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Smooth structures and Einstein metrics on $\mathbb{CP}^2\#5, 6, 7\overline{\mathbb{CP}^2}$

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Abstract

We show that each of the topological 4-manifolds $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, for $k = 5, 6, 7, 8$ admits a smooth structure which has an Einstein metric of scalar curvature $s > 0$, a smooth structure which carries an Einstein metric with $s < 0$ and infinitely many non-diffeomorphic smooth structures which do not admit Einstein metrics. We also exhibit new examples of higher dimensional manifolds carrying Einstein metrics of both positive and negative scalar curvature.

1. *Introduction*

Recently, new methods have been developed [**1**, **10**] to construct symplectic 4-manifolds with small topology and exotic smooth structures. Moreover, the method proposed by J. Park [**10**] was later refined [**8**, **11**] to produce interesting examples of minimal complex surfaces of general type. In this paper we show how these constructions can be used in regard to the existence or non-existence of Einstein metrics.

In [**4**], Catanese and LeBrun have shown that there exist homeomorphic, non-diffeomorphic manifolds such that one of them admits an Einstein metric of positive sign while the other admits an Einstein metric of negative sign. The manifold which has positive scalar curvature Einstein metric is $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, while the second manifold is a deformation of the Barlow surface. The key ingredient in their proof was to show that the Barlow surface, a simply connected minimal surface of general type with $K^2 = 1$ can be deformed to a surface with ample canonical bundle. In particular, they showed that there exists a simply connected complex surface of general type with $p_g = 0$, $K^2 = 1$ and having ample canonical bundle. The ampleness of the canonical bundle ensures the existence of a Kähler–Einstein metric of negative scalar curvature [**2**, **16**].

Based on ideas from [**10**], Lee and Park [**8**] and more recently Park, Park and Shin [**11**] constructed new examples of simply connected, minimal surfaces of general type with $p_g = 0$, $K^2 = 1, 2$ or 3. We show that their examples satisfy the ampleness condition. As far as the authors are aware, these are the examples of complex surfaces with *ample* canonical line bundles with the smallest known topological invariants.

THEOREM 1·1. *There exists a simply connected complex surface of general type, with* $p_g = 0$, $K^2 = 2$ *or* 3, *and ample canonical bundle.*

Remark 1·2. In [**12**] Park, Park and Shin found a complex surface of general type with $p_g = 0$, $K^2 = 4$. Using the same techniques as the ones in the proof of the above theorem, they were also able to show the ampleness of the canonical line bundle for this new example.

On $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, Catanese and LeBrun exhibit [**4**] two smooth structures, each manifold admitting an Einstein, but the metrics have opposite signs, while Park [**10**] constructs a smooth structure which does not admit an Einstein metric. We extend their results to $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, for $k = 5, 6, 7$, and we also exhibit an infinite family of differential structures for which no Einstein metric exists.

THEOREM 1·3. *Each of the topological* 4*-manifolds* $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, *for* $k = 5, 6, 7, 8$ *admits a smooth structure which has an Einstein metric of scalar curvature* $s > 0$, *a smooth structure which has an Einstein metric with* $s < 0$ *and infinitely many non-diffeomorphic smooth structures which do not admit Einstein metrics.*

It is a well known fact that in dimensions 2 and 3 the sign of the Einstein metric is a topological invariant. This led Besse [**3**, p.19] to consider the conjecture that no smooth compact $n-$manifold can admit Einstein metrics with different scalar curvature signs. As we've seen in the previous theorem, in dimension four, a change of the differential structure can change the sign of the Einstein metric. For higher dimensions, as far as the authors know, Catanese and LeBrun constructed [**4**] the only known counterexamples to Besse's conjecture. These examples are in dimensions $4k$, where $k \geqslant 2$. In these dimensions, we are able to provide many new examples with the same property:

PROPOSITION 1·4. *Let* $N_1 = \mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, $N_2 = \mathbb{CP}^2\#7\overline{\mathbb{CP}^2}$, $N_3 = \mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$ *and* $N_4 = \mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$. *Then the manifold* N *obtained by taking the* $k-$*fold products,* $k \geqslant 2$, *of arbitrary* N_1, N_2, N_3 *or* N_4, *admits two Einstein metrics* g_1, g_2 *such that the signs of the scalar curvature are* $s_{g_1} = -1$, $s_{g_2} = +1$. *Moreover, these metrics are Kähler-Einstein with respect to two distinct complex structures* J_1, J_2.

