KALININ EFFECTIVITY AND WONDERFUL COMPACTIFICATIONS

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Abstract. We review the definition and main properties of Kalinin effectivity and describe methods for constructing effective spaces together with several examples. We analyze the Kalinin effectivity of wonderful compactifications and prove that the wonderful compactifications of hyperplane arrangements and of configuration spaces associated to Kalinin effective compact complex manifolds are themselves Kalinin effective. As an application, we show that the Deligne–Mumford space of real rational curves with marked points is effective. Finally, we apply Kalinin effectivity to study Smith–Thom maximality for Hilbert squares.

> I would say that mathematics is the science of skillful operations with concepts and rules invented just for this purpose.
>
> E. Wigner - "The Unreasonable Effectiveness of Mathematics in the Natural Sciences"

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1. Introduction

1.1. **Motivation and Context.** The main object of interest in this paper is the topology of the real loci of Deligne-Mumford spaces $\overline{\mathcal{M}}_{0,n}$ and, more generally, the topology of the real loci of so-called wonderful compactifications [25] of various arrangements.

Most of the previously obtained results on the topology of these real manifolds concern calculations of their Betti numbers and cohomology rings. In [10], P. Etingof, A. Henriques, J. Kamnitzer, and E. Rains described the integral cohomology ring of the real locus of $\overline{\mathcal{M}}_{0,n}$ for the real model corresponding to a choice of real marked points. For the other real structures on $\overline{\mathcal{M}}_{0,n}$, the additive structure of the cohomology with $\mathbb{F}_2$-coefficients was described by Ö. Ceyhan [3] in terms of graph-homology. More recently, X. Chen, P. Georgieva, and A. Zinger [4] computed the rational cohomology ring of the Deligne-Mumford moduli space of real rational curves with conjugate marked points and showed that the integer homology has only 2-torsion. In the case of the De Concini-Procesi models associated to arrangements of real projective spaces, E. Rains computed in [26] the integral homology, modulo 2-torsion of their real loci and described its multiplicative structure.

Motivated by the importance of Steenrod squares in understanding the topology of involutions and their fixed loci, we adopt a different but related point of view. Specifically, we apply Kalinin's approach [17, 18], which relates the cohomology ring and the Steenrod algebra structure in equivariant cohomology to those of the fixed locus.

1.2. **Framework: Smith Theory, Kalinin Effectivity, and Conjugation Spaces.** The topology of the real locus of complex manifolds equipped with an anti-holomorphic involution has long been studied using the classical methods of Smith theory. A fundamental consequence of this theory is that the total Betti number of the fixed locus is bounded above by the total Betti number of the ambient complex manifold[1]. When the equality is attained, the manifold is said to be Smith-Thom maximal. Another important tool in this context is the Borel-Serre spectral sequence together with its stabilized version specific to involutions, the Kalinin spectral sequence [16, 6]. In this language, Smith-Thom maximality is equivalent to the collapse of the Kalinin spectral sequence at the first page, while another closely related condition, the Galois maximality, corresponds to collapse at the second page (see Section 2).

Kalinin's notion of effective space originated from his work [16] on the topology of real projective hypersurfaces and was further developed in [17]. Informally, effective spaces are spaces equipped with an involution for which the limiting term of the Kalinin spectral sequence provides complete and functorial control over the cohomology ring and the Steenrod algebra structure of the fixed point set (see Theorem 2.10). In particular, for Smith-Thom maximal effective spaces, this control becomes especially strong: their cohomology ring is isomorphic, as an algebra over the Steenrod algebra, to the cohomology ring of the fixed point set, as described in Theorem 2.19.

Kalinin introduced the notion of effective space in his 2002 paper [17]. Three years later, J. C. Hausmann, T. Holm, and V. Puppe [15] introduced the notion of conjugation space, with applications ranging from transformation groups to toric topology. It turns out, however, that the two theories meet in a very precise way: a conjugation space is an effective space in Kalinin's sense satisfying Smith-Thom maximality. We prove this equivalence in Proposition 2.22, thereby placing conjugation spaces within the framework of Kalinin effectivity.

The class of effective spaces is considerably broader than that of conjugation spaces. This additional flexibility makes it possible to analyze the topology of real loci in situations where Smith-Thom maximality fails. One of the main objectives of this paper is to identify natural geometric constructions that preserve effectivity and, when present, also preserve Smith-Thom or Galois maximality. In particular, we obtain new examples of Kalinin effective spaces and of Smith-Thom and Galois maximal spaces.

1.3. **Main Results and Applications.**

We begin by reviewing Kalinin's spectral sequence and the notion of Kalinin effectivity for pairs, which provides the most convenient framework for the applications considered here.We also exhibit several new examples of effective spaces. The stability of Kalinin effectivity under blow-ups is then investigated and established for pairs satisfying a cohomological condition that we call stretchedness (see Section 4.1).

[1]Throughout this paper we will only consider cohomology with coefficients in $\mathbb{F}_2$.

This stability result is subsequently applied to produce new examples of Kalinin effective and conjugation spaces among wonderful compactifications of arrangements, in the general setting introduced by Li [25]. Our main result in this direction, proved in Section 5, gives a general criterion (see Theorem 5.11) describing when the wonderful compactification of an arrangement equipped with a real structure is effective. As an application, in Section 6 we study the Kalinin effectivity of several wonderful compactifications of particular interest and obtain the following:

Theorem A. *The De Concini - Procesi wonderful compactification of any arrangement of real linear subspaces in $\mathbb{P}^n$ is a conjugation space.*

The De Concini-Procesi wonderful compactification of the braid arrangement A_{n-2} can be identified with the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ [5, page 483]. This observation shows that the method developed for proving Theorem 5.11 also applies to the moduli space of real stable rational curves with real marked points.

