;\mathbb{Z})$ (see, for example, [5, Lemma 1.2]).

Indeed, from the enumerated above properties and exactness of the $\mathbb{F}_2$-cohomology sequence of the pair $(\mathbb{P}^N, X)$ it follows that

$$\beta_*(\mathbb{P}^N, X) = \beta_*(\mathbb{P}^N) + \beta_*(X) - 2\left(\left[\frac{n}{2}\right] + 1\right).$$

On the other hand, from the exactness of the $\mathbb{F}_2$-cohomology sequence of the pair $(\mathbb{P}^N(\mathbb{R}), X(\mathbb{R}))$ we get

$$\beta_*(\mathbb{P}^N(\mathbb{R}), X(\mathbb{R})) = \beta_*(\mathbb{P}^N(\mathbb{R})) + \beta_*(X(\mathbb{R})) - 2(\ell + 1),$$

where ℓ is determined by the property that $\upsilon^r \neq 0$ for $r \leq l$ and $\upsilon^r = 0$ for $r > \ell$. Now, it remains to apply the Smith inequality in the relative setting (see [2], Chap. 3, Theorem 4.1),

$$\beta_*(\mathbb{P}^N(\mathbb{R}), X(\mathbb{R})) \leq \beta_*(\mathbb{P}^N, X)$$

and to use the maximality assumption, $\beta_*(X(\mathbb{R})) = \beta_*(X)$. □

2.3. **Elementary computations.** The following computations are used in the proof of Theorem 1.2.

Lemma 2.7. *Let X be a maximal real nonsingular projective variety of odd dimension n. Then, the following relations hold:*

$$\sum_{l=0}^{\frac{n-1}{2}} \sum_{i+j=2l} \beta_i(X(\mathbb{R}))\beta_j(X(\mathbb{R})) = \frac{1}{4}\beta_*^2\,, \tag{2.9}$$

$$\sum_{l=0}^{\frac{n-1}{2}} \beta_l(X(\mathbb{R})) = \frac{1}{2}\beta_*\,, \tag{2.10}$$

$$\sum_{l=0}^{\frac{n-1}{2}} \sum_{i=0}^{2l-1} \beta_i(X(\mathbb{R})) = \frac{n-1}{4}\beta_*. \tag{2.11}$$

Proof. By Poincaré duality, we find:

$$\begin{aligned}
&2\sum_{l=0}^{\frac{n-1}{2}} \sum_{i+j=2l} \beta_i(X(\mathbb{R}))\beta_j(X(\mathbb{R})) \\
&\quad= \sum_{l=0}^{\frac{n-1}{2}} \sum_{i+j=2l} \beta_i(X(\mathbb{R}))\beta_j(X(\mathbb{R})) + \sum_{l=0}^{\frac{n-1}{2}} \sum_{i+j=2l} \beta_{n-i}(X(\mathbb{R}))\beta_{n-j}(X(\mathbb{R})) \\
&\quad= \sum_{l=0}^{\frac{n-1}{2}} \sum_{i+j=2l} \beta_i(X(\mathbb{R}))\beta_j(X(\mathbb{R})) + \sum_{l'=\frac{n+1}{2}}^{n} \sum_{i'+j'=2l'} \beta_{i'}(X(\mathbb{R}))\beta_{j'}(X(\mathbb{R})) \\
&\quad= \sum_{k=0}^{n} \sum_{a+b=2k} \beta_a(X(\mathbb{R}))\beta_b(X(\mathbb{R})) \\
&\quad= \beta_{\text{even}}^2(X(\mathbb{R})) + \beta_{\text{odd}}^2(X(\mathbb{R})).
\end{aligned}$$

Since $X(\mathbb{R})$ is an odd-dimensional manifold, using again the Poincaré duality we see that $\beta_{\text{odd}}(X(\mathbb{R})) = \beta_{\text{even}}(X(\mathbb{R}))$, while by the maximality of X we have $\beta_{\text{odd}}(X(\mathbb{R})) + \beta_{\text{even}}(X(\mathbb{R})) = \beta_*$. Therefore

$$\beta_{\text{odd}}(X(\mathbb{R})) = \beta_{\text{even}}(X(\mathbb{R})) = \frac{1}{2}\beta_*,$$

wherefrom (2.9) follows immediately.

Again by Poincaré duality, we have

$$\begin{aligned}
2\sum_{l=0}^{\frac{n-1}{2}} \beta_l(X(\mathbb{R})) &= \sum_{l=0}^{\frac{n-1}{2}} \beta_l(X(\mathbb{R})) + \sum_{l=0}^{\frac{n-1}{2}} \beta_{n-l}(X(\mathbb{R})) \\
&= \sum_{l=0}^{\frac{n-1}{2}} \beta_l(X(\mathbb{R})) + \sum_{l'=\frac{n+1}{2}}^{n} \beta_{l'}(X(\mathbb{R})) \\
&= \beta_*(X(\mathbb{R})).
\end{aligned}$$

This implies (2.10) due to the maximality of X.

A direct computation shows that

$$\sum_{l=0}^{\frac{n-1}{2}} \sum_{i=0}^{2l-1} \beta_i(X(\mathbb{R})) = \sum_{j=0}^{n} \left\lceil \frac{n-1-j}{2} \right\rceil \beta_j(X(\mathbb{R})). \tag{2.12}$$

Arguing as before, by Poincaré duality, we obtain

$$\begin{aligned}
&2\sum_{j=0}^{n} \left\lceil \frac{n-1-j}{2} \right\rceil \beta_j(X(\mathbb{R})) \\
&= \sum_{j=0}^{n} \left\lceil \frac{n-1-j}{2} \right\rceil \beta_j(X(\mathbb{R})) + \sum_{j=0}^{n} \left\lceil \frac{n-1-j}{2} \right\rceil \beta_{n-j}(X(\mathbb{R})) \\
&= \sum_{j=0}^{n} \left\lceil \frac{n-1-j}{2} \right\rceil \beta_j(X(\mathbb{R})) + \sum_{i=0}^{n} \left\lceil \frac{i-1}{2} \right\rceil \beta_i(X(\mathbb{R})) \\
&= \sum_{a=0}^{n} \left(\left\lceil \frac{n-1-a}{2} \right\rceil + \left\lceil \frac{a-1}{2} \right\rceil \right) \beta_a(X(\mathbb{R})) \\
&= \frac{n-1}{2} \sum_{a=0}^{n} \beta_a(X(\mathbb{R})). \qquad (2.13)
\end{aligned}$$

The conclusion of the lemma follows now from (2.12), (2.13) and the maximality of X. $\square$

When X is even dimensional, we need similar identities. We will only state them, as the proof is elementary and follows the one in the odd dimensional case.

Lemma 2.8. *Let X be a maximal real nonsingular projective variety of even dimension n. Then, the following relations hold:*

$$\sum_{l=1}^{\frac{n}{2}} \sum_{\substack{a+b=2l-1 \\ a<b}} \beta_a(X(\mathbb{R}))\beta_b(X(\mathbb{R})) = \frac{1}{2}\beta_{\text{even}}(X(\mathbb{R}))\beta_{\text{odd}}(X(\mathbb{R})), \tag{2.14}$$

$$\sum_{l=1}^{\frac{n}{2}} \sum_{i=0}^{2l-2} \beta_i(X(\mathbb{R})) = \frac{n}{4}\beta_* - \frac{1}{2}\beta_{\text{odd}}(X(\mathbb{R})). \quad \square \tag{2.15}$$

3. Cut-and-Paste construction of Hilbert squares over the reals

For smooth varieties a simple, well known, construction of the Hilbert square consists in the following. Given a smooth variety X, one lifts the involution $\tau : X \times X \to X \times X$ permuting the factors to an involution $Bl(\tau)$ on the blowup $\mathrm{Bl}_\Delta(X \times X)$ of $X \times X$ along the diagonal $\Delta \subset X \times X$. The quotient of $\mathrm{Bl}_\Delta(X \times X)$ by $\mathrm{Bl}(\tau)$ is then naturally isomorphic to the

Hilbert square $X^{[2]}$. By construction, the branch locus $E \subset X^{[2]}$ of the double ramified covering $\mathrm{Bl}_\Delta(X \times X) \to X^{[2]}$ coincides with the exceptional divisor of the blowup, and, since the normal vector bundle of the blown-up locus Δ in $X \times X$ is isomorphic to the tangent vector bundle TX of X, both this exceptional divisor and E are naturally isomorphic to $\mathbb{P}(T^*X)$.[4] In other words, a point of E is identified with a point of X plus a complex line in the tangent space at that point. Pairs of distinct points in X represent the points of the complement, $X^{[2]} \setminus E$. Notice also that the normal bundle of E in $X^{[2]}$ is naturally isomorphic to the square of the tautological bundle over $\mathbb{P}(T^*X)$.

This construction is independent on the choice of the ground field. Applying it to a smooth variety X defined over the reals, it equips $X^{[2]}$ with a natural real structure that acts on points in $X^{[2]}$ represented by pairs of distinct points in X by sending a pair to the complex conjugate one, while the points represented by a point of X with a tangent line are sent to the conjugate point with the conjugate line. Thus, $X^{[2]}(\mathbb{R}) = X/\mathrm{c}$ if $X(\mathbb{R})$ is empty, and otherwise $X(\mathbb{R})$ has the following properties:

- $E(\mathbb{R}) \subset X^{[2]}(\mathbb{R})$ is naturally diffeomorphic to $\mathbb{P}_\mathbb{R}(T^*X(\mathbb{R}))$ and has codimension 1.
- The complement, $X^{[2]}(\mathbb{R}) \setminus E(\mathbb{R})$, is a disjoint union of $(X/\mathrm{c}) \setminus X(\mathbb{R})$, $F_i^{(2)} \setminus \Delta F_i$ $(1 \le i \le r)$, and $F_i \times F_j$ $(1 \le i < j \le r)$ where $F_1, \dots, F_r$ are the connected components of $X(\mathbb{R})$.
- Each of the connected components $\mathbb{P}_\mathbb{R}(T^*F_i)$ of $E(\mathbb{R})$ is a common boundary of $(X/\mathrm{c}) \setminus X(\mathbb{R})$ with $F_i^{(2)} \setminus \Delta F_i$.