As a peculiar consequence of the Kählerian property of the metrics and of our method of construction the volumes of these two Einstein metrics actually coincide: $Vol_{g_1}(N) = Vol_{g_2}(N)$.

The paper is organized as follows: in Section 2 we discuss the ampleness of the canonical line bundles of some interesting examples of complex surfaces and the relation to the existence of Einstein metrics, while in Section 3 we introduce the Seiberg-Witten invariants and use them to get results on non-existence of Einstein metrics. In Section 4 we treat the higher dimensional case.

2. *Ampleness of the (anti)canonical bundle: existence of Einstein metrics*

In general the existence or non-existence of Einstein metrics on a given manifold is hard to prove. In the case $c_1(M) > 0$, in the unobstructed situations, i.e. for certain complex surfaces whose underlying differential structure is $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, $3 \leqslant k \leqslant 8$, the existence of Kähler–Einstein metrics was proved by Siu [**13**] and by Tian and Yau [**14**]. A complete solution to the existence problem was given by Tian [**15**]:

THEOREM 2·1 (Tian). *A compact complex surface* (M^4, J) *admits a compatible Kähler–Einstein metric with* $s > 0$ *if and only if its anti-canonical line bundle* K_M^{-1} *is ample and its Lie algebra of holomorphic vector fields is reductive.*

In the case when $c_1(M) < 0$, a criterion for the existence of a Kähler–Einstein metric was independently found by Aubin [**2**] and Yau [**16**]:

THEOREM 2·2 (Aubin,Yau). *A compact complex manifold* (M^4, J) *admits a compatible Kähler-Einstein metric with* $s < 0$ *if and only if its canonical line bundle* K_M *is ample. When such a metric exists, it is unique, up to an overall multiplicative constant.*

In this section, we discuss some examples of smooth complex structures on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, $6 \leqslant k \leqslant 8$. For each such k, the natural smooth structure as the complex projective plane blown-up at k generic points, can be endowed [**15**] with a Kähler–Einstein metric of positive scalar curvature. On the other hand, Lee and Park [**8**] and more recently Park, Park and Shin [**11**] have constructed new exotic smooth structures on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, for any $6 \leqslant k \leqslant 8$. Moreover, these admit complex structures yielding interesting examples of minimal surfaces of general type. We will show that each of these surfaces has ample canonical line bundle, and therefore admits a Kähler-Einstein metric of negative scalar curvature.

The examples constructed in these papers are very similar. In this article we are going to treat the most complicated example, of a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 3$ from [**11**], in depth. The other examples can be treated similarly, and we only sketch an outline of the proofs.

We begin with a description of the example of a minimal surface of general type with $p_g = 0$ and $K^2 = 3$, as in [**11**]. We start with two degenerate cubics on $\mathbb{CP}^2$. The first cubic is given by a smooth conic B and a transverse line A. From an arbitrary point Q on the line A, we consider the tangent lines L_1, L_2 to the conic. Let P, R be the points of tangency and L_3 the line defined by these points. We consider the pencil of cubics associated to $A + B$ and $L_1 + L_2 + L_3$. We need to blow up $\mathbb{CP}^2$ at nine base points of the pencil to obtain an elliptic fibration, $g\colon Y \to \mathbb{CP}^1$. The rational elliptic surface Y has four singular fibers: two degenerate fibers of type I_8 and I_2, and two nodal fibers of type I_1. Moreover, it admits 4 sections. First we blow up the singular points of the nodal fibers to smoothen them. Then, we blow up: the singular I_8 fiber five times, where the exceptional divisors are $F_1, G_7, G_8, \ldots, G_{10}$, the I_2 fiber at two points, with exceptional divisors F_4, F_8, the first nodal fiber at two points, with exceptional divisors F_2, F_5, and the second nodal fiber at 10 points, with exceptional divisors $F_3, F_6, F_7, H_6, H_7, I_2, \ldots, I_6$, as represented in the Figure 1. We obtain a new rational surface Z, which is the blow-up of $\mathbb{CP}^2$ at $9 + 2 + 5 + 2 + 2 + 10 = 30$ points. Then, Z contains four particular disjoint chains of rational curves consisting of proper transforms of the singular fibers, proper transform of some sections and some of the exceptional divisors introduced in the blowing-up process. These chains are $G = \sum_{i=1}^{10} G_i$, $H = \sum_{i=1}^{7} H_i$, $I = \sum_{i=1}^{6} I_i$ and a chain of length one denoted by A, represented by continuous lines in Figure 1.