The Deligne-Mumford space $\overline{\mathcal{M}}_{0,n}$ admits several natural real structures c_σ, indexed by permutations $\sigma \in S_n$ of order 2 acting on the marked points (see Section 6.2). The case $\sigma = \mathrm{id}$ corresponds to real stable rational curves with all markings real. We prove:

Theorem B. *Let $n \geq 3$ and let c_σ be a real structure on the Deligne–Mumford space $\overline{\mathcal{M}}_{0,n}$ defined by an order 2 permutation $\sigma \in S_n$. Then the following hold:*

(i) *If $\sigma = \mathrm{id}$, the real pair $(\overline{\mathcal{M}}_{0,n}, \mathrm{c}_{\mathrm{id}})$ is a conjugation space.*
(ii) *If $\mathrm{Fix}(\sigma) \neq \emptyset$, the real pair $(\overline{\mathcal{M}}_{0,n}, \mathrm{c}_\sigma)$ is effective and Galois maximal.*

As a direct consequence of Theorem B and Theorem 2.19 we obtain the following improvement of Theorem 5.6 in [10]:

Corollary C. *Let $F\overline{\mathcal{M}}_{0,n}$ denote the fixed locus of the involution c_{id} on $\overline{\mathcal{M}}_{0,n}$. Then $H^*(F\overline{\mathcal{M}}_{0,n})$ with its standard and $H^{2*}(\overline{\mathcal{M}}_{0,n})$ with its intrinsic structure of algebra over Steenrod algebra are naturally isomorphic*[2].

Kalinin effective spaces that are not necessarily conjugation spaces also arise among the wonderful compactifications of configuration spaces of ordered points. Let (X, c) be a closed, nonsingular complex variety equipped with an anti-holomorphic involution and n a positive integer. Then the space of n-tuples of distinct points

$$\mathrm{Conf}_n(X) = \{(p_1, \ldots, p_n) \mid p_i \in X,\, p_i \neq p_j \text{ for every } i \neq j\}$$

has a naturally induced anti-holomorphic involution. Several wonderful compactifications of $\mathrm{Conf}_n(X)$ were constructed by Fulton-MacPherson [11], Ulyanov [27] and Kuperberg-Thurston [24] and discussed by Li in [25].

[2] See Section 2.2 for a description of this isomorphism.

Such compactifications automatically satisfy the aforementioned stretchedness condition, and we find:

Theorem D. *Let (X, c) be a closed, nonsingular complex variety equipped with an anti-holomorphic involution.*

1) *If X is Kalinin effective, then the Fulton-MacPherson, Ulyanov and Kuperberg-Thurston wonderful compactifications are Kalinin effective spaces.*
2) *If X is maximal or Galois maximal, then the Fulton-MacPherson, Ulyanov and Kuperberg-Thurston wonderful compactifications are maximal or Galois maximal spaces, respectively.*
3) *If X is a conjugation space, then the Fulton-MacPherson, Ulyanov and Kuperberg-Thurston wonderful compactifications are conjugation spaces.*

As another application of Kalinin effectivity, which was the starting point of our investigations of Kalinin effectivity, we compute the deficiency of its Hilbert square of effective Galois maximal varieties.

Theorem E. *Let X be a compact connected nonsingular complex manifold of dimension n, such that $H_{\mathrm{odd}}(X) = 0$, equipped with a real structure, which is effective and Galois maximal.If the Smith-Thom deficiency of X is $\mathfrak{D}(X) = a$, then the Smith-Thom deficiency of the Hilbert square $X^{[2]}$ equipped with the induced real structure is*

$$\mathfrak{D}(X^{[2]}) = \sum_{k=1}^{2n} (2k-1)\delta_k + a\beta_* + \frac{a(a-1)}{2}.$$

This result improves and extends Theorems 1.7 and 1.8 in [22]. As a consequence we find new examples of maximal spaces among Hilbert squares:

Corollary F. *The Hilbert square of a maximal effective space is maximal.*

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Notations and conventions:

1) By a complex manifold equipped with a real structure we mean a pair (X, c) consisting of a complex manifold X together with an anti-holomorphic involution $\mathrm{c} : X \to X$.
2) Let $G = \{\mathrm{id}, \mathrm{c}\}$ be the cyclic group of order 2. If a complex manifold X is equipped with an anti-holomorphic involution $\mathrm{c} : X \to X$, then c canonically determines an action of G on X, and we view (X, c) as a G-space with respect to this action.
3) If (X, ι) is a space equipped with an involution, we denote by $F(X)$ the set of fixed points of ι.

4) Unless explicitly stated, all homology and cohomology groups are taken with coefficients in the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. We denote by $\beta_i(\,\cdot\,)$ the i-th Betti number with $\mathbb{F}_2$-coefficients, by $\beta_*(\,\cdot\,)$ the total Betti number, and by $\beta_{\text{odd}}(\,\cdot\,)$ the sum of the Betti numbers in odd degrees.

2. Kalinin effectivity and maximality

Let G be the cyclic group of order 2.