From this we can conclude that $X^{[2]}(\mathbb{R})$ is a disjoint union of connected components

$$X^{[2]}(\mathbb{R}) = X^{[2]}_{\mathrm{main}}(\mathbb{R}) \bigsqcup X^{[2]}_{\mathrm{extra}}(\mathbb{R}), \quad X^{[2]}_{\mathrm{extra}}(\mathbb{R}) = \bigsqcup_{1 \le i < j \le r} (F_i \times F_j)$$

where $X^{[2]}_{\mathrm{main}}(\mathbb{R})$ is the component of $X^{[2]}(\mathbb{R})$ that contains $E(\mathbb{R})$ in such a way that $E(\mathbb{R})$ divides $X^{[2]}_{\mathrm{main}}(\mathbb{R})$ in $r+1$ submanifolds with boundary:

$$X^{[2]}_{\mathrm{main}}(\mathbb{R}) = \bigcup_{i=0}^{r} \mathbb{H}_i,$$

where

$$\partial \mathbb{H}_0 = E(\mathbb{R}), \quad \mathrm{Int}\, \mathbb{H}_0 \cong (X/\mathrm{c}) \setminus X(\mathbb{R})$$
$$\partial \mathbb{H}_i = \mathbb{P}_\mathbb{R}(T^*F_i), \quad \mathrm{Int}\, \mathbb{H}_i \cong F_i^{(2)} \setminus \Delta F_i, i = 1, \dots, r.$$

[4]In agreement with [12], which we use as reference, we follow the Grothendieck convention according to which the projectivization of a vector space is the space of its hyperplanes.

Here ΔF_i is the diagonal in $F_i^{(2)}$, for every $i = 1, \ldots, r$. Each manifold $\mathbb{H}_i$, $i = 1, \ldots, r$ is glued to $\mathbb{H}_0$ along their common boundary $\mathbb{P}_{\mathbb{R}}(T^*F_i) \subset \mathbb{P}_{\mathbb{R}}(T^*X(\mathbb{R}))$. For convenience, let $\mathbb{H} = \sqcup_{i=1}^r \mathbb{H}_i$, and notice that $\partial\mathbb{H} = E(\mathbb{R})$.

Denote by

$$\mathrm{in}_0 : E(\mathbb{R}) = \partial\mathbb{H}_0 \to \mathbb{H}_0, \quad \mathrm{in} : E(\mathbb{R}) = \partial(\sqcup_{i=1}^r \mathbb{H}_i) \to \bigsqcup_{i=1}^r \mathbb{H}_i$$

the inclusion maps. Let in_0^k and in^k be the induced maps between the k^{th} cohomology groups, and in_0^* and in^* denote the induced maps

$$\mathrm{in}_0^* : H^*(\mathbb{H}_0) \to H^*(E(\mathbb{R})) \quad \text{and} \quad \mathrm{in}^* : H^*(\mathbb{H}) \to H^*(E(\mathbb{R})),$$

respectively, where

$$H^*(E(\mathbb{R})) = \bigoplus_{i \geq 0} H^i(E(\mathbb{R})) \quad \text{and} \quad H^*(\mathbb{H}) = \bigoplus_{i \geq 0} H^i(\mathbb{H}).$$

We will write the corresponding induced maps in homology by lowering the degree index. We let $\mu = (\mathrm{in}_0, \mathrm{in}) : E(\mathbb{R}) \to \mathbb{H}_0 \sqcup (\sqcup_{i=1}^r \mathbb{H}_i)$ and use similarly induced notations for the induced maps in cohomology and homology.

3.1. **Compatibility of characteristic classes.** A natural bijection between the set of double coverings of a topological space M and $H^1(M)$ consists in taking the first Stiefel-Whitney class of the associated real line vector bundle. Given a double covering $\pi : N \to M$, the corresponding cohomology class $b \in H^1(M)$ is called the *characteristic class of the covering.*

In our context, we denote by $b_0 \in H^1(\mathbb{H}_0 \setminus \partial\mathbb{H}_0) = H^1(\mathbb{H}_0)$ the characteristic class of the double covering $X \setminus X(\mathbb{R}) \to \mathbb{H}_0 \setminus \partial\mathbb{H}_0$, while $b \in H^1(X(\mathbb{R})^{(2)} \setminus \Delta X(\mathbb{R}))$ is the characteristic class of the double covering $X(\mathbb{R}) \times X(\mathbb{R}) \setminus \Delta X(\mathbb{R}) \to X(\mathbb{R})^{(2)} \setminus \Delta X(\mathbb{R})$. Notice that this double covering is trivial when restricted to the connected components $F_i \times F_j$ of $X(\mathbb{R})^{(2)} \setminus \Delta X(\mathbb{R})$ with $i \neq j$. Accordingly, the restriction of b to these components is trivial, and b can be viewed as an element of $H^1(\mathbb{H} \setminus \partial\mathbb{H}) = H^1(\mathbb{H})$. Under this identification, we see that $b = (b_1, \ldots, b_r)$, where $b_i \in H^i(F_i^{(2)} \setminus \Delta F_i) = H^1(\mathbb{H}_i)$ is the characteristic class of the the double covering $F_i \times F_i \setminus \Delta F_i \to F_i^{(2)} \setminus \Delta F_i$, for every $i = 1, \ldots, r$.

Let $\gamma \in H^1(E(\mathbb{R}))$ denote the first Stiefel-Whitney class of the tautological line bundle of $E(\mathbb{R}) = \mathbb{P}_{\mathbb{R}}(T^*X(\mathbb{R}))$. Due to the cut-and-paste construction and naturality of the characteristic classes, we have the following compatibility relations:

$$\mathrm{in}_0^1(b_0) = \gamma = \mathrm{in}^1(b). \tag{3.1}$$

3.2. **The Betti numbers of $\mathbb{H}_i$.** Here, we compute the Betti numbers of the components of $X^{[2]}_{\text{main}}(\mathbb{R})$ and the rank of the corresponding inclusion maps induced in homology/cohomology. To simplify the notation, throughout the rest of the article $\beta_k(X) = \beta_k(X(\mathbb{C}))$ will always be denoted by β_k, and, similarly, we will use the notation β_* for $\beta_*(X) = \beta_*(X(\mathbb{C}))$.

Lemma 3.1. *If X is a maximal n-dimensional real nonsingular projective variety, then the following formulas hold:*

1) $\beta_k(\mathbb{H}_0) = \displaystyle\sum_{i=2n-k}^{2n} \beta_i$, *for every integer* $0 \leq k \leq n-1$.
2) $\beta_*(\mathbb{H}_0) = \dfrac{n}{2}\beta_*$.

Proof. By Poincaré-Alexander-Lefschetz duality, we find

$$\beta_r(\mathbb{H}_0) = \beta_{2n-r}(X/\operatorname{c}, X(\mathbb{R})),$$

and the proof of the claims follow from applying Lemma 2.5 to the pair (X, c). $\square$

Lemma 3.2. *If X is a maximal n-dimensional real nonsingular projective variety, then*

$$\operatorname{rank}(\operatorname{in}_0^k) = \sum_{i=0}^{k} \beta_i$$

for every $k < n$.

Proof. We argue by induction, and proceed by noticing first that $\operatorname{rank}(\operatorname{in}_0^0) = \beta_0$. Indeed, from the short exact sequence

$$0 \to H^0(\mathbb{H}_0, \partial\mathbb{H}_0) \to H^0(\mathbb{H}_0) \to \operatorname{Im}(\operatorname{in}_0^0) \to 0,$$

and Poincaré-Alexander-Lefschetz duality, we obtain

$$\begin{aligned}
\operatorname{rank}(\operatorname{in}_0^0) &= \beta_0(\mathbb{H}_0) - \beta_0(\mathbb{H}_0, \partial\mathbb{H}_0)\\
&= \beta_0(X/\operatorname{c} \setminus X(\mathbb{R})) - \beta_0(X/\operatorname{c}, X(\mathbb{R}))\\
&= \beta_{2n}(X/\operatorname{c}, X(\mathbb{R})) - \beta_0(X/\operatorname{c}, X(\mathbb{R})).
\end{aligned}$$

Since X is maximal, from (2.5) we obtain $\beta_{2n}(X/\operatorname{c}, X(\mathbb{R})) = \beta_{2n}$ and $\beta_0(X/\operatorname{c}, X(\mathbb{R})) = 0$. Therefore $\operatorname{rank}(\operatorname{in}_0^0) = \beta_{2n} = \beta_0$.