We denoted by F_i, $i = 1, \ldots, 7$ the seven smooth rational curves of self-intersection -1, which are marked by discontinuous lines in Figure 1 and by F_8 the remaining exceptional

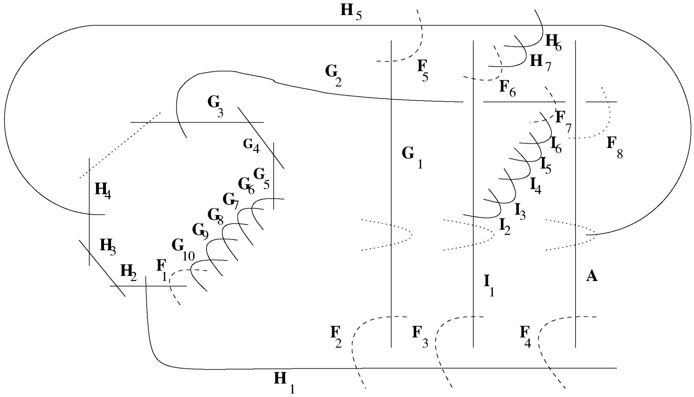

Fig. 1. The manifold $Z = \mathbb{CP}^2 \# 30\overline{\mathbb{CP}^2}$.

divisor from the last singular fiber, drawn as a dotted line. It is easy to see that the Poincaré duals of the curves $F_i, i = 1, \ldots, 7$, together with all of the irreducible components of our four chains G, H, I and A form a basis of $H^2(Z, \mathbb{Q})$, which is torsion-free and 31-dimensional.

All of the above four chains satisfy Artin's contractibility criterion [**5**]. By contracting them, we obtain a projective surface X with 4 singularities, each admitting 1-parameter $\mathbb{Q}-$Gorenstein smoothings. Let $f\colon Z \to X$ be the contraction map. Let $Exc(f) = \sum G_i + \sum H_i + \sum I_i + A$ denote the exceptional divisor of f.

Using the arguments from [**8**], it is proved in [**11**] that the projective surface X admits a $\mathbb{Q}-$Gorenstein smoothing, whose generic fiber is a smooth, simply-connected, minimal surface of general type with $K^2 = 3$. Moreover, K_X is nef and:

$$f^* K_X \equiv_{\mathbb{Q}} \sum_{i=1}^{8} a_i F_i + \sum_{i=1}^{10} b_i G_i + \sum_{i=1}^{7} c_i H_i + \sum_{i=1}^{6} d_i I_i + eA. \qquad (2{\cdot}1)$$

In [**11**], the coefficients are explicitly given, but for our purpose it is important to know that a_i, b_i, c_i, d_i and e are strictly positive rational numbers. We should point out that there is a slight difference in our notation from the one in [**11**], when it comes to labeling the curves F_i.

First, we prove the ampleness of the canonical divisor of the singular surface X. Suppose that K_X fails to be ample. Since K_X is already nef, according to the Nakai–Moishezon criterion [**5**], there exists an irreducible curve $C \subset X$ such that $(K_X \cdot C) = 0$.

The total transform of C in Z is

$$f^* C \equiv_{\mathbb{Q}} C' + \sum_{i=1}^{10} x_i G_i + \sum_{i=1}^{7} y_i H_i + \sum_{i=1}^{6} z_i I_i + tA, \qquad (2{\cdot}2)$$

where C' is the strict transform of C and x_i, y_i, z_i, t are non-negative rational numbers. We should note here that C' is not numerically equivalent to 0. To give an immediate proof for this assertion, note that $(C \cdot H) > 0$ for any ample line bundle H on the projective surface X. But, if $C' \equiv_{\mathbb{Q}} 0$ we would have

$$0 = (C' \cdot f^* H) = (f^* C \cdot f^* H) = (C \cdot H),$$

and this is a contradiction.