Definition 2.1. *A G-flag $(V_p, V_{p-1}, \dots, V_1; \mathrm{c})$ consists of a CW-complex $V = V_p$ equipped with a continuous involution c, that sends cells to cells and acting identically on each invariant cell, and c-invariant subcomplexes $V_k \subseteq V$ such that $V_k \subseteq V_{k+1}$ for all $k = 1, \dots, p-1$.*

For convenience, a G-flag will be called a *G-space* if $p = 1$, a *G-pair* if $p = 2$, and a *G-triple* if $p = 3$. We will often omit the involution c from notation of a G-flag when it is understood from the context.

The fixed point sets of a G-flag $(V_p, V_{p-1}, \dots, V_1; \mathrm{c})$ form a flag of CW-complexes

$$F(V_p, \dots, V_1) := (F(V_p), \dots, F(V_1)),$$

where $F(V_j) = F(V_p) \cap V_j$ is the fixed point set of V_j, $j = 1, \dots, p$. The flag $F(V_p, \dots, V_1)$ will be considered as a G-flag with G acting identically on it.

Remark 2.2. All G-flags considered in this article are flags of manifolds. In what follows, we will mainly use G-pairs and G-triples.

2.1. The Kalinin spectral sequence.

Theorem 2.3 (Kalinin, [16])**.** *Let (X, Y) be a G-pair. Then the following objects exist and are natural with respect to G-maps:*

1) *A group filtration $\mathcal{F} = \{\mathcal{F}_n(X, Y)\}_{n \geq -1}$, where*

$$0 = \mathcal{F}_{-1}(X, Y) \subseteq \mathcal{F}_0(X, Y) \subseteq \cdots \subseteq \mathcal{F}_n(X, Y) \subseteq \cdots H^*(F(X, Y)).$$

2) *A $\mathbb{Z}$-graded spectral sequence $({}^rH^*(X, Y), {}^rd^*)$, where*

$${}^rd^q : {}^rH^q(X, Y) \to {}^rH^{q-r+1}(X, Y), \quad {}^rd^{q-r+1} \circ {}^rd^q = 0,$$

for which we have

$${}^1H^q = H^q(X, Y), \quad {}^1d^* = 1 + \mathrm{c}^*,$$

and

$${}^2H^q = H^1(G, H^q(X, Y)).$$

3) *An isomorphism*

$$\phi_i : \mathrm{gr}^i_{\mathcal{F}}(H^*(F(X, Y))) \to {}^\infty H^i,$$

where $\mathrm{gr}^i_{\mathcal{F}}(H^*(F(X, Y)))$ is the graded group associated to the filtration $\mathcal{F}$. □

The above spectral sequence is called *Kalinin's spectral sequence.*[3] We briefly recall how Kalinin's spectral sequence is obtained from the Leray-Serre spectral sequence of the Borel fibration.

The Kalinin spectral sequence is derived from the Leray-Serre spectral sequence $\{E_r^{p,q}\}$ of the Borel fibration $\pi : (X,Y)_G = S^\infty \times_G (X,Y) \to BG$, in the following way. Since $BG = \mathbb{RP}^\infty$, the multiplication by the generator

$$u \in H^1(\mathbb{RP}^\infty, \mathbb{F}_2) \simeq \mathbb{F}_2[u]$$

produces isomorphisms ${}^r\phi_p : {}^rE^{p,*} \to {}^rE^{p+1,*}$ for $p \gg 0$ and consider the direct limits

$${}^rH^q := \varinjlim({}^rE^{p,q}, {}^r\phi_p).$$

By the Borel theorem, the inclusion homomorphisms

$$H^i((X,Y)_G) \to H^i(F(X,Y)_G)$$

are isomorphisms for each $i > \dim X$, which establishes natural isomorphisms

$$\varinjlim H^i((X,Y)_G) = \varinjlim H^i(F(X,Y)_G) = H^*(F(X,Y)). \tag{2.1}$$

The filtration $\{\mathcal{F}_n\}$ on $H^*(F(X,Y))$ is defined by

$$\mathcal{F}_n := \varinjlim(\mathcal{F}'_i \cap H^{n+i}((X,Y)_G)),$$

where $H^*((X,Y)_G) = \mathcal{F}'_0 \supseteq \mathcal{F}'_1 \supseteq \dots$ is the filtration by $\mathcal{F}'_j = \pi^*(H^{\geq j}(BG))$. In particular, by taking i sufficiently large and using the identification (2.1), we obtain

$$\mathcal{F}_n = \mathcal{F}'_{i-n} \cap H^i((X,Y)_G).$$

Notice that from functoriality of this spectral sequence it follows that

$$\mathcal{F}_i(X,Y) \subseteq H^{\leq i}(F(X,Y)) \quad \text{for each } i. \tag{2.2}$$

For the reader's convenience, we recall next several basic properties of Kalinin's spectral sequence.

Proposition 2.4 (Kalinin, [16])**.** *The cup-product in X descends to a multiplicative structure in ${}^rH^*$, so that ${}^rH^*$ is a $\mathbb{Z}_2$-algebra. Moreover:*

1) *The differentials ${}^rd^*$ are derivations for all $r \geq 2$, i.e.,*

$${}^rd^*(x \cup y) = {}^rd^*x \cup y + x \cup {}^rd^*y.$$

2) *The filtration $\mathcal{F}_*$ is multiplicative, i.e., for every $p, q \geq 0$*

$$\mathcal{F}_p \cup \mathcal{F}_q \subseteq \mathcal{F}_{p+q}.$$

3) *The maps ϕ_i, $i \geq 0$ satisfy*

$$\phi_{i+j}(x \cup y) = \phi_i(x) \cup \phi_j(y),$$

for every $x \in \mathrm{gr}^i_{\mathcal{F}}(H^*(F(X,Y)))$ and $y \in \mathrm{gr}^j_{\mathcal{F}}(H^*(F(X,Y)))$.