Consider next the cohomology long exact sequence of the pair $(\mathbb{H}_0, \partial\mathbb{H}_0)$. We find

$$0 \to \frac{H^{k-1}(\partial\mathbb{H}_0)}{\operatorname{Im}(\operatorname{in}_0^{k-1})} \to H^k(\mathbb{H}_0, \partial\mathbb{H}_0) \to H^k(\mathbb{H}_0) \to \operatorname{Im}(\operatorname{in}_0^k) \to 0, \tag{3.2}$$

for every $k \geq 1$. By Poincaré-Alexander-Lefschetz duality, Künneth formula and (2.5), from (3.2) we have:

$$\begin{aligned}
\operatorname{rank}(\mathrm{in}_0^{k-1}) &+ \operatorname{rank}(\mathrm{in}_0^k) \\
&= \beta_{k-1}(\partial \mathbb{H}_0) + \beta_k(\mathbb{H}_0) - \beta_k(\mathbb{H}_0, \partial \mathbb{H}_0) \\
&= \beta_{k-1}(E(\mathbb{R})) + \beta_k(X/\mathrm{c} \setminus X(\mathbb{R})) - \beta_k(X/\mathrm{c}, X(\mathbb{R})) \\
&= \sum_{i=0}^{k-1} \beta_i(X(\mathbb{R})) + \beta_{2n-k}(X/\mathrm{c}, X(\mathbb{R})) - \beta_k(X/\mathrm{c}, X(\mathbb{R})) \\
&= \sum_{i=0}^{k-1} \beta_i(X(\mathbb{R})) + \sum_{i=2n-k}^{2n} (\beta_i - \beta_i(X(\mathbb{R}))) - \sum_{i=k}^{2n} (\beta_i - \beta_i(X(\mathbb{R}))) \\
&= \sum_{i=0}^{k-1} \beta_i(X(\mathbb{R})) - \sum_{i=k}^{2n-k-1} \beta_i + \sum_{i=k}^{2n-k-1} \beta_i(X(\mathbb{R})) \\
&= \sum_{i=0}^{2n-k-1} \beta_i(X(\mathbb{R})) - \sum_{i=k}^{2n-k-1} \beta_i. \qquad (3.3)
\end{aligned}$$

Since $k < n$ and X is maximal, we find

$$\sum_{i=0}^{2n-k-1} \beta_i(X(\mathbb{R})) = \sum_{i=0}^{n} \beta_i(X(\mathbb{R})) = \beta_*,$$

and (3.3) can be rewritten as

$$\operatorname{rank}(\mathrm{in}_0^{k-1}) + \operatorname{rank}(\mathrm{in}_0^k) = \sum_{i=0}^{k-1} \beta_i + \sum_{i=2n-k}^{2n} \beta_i. \qquad (3.4)$$

Assume now that $\operatorname{rank}(\mathrm{in}_0^{k-1}) = \sum_{i=0}^{k-1} \beta_i$. From (3.4) we find that $\operatorname{rank}(\mathrm{in}_0^k) = \sum_{i=2n-k}^{2n} \beta_i$, which, by Poincaré duality, is equivalent to $\operatorname{rank}(\mathrm{in}_0^k) = \sum_{i=0}^{k} \beta_i$, concluding the induction argument. $\square$

Analogs of Lemmas 3.1 and 3.2 for $\mathbb{H}_i$, $i = 1, \dots, r$, are already available in the literature, in a wider context. For convenience, we collect below several results extracted from Theorem 4.2 in [12] and its proof.

Let F be a compact C^∞-manifold of real dimension m. The complement of the diagonal in its symmetric square $F^{(2)} \setminus \Delta F$ is naturally seen as the interior of a smooth compact $2m$-dimensional manifold $\mathbb{H}_F$ with boundary $\mathbb{P}_\mathbb{R}(T^*F)$. Let

$$\mathrm{in}_F^k : H^k(F^{(2)} \setminus \Delta F) = H^k(\mathbb{H}_F) \to H^k(\partial \mathbb{H}_F) = H^k(\mathbb{P}_\mathbb{R}(T^*F))$$

be the restriction homomorphism.

Theorem 3.3. *Let $z_1, \dots, z_s$ be a basis for $H^*(F)$ and let Z_i be a closed pseudomanifold in F that represents the class z_i. Let $b \in H^1(F^{(2)} \setminus \Delta F)$ be the class of the double cover $g : (F \times F) \setminus \Delta F \to F^{(2)} \setminus \Delta F$. For every integer $k \geq 0$, we have:*

1) *A basis for $H^k(F^{(2)} \setminus \Delta F)$ is given by the elements $g_*(z_i \otimes z_j)$ with $\deg z_i + \deg z_j = k$ and $i < j$, together with the elements $b^j[Z_i^{(2)} \setminus \Delta Z_i]$ such that $2 \deg z_i + j = k$, $i \geq 0$ and satisfying $0 \leq j \leq m - 1 - \deg z_i$.*
2) $\mathrm{in}_F^k\left(g_*(z_i \otimes z_j)\right) = 0$ *for all $i < j$.*
3) *The restrictions $\mathrm{in}_F^k(b^j[Z_i^{(2)} \setminus \Delta Z_i])$ satisfying $2 \deg z_i + j = k$, $i \geq 0$ and $0 \leq j \leq m - 1 - \deg z_i$ form a basis of $\mathrm{Im}(\mathrm{in}_F^k) \subseteq H^k\left(\mathbb{P}_{\mathbb{R}}(T^*F)\right)$.*

□

Adapted to our situation, we get the following corollaries.

Corollary 3.4. *If X is an n-dimensional nonsingular real projective variety, then, for every $i = 1, \dots, r$ and $k \geq 0$, the following formulas hold:*

1) $$\beta_{2k}(\mathbb{H}_i) = \sum_{\substack{a+b=2k \\ a<b}} \beta_a(F_i)\beta_b(F_i) + \frac{1}{2}\beta_k(F_i)(\beta_k(F_i) - 1) + \sum_{l=2k-n+1}^{k} \beta_l(F_i).$$
2) $$\beta_{2k+1}(\mathbb{H}_i) = \sum_{\substack{a+b=2k+1 \\ a<b}} \beta_a(F_i)\beta_b(F_i) + \sum_{l=2k+1-n+1}^{k} \beta_l(F_i).$$
3) $$\beta_*(\mathbb{H}_i) = \frac{1}{2}\beta_*(F_i)(\beta_*(F_i) - 1) + \sum_{k=0}^{n}(n-k)\beta_k(F_i).$$
4) *If, in addition, X is maximal, then* $\beta_*(\mathbb{H}) = \frac{1}{2}\sum_{i=1}^{r} \beta_*^2(F_i) + \frac{n-1}{2}\beta_*.$

Corollary 3.5. *If X is an n-dimensional real projective manifold, then*

$$\mathrm{rank}(\mathrm{in}^m) = \sum_{[\frac{m}{2}] \geq k \geq m-n+1} \beta_k(X(\mathbb{R})),$$

for every $m \in \{0, \dots, 2n\}$.

We conclude this section with the following results generalizing Proposition 3.2 in [6]. The proof presented in [6, Proposition 3.2] extends literally to higher dimensions and will be omitted.

Proposition 3.6. *Let X be a compact complex manifold of dimension n equipped with a real structure c. We have:*

1) *The relation*

$$\chi(X^{[2]}(\mathbb{R})) = \frac{1}{2}\beta_* - \beta_{\mathrm{odd}} + \frac{1}{2}\chi(X(\mathbb{R}))^2 - \chi(X(\mathbb{R})). \tag{3.5}$$

2) *If $\mathrm{Tors}_2\, H_*(X;\mathbb{Z}) = 0$, the relation*

$$\beta_*(X^{[2]}) = \frac{1}{2}\beta_*(\beta_* - 1) + n\beta_* - \beta_{\mathrm{odd}}, \tag{3.6}$$

3) *If* $\operatorname{Tors}_2 H_*(X;\mathbb{Z}) \neq 0$, *the relation*

$$\beta_*(X^{[2]}) \geq \frac{1}{2}\beta_*(\beta_* - 1) + n\beta_* - \beta_{\rm odd}. \tag{3.7}$$

3.3. **Special submanifolds of Hilbert squares.** Let X be a compact complex manifold of dimension n equipped with a real structure c, and $Y \subseteq X$ a smooth, c-invariant, complex submanifold of codimension m, with $Y(\mathbb{R}) \neq \emptyset$, and denote by $\upsilon \in H^m(X(\mathbb{R}))$ the cohomology class of $Y(\mathbb{R})$. The Hilbert square $Y^{[2]} \subseteq X^{[2]}$ is a c-invariant complex submanifold of codimension $2m$. Its real locus, $Y^{[2]}(\mathbb{R})$, is transversal to $E(\mathbb{R})$ and intersects the latter along $\mathbb{P}_{\mathbb{R}}(T^*Y(\mathbb{R}))$. The cycle $Y^{[2]}(\mathbb{R})$ defines a cohomology class Υ_Y in $H^{2m}(X^{[2]}(\mathbb{R}))$.