Now, a straightforward computation gives:

$$\begin{aligned}(K_X \cdot C) &= (f^*K_X \cdot f^*C) = (f^*K_X \cdot C') \\ &= \sum_{i=1}^{8} a_i(F_i \cdot C') + \sum_{i=1}^{10} b_i(G_i \cdot C') + \sum_{i=1}^{7} c_i(H_i \cdot C') \\ &\quad + \sum_{i=1}^{6} d_i(I_i \cdot C') + e(A \cdot C').\end{aligned} \tag{2·3}$$

The intersection number of C' with any component of $Exc(f)$ is greater or equal to 0, with equality if and only if C' does not intersect any of the irreducible components of $Exc(f)$, i.e. C does not pass through the singular points of X. Hence, it follows that

$$\sum_{i=1}^{10} b_i(G_i \cdot C') + \sum_{i=1}^{7} c_i(H_i \cdot C') + \sum_{i=1}^{6} d_i(I_i \cdot C') + e(A \cdot C') \geqslant 0.$$

Thus $\sum_{i=1}^{8} a_i(F_i \cdot C') \leqslant 0$. In this case, either there is an $i_0 \in \{1,\ldots,8\}$ such that $(C' \cdot F_{i_0}) < 0$, or $(C' \cdot F_i) = 0$ for all $i = 1,\ldots,8$, and C' does not meet any component of $Exc(f)$. In the first case C' must coincide with F_{i_0}. However, the computations in [**8**, **11**] show that $(f^*K_X \cdot F_i) > 0$ for all $i = 1,\ldots,8$, which is impossible by our assumption. Therefore C' must have vanishing intersection number with all of the F_i's and also with all of the components of $Exc(f)$. But, as the Poincaré duals of the irreducible components of $Exc(f)$ and of the F_i's generate $H^2(Z,\mathbb{Q})$, C' must be numerically trivial on Z, and this is a contradiction. Hence, X has ample canonical bundle.

For $k = 7, 8$, i.e. for complex structures of general type on $\mathbb{CP}^2\#7\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, we use the examples constructed in Sections 3 and 7 of [**8**]. In both cases, the authors start with an appropriate relatively minimal rational elliptic surface, which is further blown-up to a complex surface $\widetilde{Z}$, which contains disjoint chains of curves, satisfying Artin's contractibility criterion. Collapsing these chains, $f\colon \widetilde{Z} \to X$, Lee and Park [**8**] obtain a singular variety X which admits a 1-parameter $\mathbb{Q}-$Gorenstein smoothing. The general fiber is a smooth, simply connected, minimal complex surface of general type, homeomorphic to $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$. An important step in their proof was that both K_X and f^*K_X are nef. Moreover, f^*K_X is numerically equivalent to a rational combination of effective curves, where all the coefficients are strictly positive. The curves appearing in the decomposition have all negative self-intersection and their Poincaré duals form a set of generators for $H^2(\widetilde{Z},\mathbb{Q})$. These properties were the only needed ingredients to prove the stronger result that K_X is in fact ample. As the computations are analogous to the previous proof, we refrain from repeating them.

PROPOSITION 2·3. *On the topological space* $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, *for* $k = 6, 7, 8$, *there exists an exotic smooth structure which admits a complex structure with ample canonical line bundle.*

Proof. For each $k = 6, 7, 8$, we have associated above a singular complex surface X with ample $\mathbb{Q}-$Cartier canonical divisor. Each of these examples admits a 1-parameter $\mathbb{Q}-$Gorenstein smoothing [**8**, **11**], which provides an exotic smooth structure on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ for $k = 6, 7, 8$. Since ampleness is an open property [**5**], the canonical bundle of the general fiber of these smoothings is automatically ample.

The above proposition, for $k = 6, 7$, is a reformulation of the Theorem 1·1.