[3]A similar spectral sequence in homology exists as well [6].

Proposition 2.5 (Kalinin, [17]). *Let (X, Y) and (A, B) be two G-pairs. Then:*

1) *The multiplication*
$$\times : H^*(X, Y) \otimes H^*(A, B) \to H^*(X \times A, Y \times A \cup X \times B)$$
induces spectral sequence isomorphisms
$$\times : {}^r H^i(X, Y) \otimes {}^r H^j(A, B) \to {}^r H^{i+j}(X \times A, Y \times A \cup X \times B),$$
where ${}^r d(a \otimes b) = a \otimes {}^r d(b) + {}^r d(a) \otimes b$, for any $r > 1$.
2) *We have*
$$\times : \mathcal{F}_i(X, Y) \otimes \mathcal{F}_j(A, B) \subseteq \mathcal{F}_{i+j}(X \times A, Y \times A \cup X \times B).$$
3) *We have*
$$\phi_*(a \otimes b) = \phi_*(a) \times \phi_*(b).$$

$\square$

Proposition 2.6 (Kalinin, [17]). *Let (X, Y) be a G-pair, where X is an n-dimensional closed manifold and Y a submanifold. Then:*

1) *For $r \leq \infty$, the non-degenerate bilinear form*
$$\Phi_{(X,Y)} : H^*(X, Y) \otimes H^*(X \setminus Y) \to \mathbb{F}_2, \Phi_{(X,Y)}(x \otimes y) = (x \cup y) \cap [X, Y],$$
induces a non-degenerate bilinear form
$${}^r\Phi_{(X,Y)} : {}^r H^*(X, Y) \otimes {}^r H^*(X \setminus Y) \to \mathbb{F}_2;$$
for $r < \infty$, the differentials ${}^r d_{(X,Y)}$ and ${}^r d_{X \setminus Y}$ are mutually adjoint with respect to this form.
2) *We have*
$$\mathcal{F}_i(X, Y)^{\perp} = w^{-1}\mathcal{F}_{n-i-1}(X \setminus Y),$$
where w is the total Stiefel-Whitney class of the normal vector bundle of the submanifold $F(X \setminus Y)$ in $X \setminus Y$.
3) *We have*
$${}^\infty\Phi_{(X,Y)}(\phi_* x \otimes \phi_* y) = \Phi_{F(X,Y)}((w^{-1}x) \otimes y)$$
for any $x \in H^*(F(X, Y))$ and $y \in H^*(F(X \setminus Y))$.

$\square$

Proposition 2.7 (Kalinin, [18]). *Let (X, Y, Z) be a G-triple. Then:*

1) *The boundary homomorphism $\delta : H^*(X, Y) \to H^{*+1}(Y, Z)$ in the cohomology sequence of a triple induces a spectral sequence morphism*
$${}^r\delta : {}^r H^*(X, Y) \to {}^r H^{*+1}(Y, Z).$$
2) *$F\delta(\mathcal{F}_i(X, Y)) \subseteq \mathcal{F}_{i+1}(Y, Z)$, where $F\delta$ is the boundary homomorphism in the cohomology sequence of the triple $F(X, Y, Z)$.*
3) *$\phi_* F\delta = {}^\infty\delta\phi_*$.*

$\square$

2.2. **Effectivity.**

Definition 2.8. *Let (X, Y) be a G-pair.*

1) *The G-pair (X, Y) is said to be effective if*
$$\mathcal{F}_i(X, Y) = \operatorname{Sq} H^{\leq i/2}(F(X, Y)),$$
where $\operatorname{Sq} = 1 + \operatorname{Sq}^1 + \operatorname{Sq}^2 + \cdots$ is the total Steenrod operation.
2) *The G-space X is said to be effective if the G-pair $(X, \emptyset)$ is effective.*

Proposition 2.9. *Let (X, c) be a G-space, where X is an n-dimensional closed manifold, and Y a c-invariant submanifold. Then, the G-pair (X, Y) is effective if and only if the G-space $X \setminus Y$ is effective.*

Proof. Proposition 2.6 and the Wu formulas imply that
$$(\operatorname{Sq}^{-1} \mathcal{F}_i(X, Y))^{\perp} = \operatorname{Sq}^{-1} \mathcal{F}_{n-i-1}(X \setminus Y),$$
where $\operatorname{Sq} = 1 + \operatorname{Sq}^1 + \operatorname{Sq}^2 + \cdots$ is the total Steenrod operation. The proof of the assertion follows now directly from the definition and the Poincaré-Alexander-Lefschetz duality. □

Let (X, Y) be an effective G-pair. We denote by
$$\psi^i_{X,Y} : H^i(F(X, Y)) \to {}^{\infty}H^{2i}(X, Y).$$
the composition of the homomorphisms
$$H^i(F(X, Y)) \xrightarrow{\operatorname{Sq}} \mathcal{F}_{2i}(X, Y) \xrightarrow{\operatorname{pr}} \frac{\mathcal{F}_{2i}(X, Y)}{\mathcal{F}_{2i-1}(X, Y)} \xrightarrow{{}^{\infty}\phi_{2i}} {}^{\infty}H^{2i}(X, Y).$$

The following remarkable property was one of the main Kalinin's motivations for introducing and study of the notion of effective spaces.