Let j_0, j and κ denote the inclusions $\mathbb{H}_0 \hookrightarrow X^{[2]}(\mathbb{R})$, $\sqcup_{i=1}^r \mathbb{H}_i \hookrightarrow X^{[2]}(\mathbb{R})$, and $E(\mathbb{R}) \hookrightarrow X^{[2]}(\mathbb{R})$, respectively. For every integer $i \geq 0$, consider the induced commutative diagram of restrictions:

$$\begin{tikzcd} & H^{2i}(\mathbb{H}_0) \arrow[dr, "\mathrm{in}_0^*"] & \\ H^{2i}(X^{[2]}(\mathbb{R})) \arrow[ur, "j_0^*"] \arrow[rr, "\kappa^*"] \arrow[dr, "j^*"'] & & H^{2i}(E(\mathbb{R})) \\ & H^{2i}(\sqcup_{i=1}^r \mathbb{H}_i) \arrow[ur, "\mathrm{in}^*"'] & \end{tikzcd} \tag{3.8}$$

Let

$$\begin{aligned} \theta_Y =& j_0^*(\Upsilon_Y) = [(Y/\,\mathrm{c}) \setminus Y(\mathbb{R})] \in H^{2m}(\mathbb{H}_0) \\ \sigma_Y =& j^*(\Upsilon_Y) = [Y(\mathbb{R})^{(2)} \setminus \Delta Y(\mathbb{R})] \in H^{2m}(\sqcup_{i=1}^r \mathbb{H}_i). \end{aligned}$$

By the commutativity of diagram (3.8), we have

$$\mathrm{in}_0^{2m}(\theta_Y) = \mathrm{in}^{2m}(\sigma_Y) = \kappa^*(\Upsilon_Y). \tag{3.9}$$

As is well-known, the cohomology ring $H^*(E(\mathbb{R}))$ is the polynomial ring $H^*(X(\mathbb{R}))[\gamma]/\langle \gamma^n + w_1\gamma^{n-1} + \cdots + w_n\rangle$, where $w_1, \ldots, w_n$ are the Stiefel-Whitney classes of X, and γ is the first Stiefel-Whitney class of the tautological line bundle of $E(\mathbb{R}) = \mathbb{P}_{\mathbb{R}}(T^*X(\mathbb{R}))$ denoted by η.

Further on, to avoid excessive notations, we use (where it does not lead to a misunderstanding) the same symbol for a cohomology class, or a vector bundle, and its restrictions.

Lemma 3.7. *We have*

$$\kappa^*(\Upsilon_Y) = \gamma^m \upsilon + \gamma^{m-1}\operatorname{Sq}^1 \upsilon + \cdots + \gamma \operatorname{Sq}^{m-1}\upsilon + \operatorname{Sq}^m \upsilon$$

where $\upsilon \in H^m(X(\mathbb{R}))$ *is the cohomology class of* $Y(\mathbb{R})$.

Proof. The statement is the real analog of Lemma 6.1 in [12]. The proof below follows the same lines.

Let $V = Y^{[2]}(\mathbb{R}) \cap E(\mathbb{R})$, and notice that

$$V = \mathbb{P}_{\mathbb{R}}(T^*Y(\mathbb{R})) \subseteq \mathbb{P}_{\mathbb{R}}(T^*X(\mathbb{R}))_{|Y(\mathbb{R})} \subseteq \mathbb{P}_{\mathbb{R}}(T^*X(\mathbb{R})) = E(\mathbb{R})$$

is the zero set of a transverse section of the vector bundle $\mathrm{Hom}(\eta^\vee, N)$ over $\mathbb{P}_{\mathbb{R}}(T^*X(\mathbb{R}))_{|Y(\mathbb{R})}$, where N is the pullback to $\mathbb{P}_{\mathbb{R}}(T^*X(\mathbb{R}))_{|Y(\mathbb{R})}$ of the normal bundle $N_{Y(\mathbb{R})/X(\mathbb{R})}$ of $Y(\mathbb{R})$ in $X(\mathbb{R})$. Indeed, such a section of $\mathrm{Hom}(\eta^\vee, N)$ is provided by the subbundle η of $T^*X(\mathbb{R})_{|Y(\mathbb{R})}$. As a consequence, the cohomology class of V in $\mathbb{P}_{\mathbb{R}}(T^*X(\mathbb{R}))_{|Y(\mathbb{R})}$ is the top Stiefel-Whitney class of the rank m vector bundle $\eta \otimes N$:

$$\begin{aligned} w_m(\eta \otimes N) &= w_1^m(\eta) + w_1^{m-1}(\eta) w_1(N) + \cdots + w_m(N) \\ &= \gamma^m + \gamma^{m-1} w_1(N) + \cdots + w_m(N). \end{aligned} \tag{3.10}$$

Let now $s : Y(\mathbb{R}) \to X(\mathbb{R})$ denote the inclusion. Since by [10, La formule générale] we have

$$\mathrm{Sq}^i\, v = s_* w_i(N_{Y(\mathbb{R})/X(\mathbb{R})}),$$

the conclusion of the lemma follows by pushing forward (3.10) to $E(\mathbb{R})$. $\square$

4. Proof of Theorem 1.1

We follow here the same strategy as in the proof of [6, Theorem 1.1]. As a first step, we show the following:

Proposition 4.1. *Let X be a real nonsingular projective variety of dimension $n \geq 2$. If $X^{[2]}$ is maximal, then $X(\mathbb{R}) \neq \emptyset$.*

Proof. By contradiction, let assume that $X^{[2]}$ is maximal and $X(\mathbb{R}) = \emptyset$. Then $X^{[2]}(\mathbb{R})$ is the smooth quotient manifold $X/\,\mathrm{c}$, of real dimension $2n$, and so

$$\beta_*(X^{[2]}(\mathbb{R})) = \beta_*(X/\,\mathrm{c}).$$

Since $X(\mathbb{R}) = \emptyset$, the first relevant homology groups in the Smith sequence in Theorem 2.1 satisfy

$$0 \to H_{2n}(X/\,\mathrm{c}) \xrightarrow{\Delta_{2n}} H_{2n-1}(X/\,\mathrm{c}) \xrightarrow{\mathrm{tr}^*_{2n-1}} H_{2n-1}(X) \to \cdots$$

Since $X/\,\mathrm{c}$ is connected, we have $\beta_{2n}(X/\,\mathrm{c}) = 1$ and we find

$$\beta_{2n-1}(X/\,\mathrm{c}) = 1 + \mathrm{rank}(\mathrm{tr}^*_{2n-1}) \leq \beta_{2n} + \beta_{2n-1}$$

where β_i states for $\beta_i(X)$. We assume

$$\beta_{p+1}(X/\,\mathrm{c}) \leq \sum_{k=p+1}^{2n} \beta_k.$$

Using again the exactness of the Smith sequence, we find

$$0 \to \mathrm{Im}(\mathrm{tr}^*_{p+1}) \to H_{p+1}(X) \xrightarrow{\mathrm{pr}_{*,p}} H_{p+1}(X/\,\mathrm{c}) \xrightarrow{\Delta_{p+1}} H_p(X/\,\mathrm{c}) \to \mathrm{Im}(\mathrm{tr}^*_p) \to 0$$

Hence

$$\beta_p(X/\,\mathrm{c}) = \mathrm{rank}(\mathrm{tr}_p^*) + \beta_{p+1}(X/\,\mathrm{c}) + \mathrm{rank}(\mathrm{tr}_{p+1}^*) - \beta_{p+1} \le \sum_{k=p}^{2n} \beta_k.$$

By descending induction, we find that

$$\beta_i(X/\,\mathrm{c}) \le \sum_{k=i}^{2n} \beta_k, \quad \text{for all}\, n \le i \le 2n.$$

By Poincaré duality, or by inspecting the Smith sequence starting from the other end, we also find

$$\beta_i(X/\,\mathrm{c}) \le \sum_{k=0}^{i} \beta_k, \quad \text{for all } 0 \le i \le n.$$

Therefore, we obtain

$$\beta_*(X/\,\mathrm{c}) \le (n+1)\beta_0 + n\beta_1 + \cdots + 2\beta_{n-1} + \beta_n + 2\beta_{n+1} + \cdots + (n+1)\beta_{2n}.$$

Since X is connected, we have $\beta_0 = \beta_{2n} = 1$ and find

$$\beta_*(X^{[2]}(\mathbb{R})) = \beta_*(X/\,\mathrm{c}) \le 2 + n\beta_*. \tag{4.1}$$

Also, since X is projective, $\beta_2 \ge 1$ and we notice that

$$\beta_* \ge \beta_* - \beta_{\mathrm{odd}} \ge \beta_0 + \beta_2 + \beta_{2n} \ge 3. \tag{4.2}$$

To finish the proof, we use now the third item of Proposition 3.6, (4.2) and (4.1) to notice that

$$\begin{aligned} \beta_*(X^{[2]}) &\ge \frac{1}{2}\beta_*(\beta_* + 1) + (n-1)\beta_* - \beta_{\mathrm{odd}} \\ &\ge 2\beta_* + (n-1)\beta_* - \beta_{\mathrm{odd}} \\ &= n\beta_* + (\beta_* - \beta_{\mathrm{odd}}) \\ &> 2 + n\beta_* \\ &\ge \beta_*(X^{[2]}(\mathbb{R})), \end{aligned}$$

contradicting the maximality of $X^{[2]}$. □

Remark 4.2. The projectivity assumption can be replaced by "compact complex" with $\beta_{2k} \ge 1$ for at least one $k, 0 < k < n$".

Proof of Theorem 1.1. The non-emptiness of $X(\mathbb{R})$ being now ensured by Proposition 4.1, the remaining part of the proof of Theorem 1.1 is identical to the proof of Theorem 1.1 in [6]. We include the main ideas for convenience of the reader.