COROLLARY 2·4. *Each of the manifolds* $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ *for* $k = 5, 6, 7, 8$ *admits an exotic smooth structure on which there is an Einstein metric with negative scalar curvature.*

Proof. For $k = 6, 7, 8$ we showed the existence of a smooth structure which admits a complex structure with ample canonical line bundle. The same result is also true for $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$, see Remark 1·2. For these differential and complex structures Aubin, Yau's Theorem 2·2 proves the existence of Kähler–Einstein metrics of negative scalar curvature.

Remark 2·5. On the topological manifold $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, there exist two constructions of smooth structures supporting complex structures with ample canonical bundle, the one mentioned above and the one provided by Catanese and LeBrun [**4**]. It is not known what is the relation between these smooth structures, and hence if we have more than one differential structure which admits an Einstein metric of negative scalar curvature.

3. *Exotic structures*: *non-existence of Einstein metrics*

The classical obstruction to the existence of an Einstein metric on an oriented, compact, smooth, Riemannian, four-manifold (M, g) is the Hitchin–Thorpe Inequality:

$$(2\chi \pm 3\tau)(M) \geqslant 0 \tag{3·1}$$

with equality when the Einstein manifold (M, g) is covered by the hyperkähler $K3$ surface or by the flat four-torus, or by the above with the reversed orientations. Here $\chi(M)$, $\tau(M)$ denote the Euler characteristic and the signature of the manifold M, respectively. Using the Seiberg–Witten theory, LeBrun was able to find new obstructions to the existence of Einstein metrics [**6**, **7**]. The novelty of his results is that the existence of Einstein metrics depends on the differential structure rather then just topological data. The techniques developed by LeBrun rely on the existence of a non-trivial solution of the Seiberg–Witten equations. We briefly introduce the needed notions of the Seiberg-Witten theory and state the main obstruction theorem.

For simplicity, we consider only the case when $H_1(M, \mathbb{Z})$ has no $2-$torsion. Then there is a one-to-one correspondence between the set of $Spin^c$ structures and the set $\{c \in H^2(M, \mathbb{Z}) | c \equiv w_2(M) \mod 2\}$ of characteristic classes in $H^2(M, \mathbb{Z})$. The Seiberg–Witten invariant is defined as an integer valued function

$$SW_M : \{c \in H^2(M, \mathbb{Z}) \mid c \equiv w_2(M) \mod 2\} \longrightarrow \mathbb{Z}. \tag{3·2}$$

This counts the number of solutions of a generic perturbation of the Seiberg–Witten equations modulo gauge transformations and counted with orientations (see for example [**9**]). We call c a basic class if $SW_M(c) \neq 0$.

The Seiberg–Witten invariant is a diffeomorphism invariant in the case $b_+ > 1$. In the case of $b_+ = 1$, the invariant depends on the choice of an orientation of $H^2(M, \mathbb{Z})$ and $H^1(M, \mathbb{R})$. But, as we are interested in manifolds for which the existence of an Einstein metric is topologically unobstructed, the inequality (3·1) must be strictly satisfied. Hence, in the case when the $SW_M(c) \neq 0$, we have $c^2 \geqslant (2\chi + 3\tau)(M) > 0$. But for any arbitrary metric g, $c^2 = c_+^2 - c_-^2 > 0$ where we denote by $c_\pm$ the (anti)-self-dual part of the harmonic representative 2-form. Thus $c_+ \neq 0$, and in this special situation (see [**9**, theorem 6·9·2]) the Seiberg–Witten invariant does not depend on the choice of metric.

We are now ready to state LeBrun's obstruction, see [**7**, theorem 3·3]:

THEOREM 3·1 (LeBrun). *Let X be a compact oriented 4−manifold with a non-trivial Seiberg-Witten invariant and with $(2\chi + 3\tau)(X) > 0$. Then*

$$M = X\#k\overline{\mathbb{CP}^2}$$

does not admit an Einstein metric if $k \geqslant (2\chi + 3\tau)(X)/3$.

There has recently been done a lot of work on constructing exotic structures with small topology. In this paper we will use a construction due to Akhmedov, Baykur and Park [**1**].