Theorem 2.10 (Kalinin [17, 18])**.** *Let (X, Y) be an effective G-pair. Then:*

1) *$\psi^i_{X,Y}$ is an isomorphism.*
2) *The homomorphism*
$$\psi^*_{X,Y} = \bigoplus_{i \geq 0} \psi^i_{X,Y} : H^*(F(X, Y)) \to {}^{\infty}H^*(X, Y)$$
is a ring isomorphism.
3) *$\psi^*_{X,Y}(\operatorname{Sq}^i x) = \operatorname{Sq}^{2i}(\psi^*_{X,Y}(x))$ for any $x \in H^*(F(X, Y))$.* □

Proposition 2.11. ([18, Proposition 3.6]) *Let (X, Y, Z) be a G-triple such that the G-pairs (X, Y) and (Y, Z) are effective. Then the boundary homomorphism*
$$\delta : H^*(F(Y, Z)) \to H^{*+1}(F(X, Y))$$
is trivial.

Proof. Let $x \in H^i(F(Y, Z))$. Using item 2) of Proposition 2.7, we find
$$\operatorname{Sq}(\delta x) = (F\delta)(\operatorname{Sq} x) \in \mathcal{F}_{2i+1}(X, Y) \subseteq \mathcal{F}_{2i+2}(X, Y).$$
Therefore,
$$\psi^{i+1}_{X,Y}(\delta x) = \phi_{2i+2}(\operatorname{pr}(\operatorname{Sq}(\delta x))) = \phi_{2i+2}(0) = 0.$$

Since the pair (X,Y) is effective, the map $\psi^{i+1}_{X,Y}$ is an isomorphism, and so $\delta x = 0$. $\square$

2.3. **Maximality.** Let $(X,Y;\mathrm{c})$ be G-pair, where $\dim H^*(X,Y)<\infty$, and denote by $\bar X$ and $\bar Y$ the quotient spaces X/c and Y/c, respectively. There exists [6, Section 1.2.8] a natural exact sequence

$$\begin{gathered}\cdots \to H^k(\bar X, F(X)\cup \bar Y) \xrightarrow{\mathrm{pr}^*} H^k(X,Y) \xrightarrow{\mathrm{tr}_*\oplus \mathrm{in}^*} \\ H^k(\bar X, F(X)\cup\bar Y)\oplus H^k(F(X),F(Y)) \xrightarrow{\Delta_k} H^{k+1}(\bar X, F(X)\cup\bar Y)\to\cdots,\end{gathered} \tag{2.3}$$

and a similar one in homology. The connecting homomorphisms Δ_k are given by $x\oplus f\mapsto x\cup\omega+\delta x$, where $\omega\in H^1(\bar X\setminus F)$ is the characteristic class of the double covering $X\setminus F\to\bar X\setminus F$. The sequence (2.3) is called the *Smith sequence* in cohomology.

As a consequence, the following inequalities hold:

1) $\dim H^*(F(X,Y))\le \dim H^*(X,Y)$ (*Smith inequality*).
2) $\dim H^*(F(X,Y))\le \dim H^1(G;H^*(X,Y))$ (*Borel-Swan inequality*),

where $H^1(G;H^*(X,Y))=\ker(1+\mathrm{c}_*)/\operatorname{Im}(1+\mathrm{c}_*)$. The difference

$$\mathfrak{D}(X,Y)=\dim H^*(X,Y)-\dim H^*(F(X,Y))$$

is called *the Smith-Thom deficiency of* (X,Y). We will use the simplified notation $\mathfrak{D}(X)$ to denote $\mathfrak{D}(X,\emptyset)$.

Definition 2.12. *Let* $(X,Y;\mathrm{c})$ *be a* G*-pair, where* $\dim H^*(X,Y)<\infty$.

1) *The pair* (X,Y) *is called Smith-Thom maximal if its Smith-Thom deficiency vanishes.*
2) *The pair* (X,Y) *is called Galois maximal, or a GM-pair, if*

$$\dim H^*(F(X,Y))=\dim H^1(G;H^*(X,Y)).$$

Remark 2.13. Throughout this article, for convenience, by "maximal" we will mean "Smith-Thom maximal", whenever the context allows.

The Smith-Thom or Galois maximality conditions control the stage at which Kalinin's spectral sequence degenerates and therefore quantify how closely the cohomology of the fixed locus reflects that of the ambient space. The following criteria are consequences of Theorem 2.3 (*cf.* [6] and [23], respectively).

Proposition 2.14. *Let* (X,Y) *be a* G*-pair.*

1) *The pair* (X,Y) *is a maximal pair if and only if Kalinin's spectral sequence collapses at the first page.*
2) *The pair* (X,Y) *is a Galois maximal pair if and only if Kalinin's spectral sequence collapses at the second page.* $\square$

We recall next the following criteria of maximality and Galois maximality, respectively.