Pick a point $p \in X(\mathbb{R})$, whose existence is ensured by Proposition 4.1, and consider the map $f : X \to X^{(2)}$ given by

$$f(x) = \{p, x\}.$$

Since the Hilbert-Chow map $\pi : X^{[2]} \to X^{(2)}$ is an isomorphism when restricted to $X^{[2]} \setminus E$ and $f(X \setminus \{p\}) \cap \pi(E) = \emptyset$, the restriction of f to $X \setminus \{p\}$ induces a map

$$\phi : X \setminus \{p\} \to X^{[2]}.$$

The map ϕ extends to the blowup $q : \mathrm{Bl}_p X \to X$ of X at the point p, and so we have a commutative diagram

$$\begin{tikzcd} \mathrm{Bl}_p X \arrow[r, "\phi"] \arrow[d, "q"'] & X^{[2]} \arrow[d, "\pi"] \\ X \arrow[r, "f"] & X^{(2)}. \end{tikzcd}$$

By [6, Lemma 4.2] [5], the map

$$\phi^* : H^*(X^{[2]}) \to H^*(\mathrm{Bl}_p X)$$

is surjective. Consider next the commutative diagram

$$\begin{tikzcd} H^*_G(X^{[2]}) \arrow[r] \arrow[d, "R^{[2]}"'] & H^*_G(\mathrm{Bl}_p X) \arrow[d, "R"] \\ H^*(X^{[2]}) \arrow[r, "\phi^*"] & H^*(\mathrm{Bl}_p X), \end{tikzcd}$$

where the notation $H^*_G(Y)$ stands for the equivariant cohomology with $\mathbb{F}_2$-coefficients of a topological space Y equipped with the action of a group G. In our case $G = \mathbb{F}_2$ acting by complex conjugation on the corresponding complex variety.

By [6, Proposition 2.6], since $X^{[2]}$ is maximal, the restriction map

$$R^{[2]} : H^*_G(X^{[2]}) \to H^*(X^{[2]})$$

is surjective, implying the surjectivity of the restriction map

$$R : H^*_G(\mathrm{Bl}_p X) \to H^*(\mathrm{Bl}_p X).$$

Applying once again [6, Proposition 2.6], we find that $\mathrm{Bl}_p(X)$ is maximal, and so there remains to notice that $\mathrm{Bl}_p(X)$ is maximal if and only if X is maximal. □

5. Projective complete intersections

This section is devoted to the proof of Theorem 1.2. We assume throughout this section that X *is a maximal real nonsingular projective complete intersection of dimension* n*.*

5.1. A reduction in the computation of the deficiency. Our approach varies according to the parity of the dimension of X.

[5] Lemma 4.2 in [6] is stated and proved for surfaces, but the same proof can be easily adapted in arbitrary dimension.

5.1.1. *Odd dimension.* We assume first that the dimension n of X is odd. In this case, $\chi(X(\mathbb{R})) = 0$, since $X(\mathbb{R})$ is a closed odd dimensional manifold. Moreover, $\beta_{\text{odd}} = \beta_n = \beta_* - (n+1)$, as it follows from the Lefschetz theorem on hyperplane sections. Therefore, using (3.5) and (3.6), the deficiency of $X^{[2]}$ can be computed as follows:

$$\begin{aligned}
\mathfrak{D}(X^{[2]}) &= \beta_*(X^{[2]}) - \beta_*(X^{[2]}(\mathbb{R})) \\
&= \beta_*(X^{[2]}) - 2\beta_{\text{even}}(X^{[2]}(\mathbb{R})) + \chi(X^{[2]}(\mathbb{R})) \\
&= \frac{1}{2}\beta_*^2 + n\beta_* - 2\beta_{\text{odd}} - 2\beta_{\text{even}}(X^{[2]}(\mathbb{R})) \\
&= \frac{1}{2}\beta_*^2 + (n-2)\beta_* + 2n + 2 - 2\beta_{\text{even}}(X^{[2]}(\mathbb{R})).
\end{aligned} \tag{5.1}$$

Notice now that

$$\begin{aligned}
\beta_{\text{even}}(X^{[2]}(\mathbb{R})) &= \beta_{\text{even}}(X^{[2]}_{\text{main}}(\mathbb{R})) + \beta_{\text{even}}(X^{[2]}_{\text{extra}}(\mathbb{R})) \\
&= 2\sum_{l=0}^{\frac{n-1}{2}} \Big(\beta_{2l}(X^{[2]}_{\text{main}}(\mathbb{R})) + \beta_{2l}(X^{[2]}_{\text{extra}}(\mathbb{R}))\Big).
\end{aligned} \tag{5.2}$$

To compute $\beta_{2l}(X^{[2]}_{\text{main}}(\mathbb{R}))$ for every integer l such that $2l \le n-1$, we use the Mayer-Vietoris sequence

$$\cdots \to H_{2l}(E(\mathbb{R})) \xrightarrow{\mu_{2l}} \bigoplus_{i=0}^{r} H_{2l}(\mathbb{H}_i) \to H_{2l}(X^{[2]}_{\text{main}}(\mathbb{R})) \to H_{2l-1}(E(\mathbb{R})) \xrightarrow{\mu_{2l-1}} \dots,$$

where μ is the inclusion map $\mu : E(\mathbb{R}) \to \sqcup_{i=0}^{r}\mathbb{H}_i$ (see Section 3). From the resulting short exact sequence

$$0 \to \operatorname{Coker}(\mu_{2l}) \to H_{2l}(X^{[2]}_{\text{main}}(\mathbb{R})) \to \operatorname{Ker}(\mu_{2l-1}) \to 0,$$

we find

$$\begin{aligned}
\beta_{2l}(X^{[2]}_{\text{main}}(\mathbb{R})) &= \dim\operatorname{Coker}(\mu_{2l}) + \dim\operatorname{Ker}(\mu_{2l-1}) \\
&= \beta_{2l}(\mathbb{H}_0) + \sum_{i=1}^{r}\beta_{2l}(\mathbb{H}_i) + \beta_{2l-1}(E(\mathbb{R})) \\
&\quad - \operatorname{rank}(\mu_{2l}) - \operatorname{rank}(\mu_{2l-1}).
\end{aligned} \tag{5.3}$$

Therefore, for every integer l such that $0 \le 2l \le n-1$, we have

$$\begin{aligned}
\beta_{2l}(X^{[2]}(\mathbb{R})) &= \beta_{2l}(X^{[2]}_{\text{main}}(\mathbb{R})) + \beta_{2l}(X^{[2]}_{\text{extra}}(\mathbb{R})) \\
&= \beta_{2l}(X^{[2]}_{\text{extra}}(\mathbb{R})) + \beta_{2l}(\mathbb{H}_0) + \sum_{i=1}^{r}\beta_{2l}(\mathbb{H}_i) + \beta_{2l-1}(E(\mathbb{R})) \\
&\quad - \operatorname{rank}(\mu_{2l}) - \operatorname{rank}(\mu_{2l-1}).
\end{aligned} \tag{5.4}$$

Notice that by the Künneth formula we have

$$\begin{aligned}\beta_{2l}(X^{[2]}_{\text{extra}}(\mathbb{R})) &= \sum_{1\le s<t\le r} \beta_{2l}(F_s\times F_t)\\ &= \sum_{i+j=2l}\sum_{1\le s<t\le r}\beta_i(F_s)\beta_j(F_t).\end{aligned} \tag{5.5}$$

5.1.2. *Even dimension.* We assume here that the dimension n of X is even. As it follows from the Lefschetz theorem on hyperplane sections, we have $\beta_{\text{odd}} = 0$. Therefore, using (3.5) and (3.6), the deficiency of $X^{[2]}$ can be computed as follows:

$$\begin{aligned}\mathfrak{D}(X^{[2]}) &= \beta_*(X^{[2]}) - \beta_*(X^{[2]}(\mathbb{R}))\\ &= \beta_*(X^{[2]}) - 2\beta_{\text{odd}}(X^{[2]}(\mathbb{R})) - \chi(X^{[2]}(\mathbb{R}))\\ &= \frac{1}{2}\beta_*^2 + (n-1)\beta_* - 2\beta_{\text{odd}}(X^{[2]}(\mathbb{R})) - \frac{1}{2}\chi(X(\mathbb{R}))^2 + \chi(X(\mathbb{R})).\end{aligned}$$

Since X is maximal, we have $\chi(X(\mathbb{R})) = \beta_* - 2\beta_{\text{odd}}(X(\mathbb{R}))$, and we find

$$\begin{aligned}\mathfrak{D}(X^{[2]}) &= n\beta_* + 2\beta_*\beta_{\text{odd}}(X(\mathbb{R})) - 2\beta_{\text{odd}}^2(X(\mathbb{R})) - 2\beta_{\text{odd}}(X(\mathbb{R}))\\ &\quad - 2\beta_{\text{odd}}(X^{[2]}(\mathbb{R}))\\ &= n\beta_* + 2\beta_{\text{even}}(X(\mathbb{R}))\beta_{\text{odd}}(X(\mathbb{R})) - 2\beta_{\text{odd}}(X(\mathbb{R}))\\ &\quad - 2\beta_{\text{odd}}(X^{[2]}(\mathbb{R})).\end{aligned} \tag{5.6}$$

By the same arguments as in the odd dimensional case, we get:

$$\begin{aligned}\beta_{\text{odd}}(X^{[2]}(\mathbb{R})) &= \beta_{\text{odd}}(X^{[2]}_{\text{main}}(\mathbb{R})) + \beta_{\text{odd}}(X^{[2]}_{\text{extra}}(\mathbb{R}))\\ &= 2\sum_{l=1}^{\frac{n}{2}}\Big(\beta_{2l-1}(X^{[2]}_{\text{main}}(\mathbb{R})) + \beta_{2l-1}(X^{[2]}_{\text{extra}}(\mathbb{R}))\Big)\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}\beta_{2l-1}(X^{[2]}_{\text{main}}(\mathbb{R})) &= \dim\operatorname{Coker}(\mu_{2l-1}) + \dim\operatorname{Ker}(\mu_{2l-2})\\ &= \beta_{2l-1}(\mathbb{H}_0) + \sum_{i=1}^{r}\beta_{2l-1}(\mathbb{H}_i) + \beta_{2l-2}(E(\mathbb{R}))\\ &\quad - \operatorname{rank}(\mu_{2l-1}) - \operatorname{rank}(\mu_{2l-2})\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\beta_{2l-1}(X^{[2]}_{\text{extra}}(\mathbb{R})) &= \sum_{1\le s<t\le r}\beta_{2l-1}(F_s\times F_t)\\ &= \sum_{i+j=2l-1}\sum_{1\le s<t\le r}\beta_i(F_s)\beta_j(F_t).\end{aligned} \tag{5.9}$$

As a result we have now reduced the computation of the deficiency of the Hilbert square is reduced to the computation of the ranks of the restriction maps in the Mayer-Vietoris sequence.