Here we are interested in exotic structures on $M = \mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$. In [**1**, section 3·4], the authors construct a minimal symplectic manifold X homeomorphic, but not diffeomorphic, to M. To give the reader a better understanding of this manifold, we sketch its construction. We start with two product manifolds $Y = T^2 \times \Sigma_2$ and T^4, where T^n is a torus of dimension n and Σ_2 is a Riemann surface of genus 2. If we consider the diagonal class $[T^2 \times point] + [point \times T^2] \in H_2(T^4, \mathbb{Z})$, then a smooth representative in this class is a Riemann surface of genus 2 with self-intersection 2. Blowing up two points on a smooth representative gives $\Sigma' \subset (T^4\#2\overline{\mathbb{CP}^2})$ a smooth Riemann surface of genus 2 and 0−self-intersection. If we endow the manifolds Y and $T^4\#2\overline{\mathbb{CP}^2}$ with the canonical symplectic structures, we can choose the representative Σ' to be a symplectic submanifold. We can now take the symplectic sum $X' = Y\#_{\Sigma_2}(T^4\#2\overline{\mathbb{CP}^2})$ along the two genus 2 symplectic submanifolds $point \times \Sigma_2 \subset Y$ and $\Sigma' \subset (T^4\#2\overline{\mathbb{CP}^2})$. X' is a symplectic manifold with topological invariants: Euler characteristic $\chi(X') = 6$ and signature $\tau(X') = -2$. As X' is not simply connected we need to make six Luttinger surgeries along well chosen tori (see [**1**]). We obtain a new simply connected, minimal, symplectic manifold X, with the same topological invariants as X'.

Notice that the definition of the Seiberg–Witten invariant in [**1**] is for the Poincaré dual of the characteristic class. The results are nevertheless the same. The manifold X has a unique basic class $\beta = c_1(K_X) \in H^2(X, \mathbb{Z})$, such that its Seiberg–Witten invariant $SW_X(\beta) = 1$. Moreover, X contains a null-homologous torus Λ, with a preferred simple loop λ. In [**1**, section 4], it is shown that 0−surgery on Λ with respect to λ yields a symplectic manifold X_0, which also has an unique basic class $\beta_0 = c_1(K_{X_0})$. Then $1/n$−surgery, $n \geqslant 1$ on Λ with respect to λ generates a family of manifolds X_n which are homeomorphic to $X_1 = X$. Corresponding to the basic class β of X there is a unique basic class β_n of X_n for which the Seiberg–Witten invariant is non-zero, and it can be computed as follows:

$$SW_{X_n}(\beta_n) = SW_X(\beta) + (n-1)SW_{X_0}(\beta_0) = 1 + (n-1) = n.$$

Hence, the family $X_n, n \geqslant 1$ consists of homeomorphic, pairwise non-diffeomorphic manifolds. As β_n is the unique basic class on X_n, and the Seiberg–Witten invariant of $X_n, n > 1$ is not ± 1, the manifold X_n does not admit a symplectic structure. Of course, its basic class satisfies $\beta_n^2 = \beta^2 = 6 = (2\chi + 3\tau)(X) = (2\chi + 3\tau)(X_n)$.

Let $M_{n,k} = X_n\#k\overline{\mathbb{CP}^2}$, $k = 2, 3, 4, 5$. Then, the conditions of Theorem 3·1 are satisfied and we have the following:

PROPOSITION 3·2. *The manifolds $\mathbb{CP}^2\#l\overline{\mathbb{CP}^2}$ for $l = 5, 6, 7, 8$ support infinitely many non-diffeomorphic exotic smooth structures none of which admits an Einstein metric.*