Proposition 2.15. [1, 13] *Let (X, Y) be a G-pair. The following conditions are equivalent:*

1) *(X, Y) is a maximal pair.*
2) *The action of G on $H^*(X, Y)$ is trivial and the Leray-Serre spectral sequence associated with the fibration $(X, Y) \hookrightarrow (X_G, Y_G) \to BG$ degenerates at the second page.*
3) *$H^*_G(X, Y)$ is a free $H^*(BG) = H^*(\mathbb{RP}^\infty)$-module.*
4) *The restriction homomorphism $\rho : H^*_G(X, Y) \to H^*(X, Y)$ is surjective.*
5) *The restriction homomorphism $r_G : H^*_G(X, Y) \to H^*_G(F(X, Y))$ is injective.* □

Proposition 2.16. *Let (X, Y) be a G-pair. The following conditions are equivalent:*

1) *(X, Y) is a Galois maximal pair.*
2) *Any c_*-invariant homology class $\alpha \in H_*(X, Y)$ can be represented by a c-invariant cycle.*

Proof. This is a known easy consequence of the Smith exact sequence in homology (see [6, item 1.2.4]). □

Proposition 2.17. *If a G-pair (X, Y) is Galois maximal, then Kalinin's filtration $\{\mathcal{F}_n\}$ coincides with the filtration of $H^*(F(X, Y))$ by the images $\operatorname{Im} \beta^n$ of the homomorphisms*

$$\beta^n : H^n_G(X, Y) \to H^n_G(F(X, Y)) = \bigoplus_{i \le n} H^i(F(X, Y)).$$

Proof. Since the pair (X, Y) is Galois maximal, the Borel-Serre spectral sequence degenerates at the second page, so that

$$\begin{aligned} {}^\infty E^{p,q} &= {}^2 E^{p,q} = H^1(G; H^q(X, Y)),\ p > 0, q \ge 0, \\ {}^\infty E^{0,q} &= {}^2 E^{0,q} = H^q(X, Y)^G. \end{aligned}$$

Furthermore, for every $r \ge 0$, the ring homomorphism

$$H^*_G(X, Y) \to H^{*+1}_G(X, Y)$$

given by multiplication by the generator u of $H^*_G(pt) = \mathbb{F}_2[u]$ acts on the spectral sequence as the simple shift

$${}^r E^{p,q} = H^1(G, H^q(X, Y)) \to {}^r E^{p+1,q} = H^1(G, H^q(X, Y))$$

if $p > 0$ and as the quotient epimorphism

$${}^r E^{0,q} = H^q(X, Y)^G \to {}^r E^{1,q} = H^1(G, H^q(X, Y))$$

with $\operatorname{Im}\{1 + \mathrm{c}^* : H^q(X, Y) \to H^q(X, Y) \subseteq H^q(X, Y)^G\}$ as the kernel. This implies that

$$\beta^n(H^n_G(X, Y)) = \beta^{n+i}(H^{n+i}_G(X, Y) \cap \mathcal{F}'_n) = \mathcal{F}_n$$

for every $n, i \ge 0$. □

We conclude this section with several results concerning the cohomology of maximal Kalinin effective spaces. Let (X, Y) be a Smith-Thom maximal Kalinin effective G-pair.

Proposition 2.18. *The odd dimensional cohomology groups $H^{2i+1}(X, Y)$, $i \geq 0$, vanish for any Smith-Thom maximal Kalinin effective G-pair.*

Proof. According to Proposition 2.14 and Theorem 2.10, if (X, Y) is a maximal pair, then $\mathcal{F}_r(X, Y)$ is isomorphic to $H^{\leq r}(X, Y)$ for any $r \geq 0$. It remains to notice that, due to the definition of an effective pair,

$$\mathcal{F}_{2i+1}(X, Y) = \operatorname{Sq} H^{\leq i}(F(X, Y)) = \mathcal{F}_{2i}(X, Y)$$

for any $i \geq 0$. $\square$

As a consequence of Proposition 2.18, the odd Steenrod squares of (X, Y) vanish. This implies that $H^{2*}(X, Y)$ inherits an *intrinsic* structure of an algebra over (mod 2) Steenrod algebra with the action of the i-th standard generator of the Steenrod algebra by Sq^{2i} (instead of Sq^i). On the other hand, applying Theorem 2.10, under assumptions of Proposition 2.18 we have a "doubling" ring isomorphism

$$\psi^*_{X,Y} = \bigoplus_{i \geq 0} \psi^i_{X,Y} : H^*(F(X, Y)) \to {}^\infty H^{2*}(X, Y) = H^{2*}(X, Y)$$

satisfying

$$\psi^*_{X,Y}(\operatorname{Sq}^i x) = \operatorname{Sq}^{2i}(\psi^*_{X,Y}(x)), \tag{2.4}$$

for any $x \in H^*(F(X, Y))$. Thus, we get:

Theorem 2.19. *Let (X, Y) a Smith-Thom maximal effective G-pair. Then, $H^{2*}(X, Y)$ with its intrinsic structure of algebra over Steenrod algebra is naturally isomorphic to $H^*(F(X, Y))$ with its standard structure of algebra over Steenrod algebra.* $\square$

Remark 2.20. The conclusion of Theorem 2.19 was one of motivations for Kalinin's introduction of the concept of effective spaces in 2002 (see Remark 2.2 in [18]). The key result (2.4) was rediscovered later, in 2006, by M. Franz and V. Puppe [12] in the context of conjugation spaces.

2.4. **Conjugation Spaces.**

Definition 2.21 (Hausmann-Holm-Puppe [15])**.** *A G-pair (X, Y) is a conjugation G-pair if the following conditions are satisfied:*

1) $H^{\text{odd}}(X, Y) = 0$,
2) *There exist a section $\sigma : H^*(X, Y) \to H^*_G(X, Y)$ of the restriction homomorphism $\rho : H^*_G(X, Y) \to H^*(X, Y)$ together with a degree-halving isomorphism $\kappa : H^{2*}(X, Y) \to H^*(F(X, Y))$ with the property that for every $x \in H^{2n}(X, Y), n \in \mathbb{N}$ there exist elements $y_1, \dots, y_n \in H^*(F(X, Y))$ such that*

$$r_G(\sigma(x)) = \kappa(x)u^n + y_1 u^{n-1} + \cdots + y_n \in H^*_G(F(X, Y)) = H^*(F(X, Y)) \otimes \mathbb{F}_2[u],$$

where $r_G : H^*_G(X,Y) \to H^*_G(F(X,Y))$ *is the restriction homomorphism in equivariant cohomology.*

Furthermore, as it is proved in [12], σ and κ are unique, multiplicative, and $y_i = \mathrm{Sq}^i(\kappa(x))$ for each $i = 1, \dots, n$.