5.2. **The rank of Mayer-Vietoris restriction maps.** Let H be a smooth, c-invariant, hyperplane section of X, and denote by H_ℓ, $1 \leq \ell \leq n$, the iterated hyperplane sections. Without loss of generality, we can and will assume that each H_ℓ is smooth and c-invariant. Let $\upsilon \in H^1(X(\mathbb{R}))$ denote the cohomology class of $H(\mathbb{R})$. For every $1 \leq \ell \leq n$, the Hilbert square $H_\ell^{[2]}$ is a c-invariant, closed complex submanifold of complex codimension 2ℓ in $X^{[2]}$. The cycles $H_\ell^{[2]}(\mathbb{R})$, $1 \leq \ell \leq n$, define cohomology classes Υ_l in $H^{2\ell}(X^{[2]}(\mathbb{R}))$.

Let $b_0 \in H^1(\mathbb{H}_0 \setminus \partial\mathbb{H}_0) = H^1(\mathbb{H}_0)$ be the characteristic class of the double covering $X \setminus X(\mathbb{R}) \to \mathbb{H}_0 \setminus \partial\mathbb{H}_0$, discussed in Section 3.1. Using the notations from Section 3.3, we define

$$\theta_\ell = j_0^*(\Upsilon_l) \in H^{2\ell}(\mathbb{H}_0),\ 1 \leq \ell \leq n,$$

and set $\theta_0 = 1 \in H^0(\mathbb{H}_0) \simeq \mathbb{F}_2$.

Lemma 5.1. *Let $X \subseteq \mathbb{P}^N$ be a real, maximal, n-dimensional complete intersection. For every $k = 0, \dots, n-1$, the collection of classes $\left\{b_0^j\theta_\ell\right\}$ with $j + 2\ell = k$ form a basis of $H^k(\mathbb{H}_0)$.*

Proof. By Poincaré-Alexander-Lefschetz duality and Lemma 2.5, it suffices to show that for every $k = 0, \dots, n-1$, the elements in the set $\left\{b_0^j\theta_\ell\right\}$ with $j + 2\ell = k$ are linearly independent. Let $c_\ell \in \mathbb{F}_2$, $0 \leq \ell \leq [k/2]$ such that

$$\sum_{\ell=0}^{[k/2]} c_\ell b_0^{k-2\ell}\theta_\ell = 0. \tag{5.10}$$

Restricting (5.10) to $H^k(E(\mathbb{R}))$ we find:

$$0 = \sum_{\ell=0}^{[k/2]} c_\ell \operatorname{in}_0^*(b_0^{k-2\ell}\theta_\ell) = \sum_{\ell=0}^{[k/2]} c_\ell \gamma^{k-2\ell} \operatorname{in}_0^* j_0^*(\Upsilon_\ell) = \sum_{\ell=0}^{[k/2]} c_\ell \gamma^{k-2\ell} \kappa^*(\Upsilon_\ell).$$

Using now Lemma 3.7, we obtain the following relation in $H^k(E(\mathbb{R}))$:

$$\sum_{\ell=0}^{[k/2]} c_\ell \gamma^{k-2\ell}(\gamma^\ell \upsilon^\ell + \gamma^{\ell-1} \operatorname{Sq}^1 \upsilon^\ell + \dots + \gamma \operatorname{Sq}^{\ell-1} \upsilon^\ell + \operatorname{Sq}^\ell \upsilon^\ell) = 0.$$

Notice that the left-hand side is a degree k polynomial $P(\gamma)$ in the variable γ, with coefficients in $H^*(X(\mathbb{R}))$. Thus, according to the Leray-Hirsch theorem, the coefficients of each γ^{k-r}, $0 \leq r \leq k$ must vanish, and we find:

$$\sum_{j=0}^{[r/2]} c_{r-j} \operatorname{Sq}^j \upsilon^{r-j} = 0, \text{ for every } r = 0, \dots, k. \tag{5.11}$$

By Lemma 2.6, $\upsilon^r \neq 0$ for every $r \leq \left[\frac{n}{2}\right]$, and the same holds for the pullbacks $\pi^*\upsilon^r \neq 0$, as it follows, once more, from the Leray-Hirsch theorem.

Using this, from (5.11) we inductively obtain $c_{[k/2]-\ell} = 0$ for all $0 \leq \ell \leq [k/2]$. $\square$

Proposition 5.2. *Let $X \subseteq \mathbb{P}^N$ be a real, maximal, non-singular complete intersection of dimension $n \geq 1$. Then*

$$\operatorname{rank}(\mu_k) = \sum_{i=0}^{[k/2]} \beta_i(X(\mathbb{R})).$$

for every $0 \leq k \leq n-1$.

Proof. By (3.1) and (3.9), we find $\mathrm{in}_0^*(b_0^j) = \mathrm{in}^*(b^j) = \gamma^j$ and $\mathrm{in}_0^*(\theta_\ell) \in \mathrm{Im}(\mathrm{in}^*)$ for every $j \geq 0$ and $0 \leq \ell \leq \left[\frac{n}{2}\right]$, respectively. Therefore $\mathrm{in}_0^*(b_0^j\theta_\ell) \in \mathrm{Im}(\mathrm{in}^*)$ for every integers $j, \ell \geq 0$ such that $j + 2\ell = k$. In particular, we find $\mathrm{Im}(\mathrm{in}_0^k) \subseteq \mathrm{Im}(\mathrm{in}^k)$, for every $0 \leq k \leq n-1$. The proof now follows from Corollary 3.5. $\square$

5.3. **Proof of Theorem 1.2.**

5.3.1. *Odd dimension.* For every integer l such that $0 \leq 2l \leq n-1$, a direct computation using (5.5) and Corollary 3.4 shows that

$$\begin{aligned}
\beta_{2l}(X^{[2]}_{\text{extra}}(\mathbb{R})) + \sum_{i=1}^{r} \beta_{2l}(\mathbb{H}_i) &= \sum_{i+j=2l} \sum_{1\leq s<t\leq r} \beta_i(F_s)\beta_j(F_t) + \sum_{i=1}^{r} \sum_{\substack{a+b=2l\\ a<b}} \beta_a(F_i)\beta_b(F_i) \\
&\quad + \sum_{i=1}^{r} \frac{1}{2}\beta_l(F_i)(\beta_l(F_i)-1) + \sum_{i=1}^{r}\sum_{a=0}^{l} \beta_a(F_i) \\
&= \frac{1}{2}\sum_{a+b=2l} \beta_a(X(\mathbb{R}))\beta_b(X(\mathbb{R})) \\
&\quad + \sum_{a=0}^{l-1} \beta_a(X(\mathbb{R})) + \frac{1}{2}\beta_l(X(\mathbb{R})). \qquad (5.12)
\end{aligned}$$

According to Lemma 3.1,

$$\beta_{2l}(\mathbb{H}_0) = \sum_{i=2n-2l}^{2n} \beta_i(X) = \sum_{j=0}^{2l} \beta_j(X) = l+1,$$

so that, from (5.12) and Proposition 5.2, we infer that (5.4) can be written as follows:

$$\begin{aligned}\beta_{2l}(X^{[2]}(\mathbb{R})) &= (l+1) + \frac{1}{2}\sum_{i+j=2l}\beta_i(X(\mathbb{R}))\beta_j(X(\mathbb{R})) + \sum_{a=0}^{l-1}\beta_a(X(\mathbb{R}))\\ &\quad + \frac{1}{2}\beta_l(X(\mathbb{R})) + \sum_{i=0}^{2l-1}\beta_i(X(\mathbb{R})) - \sum_{i=0}^{l}\beta_i(X(\mathbb{R})) - \sum_{i=0}^{l-1}\beta_i(X(\mathbb{R}))\\ &= (l+1) + \frac{1}{2}\sum_{i+j=2l}\beta_i(X(\mathbb{R}))\beta_j(X(\mathbb{R}))\\ &\quad + \sum_{i=0}^{2l-1}\beta_i(X(\mathbb{R})) - \frac{1}{2}\beta_l(X(\mathbb{R})) - \sum_{i=0}^{l-1}\beta_i(X(\mathbb{R})). \qquad (5.13)\end{aligned}$$

From Lemma 2.7 and formula (5.13) we find:

$$\begin{aligned}\beta_{\text{even}}(X^{[2]}(\mathbb{R})) &= 2\sum_{l=0}^{\frac{n-1}{2}}\beta_{2l}(X^{[2]}(\mathbb{R}))\\ &= 2\sum_{l=0}^{\frac{n-1}{2}}(l+1) + \sum_{l=0}^{\frac{n-1}{2}}\sum_{i+j=2l}\beta_i(X(\mathbb{R}))\beta_j(X(\mathbb{R}))\\ &\quad + 2\sum_{l=1}^{\frac{n-1}{2}}\sum_{i=0}^{2l-1}\beta_i(X(\mathbb{R})) - \sum_{l=0}^{\frac{n-1}{2}}\beta_l(X(\mathbb{R})) - 2\sum_{l=1}^{\frac{n-1}{2}}\sum_{i=0}^{l-1}\beta_i(X(\mathbb{R}))\\ &= \frac{(n+1)(n+3)}{4} + \frac{\beta_*^2}{4} + \frac{(n-2)\beta_*}{2} - 2\sum_{l=1}^{\frac{n-1}{2}}\sum_{i=0}^{l-1}\beta_i(X(\mathbb{R})).\end{aligned}$$

Therefore, the deficiency of $X^{[2]}$ is given by:

$$\begin{aligned}\mathfrak{D}(X^{[2]}) &= \frac{1}{2}\beta_*^2 + (n-2)\beta_* + 2n + 2 - 2\beta_{\text{even}}(X^{[2]}(\mathbb{R}))\\ &= \frac{1}{2}\beta_*^2 + (n-2)\beta_* + 2n + 2\\ &\quad - \frac{(n+1)(n+3)}{2} - \frac{\beta_*^2}{2} - (n-2)\beta_* + 4\sum_{l=1}^{\frac{n-1}{2}}\sum_{i=0}^{l-1}\beta_i(X(\mathbb{R}))\\ &= 4\left(\sum_{l=1}^{\frac{n-1}{2}}\sum_{i=0}^{l-1}\beta_i(X(\mathbb{R})) - \frac{n^2-1}{8}\right),\end{aligned}$$

which concludes the proof of Theorem 1.2 when the dimension of X is odd.