Proof. Using Theorem 3·1, the only thing that remains to be argued is that the collection $\{M_{n,k}\}_n$ contains infinitely many, homeomorphic, but pairwise not diffeomorphic, manifolds. For this, we use again the Seiberg–Witten invariant and the commutativity of our construction. We fix the number l and blow-up $k = l - 3$ points on $X = X_1$. We denote by $E_1, \dots E_k$ the exceptional divisors. This gives a new symplectic manifold, which has basic classes $\beta \pm PD(E_1) \pm \cdots \pm PD(E_k)$, where $PD(E_i)$ is the Poincare dual of E_i. $0-$surgery on Λ with respect to λ gives the manifold $X_0 \# k\overline{\mathbb{CP}^2}$ with corresponding basic classes $\beta_0 \pm PD(E_1) \pm \cdots \pm PD(E_k)$. We have a similar relation for the Seiberg–Witten invariant of $M_{n,k}$:

$$\begin{aligned} SW_{M_{n,k}}(\beta_n \pm PD(E_1) \pm \cdots \pm PD(E_k)) &= SW_X(\beta \pm PD(E_1) \pm \cdots \pm PD(E_k)) \\ &+ (n-1)SW_{X_0}(\beta_0 \pm PD(E_1) \pm \cdots \pm PD(E_k)) \\ &= 1 + (n-1) = n. \end{aligned}$$

Any manifold has a finite number of basic classes. As $l_n = \max\{SW_{M_{n,k}}(c) | c \in H^2(M_{n,k}, \mathbb{Z})\}$ goes to infinity as n increases, we can always choose a subsequence of manifolds, Y_n, such that any manifold in the subsequence has the property that it is not diffeomorphic to any of the manifolds that precede it. Hence the manifolds Y_n are homeomorphic to $\mathbb{CP}^2 \# (k+3)\overline{\mathbb{CP}^2}$, but pairwise not diffeomorphic. Moreover, when $n > 1$ they don't support a symplectic structure.

The question of the number of smooth structure which don't admit Einstein metrics was addressed in the literature before. Examples of manifolds which admit infinitely smooth structures which don't support an Einstein metric where exhibited in different papers by Kotschick and LeBrun, among others. Their examples have large topological numbers.

Remark 3·3. We would like to note that since for each of the above smooth structures the Seiberg–Witten invariant is non-trivial, the manifolds don't admit Riemannian metrics of positive constant scalar curvature. Moreover, as $c_1^2(X_n) > c_1^2(M_{n,k}) > 0$ there is a bound [**6**] on the scalar curvature:

$$\int_{M_{n,k}} s_g^2 d\mu > 32\pi^2 c_1^2(X_n) > 0$$

where $d\mu$ is the volume form with respect to g. Hence the manifolds $M_{n,k}$ don't admit non-negative constant scalar curvature metrics, and their Yamabe invariant is negative.

The results of Theorem 1·3 are immediate by combining the results from 2·2, 2·4, 3·2.

4. *Higher dimensional manifolds*

Let S_1 be a complex surface with ample canonical bundle and $c_1^2 = 1$. As an example of such surface, we can either take the one provided by Corollary 2·3, or the example constructed by Catanese and LeBrun [**4**]. Let S_2, S_3, S_4 be the complex surfaces which admit Kähler–Einstein metrics of negative scalar curvature and have $c_1^2 = 2, 3, 4,$ from Corollary 2·4. We are now ready to give the proof of Proposition 1·4.

Proof. Let $N = N_{i_1} \times \cdots \times N_{i_k}$ and let $S = S_{i_1} \times \cdots \times S_{i_k}$, where the i_j are either 1, 2, 3 or 4. The manifolds S_i and N_i are homeomorphic, hence by a theorem of Wall [**17**], they are h-cobordant. These h-cobordisms can be used to construct an h-cobordism between N and S. But as N, S are simply connected manifolds of dimension greater than 5, Smale's h-cobordism theorem tells us that they are diffeomorphic. We know that N_i and S_i admit Kähler–Einstein metrics of positive, negative scalar curvature, respectively. We can rescale these metrics such that the scalar curvatures are ± 1. On N, S we consider the product metrics associated to the corresponding i_j. As we take products of Kähler–Einstein with the same scalar curvature, the new metrics are going to be Kähler–Einstein. The product complex structures on N, S are of Kodaira dimension $-\infty$ and $4k$, respectively.

Remark 4·1. For dimension $4k$, the number of manifolds constructed in Proposition 1·4 is $(k+1)(k+2)(k+3)/6$. Hence the number of topological types which admit Einstein metrics of both positive and negative sign can be made arbitrary large.

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