Proposition 2.22. *(X,Y) is a conjugation G-pair if and only if (X,Y) is a maximal and effective G-pair.*

Proof. Recall, first, that (X,Y) is a maximal pair if and only if $H^*_G(X,Y)$ is a free $H^*(BG) = H^*(\mathbb{RP}^\infty)$-module $H^*(\mathbb{RP}^\infty) \otimes H^*(X,Y) = H^*(X,Y)[u]$ (see Proposition 2.15). Thus, if a G-pair (X,Y) is maximal and effective, the inverse of Kalinin's isomorphism ψ^* can be taken as a halving isomorphism κ. Moreover, since substituting $u = 1$ into

$$r_G : H^*(X,Y)[u] \to H^*(F(X,Y))[u]$$

transforms the filtration $H^{\leq i}(X,Y) \subseteq H^*(X,Y)$ into Kalinin's filtration $\mathcal{F}_i(X,Y) = \mathrm{Sq}\, H^{\leq i/2}(F(X,Y))$, t he required homomorphism σ becomes well defined by an equation

$$r_G(\sigma(x)) = \kappa(x)u^n + \mathrm{Sq}^1(\kappa(x))u^{n-1} + \cdots + \mathrm{Sq}^n(\kappa(x)). \tag{2.5}$$

Conversely, if (X,Y) is a conjugation G-pair, then due to existence of a halving isomorphism it is maximal, while due to [12] the conjugation equation is necessary of the form (2.5), which implies Kalinin's effectivity condition $\mathcal{F}_i(X,Y) = \mathrm{Sq}\, H^{\leq i/2}(F(X,Y))$. □

To conclude this section, we recall without proof a result of Kalinin [18, Lemma 3.10] which we will use in the next section.

Lemma 2.23. *Let (Y,Z) be an effective maximal pair, and let Z be an effective maximal space. Then:*

1) *Y is a maximal space.*
2) *There exists a short exact sequence*

$$0 \to {}^\infty H^*(Y,Z) \to {}^\infty H^*(Y) \to {}^\infty H^*(Z) \to 0.$$

□

3. Examples and constructions

3.1. **Examples of Kalinin effective spaces.** Examples of effective G-spaces can be constructed by successively attaching *G-cells*.

Definition 3.1. *A G-*cell *of complex dimension n is the pair* $(\mathbb{C}^n, \mathrm{conj})$ *where* conj *is the standard complex conjugation.*

Definition 3.2. (*cf.* [18, 15]) *We say that a G-space (X, c) admits a G-cellular decomposition if there exists a finite G-flag $(A_r, \dots, A_0)$ such that:*

1) *For every $i \geq 0$, $A_i \subseteq X$ is a closed subset, and $A_0 = \emptyset$;*
2) *For every $i \geq 1$, the G-space $(A_i \setminus A_{i-1}, \mathrm{c})$ is G-equivariantly homeomorphic to a G-cell.*

Lemma 3.3. *If (X, Y) is a G-pair such that Y is an effective space and $X \setminus Y$ is a G-cell, then X is an effective space.*

Proof. Since $X \setminus Y$ is homeomorphic to a G-cell, it is effective. Hence, by Proposition 2.9, the G-pair (X, Y) is effective, and, according to Proposition 2.11, for each $i \geq 0$, we have a short exact sequence

$$0 \to \mathcal{F}_i(X, Y) \to \mathcal{F}_i(X) \to \mathcal{F}_i(Y) \to 0.$$

Therefore, we get a short exact sequence

$$0 \to \operatorname{Sq}^{-1} \mathcal{F}_i(X, Y) \to \operatorname{Sq}^{-1} \mathcal{F}_i(X) \to \operatorname{Sq}^{-1} \mathcal{F}_i(Y) \to 0.$$

From the effectivity of (X, Y) and Y, we have

$$\operatorname{Sq}^{-1} \mathcal{F}_i(X, Y) = H^{\leq \frac{i}{2}}(F(X), F(Y)) \quad \text{and} \quad \operatorname{Sq}^{-1} \mathcal{F}_i(Y) = H^{\leq \frac{i}{2}}(F(Y))$$

Since the inclusion and relative homomorphisms respect the dimension grading, we find that

$$\operatorname{Sq}^{-1} \mathcal{F}_i(X) = H^{\leq \frac{i}{2}}(F(X)),$$

and so X is effective. □

Theorem 3.4. [18, Theorem 1.2] *Let (X, c) be a G-space admitting a G-cellular decomposition*[4]. *Then (X, c) is a conjugation space.*

Proof. The proof follows immediately by induction on the number of G-cells, from Proposition 2.22, Lemma 2.23 and Lemma 3.3. □

Corollary 3.5. *Real projective spaces, real Grassmannians, real flag varieties, and nonsingular toric varieties are conjugation spaces.*

Further examples of conjugation spaces are obtained as a direct consequence of the following result of van Hamel [28]:

Theorem 3.6. *Let (X, c) be a compact, complex manifold equipped with an anti-holomorphic involution,such that $H^*_{\mathrm{odd}}(X) = 0$. If each element of $H^*(X)$ can be represented by a* c*-invariant complex analytic cycle, then X is a conjugation space.* □