5.3.2. *Even dimension.* For every integer l such that $1 \leq l \leq \frac{n}{2}$, a direct computation using (5.9) and Corollary 3.4 shows that

$$\beta_{2l-1}(X^{[2]}_{\text{extra}}(\mathbb{R})) + \sum_{i=1}^{r} \beta_{2l-1}(\mathbb{H}_i) =$$

$$\sum_{i+j=2l-1} \sum_{1\leq s<t\leq r} \beta_i(F_s)\beta_j(F_t) + \sum_{i=1}^{r} \sum_{\substack{a+b=2l-1\\ a<b}} \beta_a(F_i)\beta_b(F_i) + \sum_{i=1}^{r}\sum_{c=0}^{l-1} \beta_c(F_i) =$$

$$\sum_{\substack{a+b=2l-1\\ a<b}} \beta_a(X(\mathbb{R}))\beta_b(X(\mathbb{R})) + \sum_{c=0}^{l-1} \beta_c(X(\mathbb{R})). \tag{5.14}$$

Notice from Lemma 3.1 and the Lefschetz theorem on hyperplane sections that $\beta_{2l-1}(\mathbb{H}_0) = l$. By (5.14) and Proposition 5.2, we now infer that (5.8) can be written as follows:

$$\begin{aligned}\beta_{2l-1}(X^{[2]}(\mathbb{R})) = l + &\sum_{\substack{a+b=2l-1\\ a<b}} \beta_a(X(\mathbb{R}))\beta_b(X(\mathbb{R})) \\ &+ \sum_{i=0}^{2l-2} \beta_i(X(\mathbb{R})) - \sum_{i=0}^{l-1} \beta_i(X(\mathbb{R})).\end{aligned} \tag{5.15}$$

From Lemma 2.8 and formula (5.15) we find:

$$\begin{aligned}\beta_{\text{odd}}(X^{[2]}(\mathbb{R})) &= 2\sum_{l=1}^{\frac{n}{2}} \beta_{2l-1}(X^{[2]}(\mathbb{R})) \\ &= 2\sum_{l=1}^{\frac{n}{2}} l + \sum_{l=1}^{\frac{n}{2}} \sum_{\substack{a+b=2l-1\\ a<b}} \beta_a(X(\mathbb{R}))\beta_b(X(\mathbb{R})) \\ &\quad + 2\sum_{l=1}^{\frac{n}{2}} \sum_{i=0}^{2l-2} \beta_i(X(\mathbb{R})) - 2\sum_{l=1}^{\frac{n}{2}} \sum_{i=0}^{l-1} \beta_i(X(\mathbb{R})) \\ &= \frac{n(n+2)}{4} + \beta_{\text{even}}(X(\mathbb{R}))\beta_{\text{odd}}(X(\mathbb{R})) + \frac{n}{2}\beta_* - \beta_{\text{odd}}(X(\mathbb{R})) \\ &\quad - 2\sum_{l=1}^{\frac{n}{2}} \sum_{i=0}^{l-1} \beta_i(X(\mathbb{R})).\end{aligned} \tag{5.16}$$

Using (5.6) and (5.16), we compute now the deficiency of $X^{[2]}$:

$$\begin{aligned}\mathfrak{D}(X^{[2]}) &= n\beta_* + 2\beta_{\text{even}}(X(\mathbb{R}))\beta_{\text{odd}}(X(\mathbb{R})) - 2\beta_{\text{odd}}(X(\mathbb{R})) - 2\beta_{\text{odd}}(X^{[2]}(\mathbb{R})) \\ &= n\beta_* + 2\beta_{\text{even}}(X(\mathbb{R}))\beta_{\text{odd}}(X(\mathbb{R})) - 2\beta_{\text{odd}}(X(\mathbb{R})) \\ &\quad - \frac{n(n+2)}{2} - 2\beta_{\text{even}}(X(\mathbb{R}))\beta_{\text{odd}}(X(\mathbb{R})) - n\beta_* + 2\beta_{\text{odd}}(X(\mathbb{R})) \\ &\quad + 4\sum_{l=1}^{\frac{n}{2}}\sum_{i=0}^{l-1}\beta_i(X(\mathbb{R}) \\ &= 4\left(\sum_{l=1}^{\frac{n}{2}}\sum_{i=0}^{l-1}\beta_i(X(\mathbb{R})) - \frac{n(n+2)}{8}\right),\end{aligned}$$

which concludes the proof of Theorem 1.2 in the case when the dimension of X is even. □

5.4. **Proof of Theorem 1.5.** We assume first that $X^{[2]}$ is maximal. Then, by Theorem 1.1, X is maximal, and thus applying Corollary 1.3 we conclude that $\beta_i(X(\mathbb{R})) = 1 = \beta_{2i}$ for every i such that $0 \leq i \leq k-1$ with $k = \frac{1}{2}\dim X$. From the maximality of X and Poincaré duality we find that $\beta_k(X(\mathbb{R})) = \beta_{2k}$. This implies

$$\begin{aligned}\chi(X(\mathbb{R})) &= \sum_{i=0}^{k-1}(-1)^i\beta_i(X(\mathbb{R})) + \beta_k(X(\mathbb{R})) + \sum_{i=k+1}^{2k}(-1)^i\beta_i(X(\mathbb{R})) \\ &= \sum_{i=0}^{k-1}(-1)^i + \beta_{2k} + \sum_{i=k+1}^{2k}(-1)^i.\end{aligned}$$

On the other hand, according to Lefschetz trace formula we have

$$\chi(X(\mathbb{R})) = \sum_{i=0}^{k-1}(-1)^i + \operatorname{tr} H^{k,k}(X(\mathbb{C})) + \sum_{i=k+1}^{2k}(-1)^i.$$

Taking the difference we obtain

$$\beta_{2k} = \operatorname{tr} H^{k,k}(X(\mathbb{C})).$$

However, the inequality $h^{k,k}(X(\mathbb{C})) < \beta_{2k}$ holds for any nonsingular projective complete intersection of dimension $2k \geq 2$ except cubic surfaces in $\mathbb{P}^3$, quadrics, and intersections of two quadrics (see [9]).

Conversely, notice that linear subspaces $X \subset \mathbb{P}^N$ satisfy $\beta_i(X(\mathbb{R})) = 1$ for all i such that $1 \leq i \leq \dim X$. In the case when X is a maximal real nonsingular quadric or a maximal real nonsingular intersection of two quadrics, then $\beta_i(X(\mathbb{R})) = 1$ f or every integer i such that $0 \leq i \leq \left[\frac{n}{2}\right]-1$ (*cf.* [8, Section 7.1] or [11, Proposition 1.10] and [11, Theorem 4.9]). Likewise, if X is a maximal real nonsingular cubic surface in $\mathbb{P}^3$, then X is the blow-up of $\mathbb{P}^2$ at 6 real points, and so $\beta_0(X(\mathbb{R})) = 1$. Thus, in all the four cases the conditions in Corollary 1.3 are fulfilled. Hence, in all the four cases the Hilbert square is maximal. □

Remark 5.3. The proof given above shows the maximality of the Hilbert square for real linear spaces and maximal real quadrics of any dimension, not only of even-dimensional ones.

6. Maximality criteria and the proof of Theorems 1.7 and 1.8

We start by proving first the following maximality criterium.

Proposition 6.1. *Let X be a maximal real nonsingular n-dimensional algebraic variety with $\operatorname{Tors}_2 H_*(X,\mathbb{Z}) = 0$. Then the defect of $X^{[2]}$ is given by*

$$\mathfrak{D}(X^{[2]}) = 2\left(\operatorname{rank}\mu_* - \frac{n}{2}\beta_* - \frac{1}{2}\beta_{\text{odd}}\right).$$[6]

In particular, $X^{[2]}$ is maximal if and only if $\operatorname{rank}\mu_* = \dfrac{n}{2}\beta_* + \dfrac{1}{2}\beta_{\text{odd}}$.

Proof. The Mayer-Vietoris sequence

$$\cdots \to H_k(E(\mathbb{R})) \xrightarrow{\mu_k} \bigoplus_{i=0}^{r} H_k(\mathbb{H}_i) \to H_k(X^{[2]}_{\text{main}}(\mathbb{R})) \to H_{k-1}(E(\mathbb{R})) \xrightarrow{\mu_{k-1}} \cdots$$

gives rise to a short exact sequence

$$0 \to \operatorname{Coker}(\mu_k) \to H_k(X^{[2]}_{\text{main}}(\mathbb{R})) \to \operatorname{Ker}(\mu_{k-1}) \to 0,$$

and so

$$\beta_k(X^{[2]}_{\text{main}}(\mathbb{R})) = \dim\operatorname{Coker}(\mu_k) + \dim\operatorname{Ker}(\mu_{k-1}).$$

As a consequence, we have

$$\begin{aligned}
\beta_*(X^{[2]}_{\text{main}}(\mathbb{R})) &= \sum_{k=0}^{2n} \beta_k(X^{[2]}_{\text{main}}(\mathbb{R})) \\
&= \sum_{k=0}^{2n} \dim\operatorname{Coker}(\mu_k) + \dim\operatorname{Ker}(\mu_{k-1}) \\
&= \sum_{i=0}^{r} \beta_*(\mathbb{H}_i) + \beta_*(E(\mathbb{R})) - 2\operatorname{rank}(\mu_*).
\end{aligned}$$

Furthermore, by Leray-Hirsch, we have $\beta_*(E(\mathbb{R})) = \beta_*(\mathbb{P}^{n-1}(\mathbb{R}))\beta_*(X(\mathbb{R}))$. In our case, X is maximal, and so

$$\beta_*(E(\mathbb{R})) = n\beta_*. \tag{6.1}$$

[6]It is not difficult to check that in the case of surfaces this formula for the Smith-Thom deficiency of $X^{[2]}$ concords with that given in [6, Remark 5.4].

Using now Lemma 3.1 and Corollary 3.4, we find

$$\begin{aligned}\beta_*(X^{[2]}_{\text{main}}(\mathbb{R})) &= \sum_{i=0}^{r} \beta_*(\mathbb{H}_i) + \beta_*(E(\mathbb{R})) - 2\operatorname{rank}(\mu_*)\\ &= \frac{1}{2}\sum_{i=1}^{r} \beta_*^2(F_i) - \frac{1}{2}\beta_* + 2n\beta_* - 2\operatorname{rank}(\mu_*). \qquad (6.2)\end{aligned}$$

Notice now that by the Künneth formula we get

$$\beta_*(X^{[2]}_{\text{extra}}(\mathbb{R})) = \sum_{1\le i<j\le r} \beta_*(F_i \times F_j) = \frac{1}{2}(\beta_*^2 - \sum_{1\le i\le r} \beta_*^2(F_i)). \qquad (6.3)$$

From (6.2) and (6.3), we find

$$\begin{aligned}\beta_*(X^{[2]}(\mathbb{R})) &= \beta_*(X^{[2]}_{\text{main}}(\mathbb{R})) + \beta_*(X^{[2]}_{\text{extra}}(\mathbb{R}))\\ &= \frac{1}{2}\sum_{i=1}^{r} \beta_*^2(F_i) - \frac{1}{2}\beta_* + 2n\beta_* - 2\operatorname{rank}(\mu_*) + \frac{1}{2}\beta_*^2 - \frac{1}{2}\sum_{1\le i\le r} \beta_*^2(F_i)\\ &= \frac{1}{2}\beta_*(\beta_* - 1) + 2n\beta_* - 2\operatorname{rank}(\mu_*). \qquad (6.4)\end{aligned}$$

Invoking now the second item of Proposition 3.6, from (6.4) we find that the maximality defect of $X^{[2]}$ is

$$\begin{aligned}\mathfrak{D}(X^{[2]}) &= \beta_*(X^{[2]}) - \beta_*(X^{[2]}(\mathbb{R}))\\ &= 2\left(\operatorname{rank}(\mu_*) - \frac{n}{2}\beta_* - \frac{1}{2}\beta_{\text{odd}}\right).\end{aligned}$$

□

Proposition 6.1 yields an effective criterion for the maximality of Hilbert squares in the absence of cohomology in odd degree. Before we state this criterion, we recall the following well known observation which goes back to Rokhlin and Thom.

Let M be a smooth compact n-dimensional manifold with boundary (*neither ∂M or M is assumed connected*). Denote by $\mathfrak{i} : \partial M \to M$ the inclusion map, and by $\mathfrak{i}_k$ and $\mathfrak{i}_*$ the induced maps, $H_k(\partial M) \to H_k(M)$ and $H_*(\partial M) \to H_*(M)$, respectively.

Lemma 6.2. $\dim \operatorname{Ker}(\mathfrak{i}_*) = \operatorname{rank}(\mathfrak{i}_*) = \frac{1}{2}\dim H_*(\partial M)$.

Proof. It suffice to show that $\dim \operatorname{Ker}(\mathfrak{i}_*) = \frac{1}{2}\dim H_*(\partial M)$. The Poincaré-Lefschetz duality implies that for every $0 \le k \le n-1$ the k-dimensional part of $\operatorname{Ker}(\mathfrak{i}_k)$ is the orthogonal complement of $\operatorname{Ker}(\mathfrak{i}_{n-1-k})$. Herefrom,

$$\dim \operatorname{Ker}(\mathfrak{i}_k) + \dim \operatorname{Ker}(\mathfrak{i}_{n-1-k}) = \beta_k(\partial M) = \beta_{n-1-k}(\partial M).$$

The result follows by summing over k from 0 to $n-1$. □

Proposition 6.3. *Let X be a maximal n-dimensional real nonsingular projective variety with $H_{\text{odd}}(X) = 0$. Then, $X^{[2]}$ is maximal if and only if one of the following two equivalent conditions holds*

1) $\operatorname{Ker}(\text{in}_{0*}) = \operatorname{Ker}(\text{in}_*)$,
2) $\operatorname{Im}(\text{in}_0^*) = \operatorname{Im}(\text{in}^*)$.

Proof. By universal coefficient theorem, the assumption $H_{\text{odd}}(X) = 0$ implies $\operatorname{Tors}_2 H_*(X, \mathbb{Z}) = 0$. Thus, Proposition 6.1 applies and shows that $X^{[2]}$ is maximal if and only if $\operatorname{rank}(\mu_*) = \frac{n}{2}\beta_*$. As $\dim H_*(E(\mathbb{R})) = n\beta_*$, this is equivalent to $\dim \operatorname{Ker}(\mu_*) = \frac{n}{2}\beta_*$. Notice now that $\operatorname{Ker}(\mu_*) = \operatorname{Ker}(\text{in}_{0*}) \cap \operatorname{Ker}(\text{in}_*)$ and, by Lemma 6.2, $\dim \operatorname{Ker}(\text{in}_{0*}) = \dim \operatorname{Ker}(\text{in}_*) = \frac{n}{2}\beta_*$. Therefore, the condition $\dim \operatorname{Ker}(\mu_*) = \frac{n}{2}\beta_*$ is equivalent to $\operatorname{Ker}(\text{in}_{0*}) = \operatorname{Ker}(\text{in}_*)$. By duality, this is equivalent to $\operatorname{Im}(\text{in}_0^*) = \operatorname{Im}(\text{in}^*)$. $\square$

Proof of Theorem 1.7. Since $X^{[2]}$ is maximal, by Proposition 6.3 we have $\operatorname{Im}(\text{in}_0^*) = \operatorname{Im}(\text{in}^*)$. In particular, we find $\operatorname{Im}(\text{in}_0^k) = \operatorname{Im}(\text{in}^k)$, for every $k = \{0, \dots, 2n-1\}$. If $k < n$, using Corollary 3.5 and Lemma 3.2 we find

$$\sum_{\ell=0}^{[k/2]} \beta_\ell(X(\mathbb{R})) = \sum_{\ell=0}^{k} \beta_\ell.$$

By induction, since $\beta_{2i+1} = 0$ for all $i \geq 1$, we find that

$$\beta_k(X(\mathbb{R})) = \beta_{2k} \quad \text{for all} \quad k < \frac{n}{2}.$$

By Poincaré duality, this extends to $k > n/2$. Finally, by Theorem 1.1, X is maximal, and so $\sum_{k=0}^{n} \beta_k(X(\mathbb{R})) = \sum_{i=0}^{n} \beta_{2k}$, which implies $\beta_k(X(\mathbb{R})) = \beta_{2k}$ if $n = 2k$. $\square$

Proof of Theorem 1.8. By Proposition 6.3 and Lemma 6.2, it suffices to show that $\operatorname{Im}(\text{in}_0^*) \supseteq \operatorname{Im}(\text{in}^*)$.

Let $\{Y_\alpha\}_{\alpha \in I}$ be a finite collection of c-invariant, smooth submanifolds of X, such that the group $H^*(X(\mathbb{R}))$ is generated by the Poincaré dual of the fundamental classes of $Y_\alpha(\mathbb{R}), \alpha \in I$. Without loss of generality, we can assume $Y_\alpha(\mathbb{R}) \neq \emptyset$ for every $\alpha \in I$. By Theorem 3.3, $\operatorname{Im}(\text{in}^*)$ is generated by the elements $\text{in}^*(b^j[Y_\alpha^{(2)}(\mathbb{R}) \setminus \Delta Y_\alpha])$, $\alpha \in I$. It remains to notice that (3.9) in Section 5.2 implies that for every $\alpha \in I$ and $j \geq 0$

$$\text{in}^*(b^j[Y_\alpha(\mathbb{R})^{(2)} \setminus \Delta Y(\mathbb{R})]) = \gamma^j \kappa^*(\Upsilon_{Y_\alpha}) = \text{in}_0^*(b_0^j \theta_{Y_\alpha}).$$

$\square$

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