Examples of effective spaces that are not conjugation spaces can be obtained as a consequence of the following results:

Proposition 3.7. *Every real nonsingular quadric hypersurface is an effective space.*

Proof. Even-dimensional maximal quadric hypersurfaces X admit a real G-cellular decomposition, and thus for them the result follows from Theorem 3.4. Otherwise, there exists an integer s with $0 \leq s < \frac{1}{2} \dim X$ such that $H^*(F(X))$ is generated by the duals of the fundamental classes of iterated

[4]Kalinin states this result in [18] for real algebraic varieties, but his proof does not use the algebraicity assumption. This theorem can also be used to produce non-algebraic examples of conjugation spaces. One such example is S^6 equipped with the standard almost-complex structure and its naturally associated real structure.

hyperplane sections $[F(X) \cap F(H^r)]$ with $0 \leq r \leq s$ and the duals of the fundamental classes of linear subspaces $[F(L^r)]$ with $0 \leq r \leq s$. Since the Poincaré duality descends to Kalinin's spectral sequence (see [6]), the classes $[X \cap H^r]$ and $[L^r]$ yield non-zero classes in ${}^{\infty}H^*(X)$. In particular, we find $\mathcal{F}_{2i}(X) = \mathcal{F}_{2i+1}(X)$ for each i, $\mathcal{F}_{2i}(X)/\mathcal{F}_{2i-1}(X) = \mathbb{F}_2$ for each $i \leq r$ and each $i \geq n-r$, while $\mathcal{F}_{2r}(X) = \cdots = \mathcal{F}_{2n-2r-1}(X)$. It remains to observe that for each of the classes $u = [Z]^* \in H^*(X)$, we have $\phi^{-1}(u) = \mathrm{Sq}([F(Z)]^*)$, where Z is any of the submanifolds $X \cap H^r$ or L^r, $0 \leq r \leq s$, which follows from [7, Corollary A.2.10]. Thus, every graded piece $\mathcal{F}_{2i}(X)/\mathcal{F}_{2i-1}(X)$ is generated by Steenrod squares of elements in $H^{\leq i/2}(F(X))$, and hence X is effective by definition. □

Proposition 3.8. *If X is either a nonsingular real curve with nonempty connected real locus or a nonsingular real surface X with nonempty connected real locus and $H^1(X) = 0$, then X is effective and Galois maximal.*

Proof. When $F(X)$ is nonempty and connected, we find $\dim \mathcal{F}_0(X) = 1$. Since $\mathcal{F}_0(X) \subseteq H^0(F(X))$, we conclude that $\mathcal{F}_0(X) = H^0(F(X))$.

If X is a curve with a nonempty connected real locus, then ${}^{\infty}H^i(X) = \mathbb{F}_2$ for $i = 0, 2$ and it is 0 for $i = 1$. Therefore, in this case, $\mathcal{F}_0(X) = H^0(F(X))$ is the only nontrivial term in Kalinin's filtration. This implies effectivity in the case of curves with a connected real locus.

If X is a surface with $H^1(X) = 0$, we find ${}^{\infty}H^i(X) = \mathbb{F}_2$ for $i = 0, 4$ and it is 0 for $i = 1, 3$. By the same arguments as above we get

$$\begin{aligned}
\mathcal{F}_0(X) &= H^0(F(X)), \\
\mathcal{F}_1(X) &= \mathcal{F}_0(X), \\
\mathcal{F}_3(X) &= \mathcal{F}_2(X), \\
\mathcal{F}_4(X) &= H^*(F(X)), \\
\mathcal{F}_4(X)/\mathcal{F}_3(X) &= \mathbb{F}_2.
\end{aligned}$$

Finally, to prove the effectivity of X there remain to check that $\mathcal{F}_2^{\perp} \subset H_*(F(X))$ is generated by W_1 and $[F(X)]$ where $W_1 \in H_1(F(X))$ is the Poincaré dual to $w_1(F(X))$ and $[F(X)] \in H_2(F(X))$ is the fundamental class. The latter property is a direct consequence of [6, Proposition 2.4.9(1)] and the duality between Kalinin's spectral sequences in homology and cohomology (see [6, Subsection 1.5.3]).

The Galois maximality in both cases discussed above holds in more generality, without assumption of connectedness of the real locus (see also [23, page 262]). The assumption $F(X) \neq \emptyset$ and, in the case of surfaces, $H^1(X) = 0$ imply, respectively, ${}^{r}H^0(X) = \mathbb{F}_2$ and, in the case of surfaces, ${}^{r}H^1(X) = 0$ for any $r \geq 1$. Then, by duality (see Proposition 2.6 (1)), ${}^{r}H^2(X) = \mathbb{F}_2$ for curves, and ${}^{r}H^3(X) = 0$, ${}^{r}H^4(X) = \mathbb{F}_2$ for surfaces. As a consequence, ${}^{r}d^q = 0$ for any $r \geq 2, q \geq 0$, which is equivalent to Galois maximality according to Proposition 2.14. □

We conclude this subsection by recalling, without proof, the following result of Kalinin, which is one of the main ingredients in the proof of Theorem E:

Proposition 3.9. [18, Proposition 1.3] *Let X be a finite CW-complex and let $\tau : X \times X \to X \times X$, $\tau(x, y) = (y, x)$. Then $(X \times X, \tau)$ is a Galois maximal, effective space.* □

3.2. **Kalinin effectivity under standard constructions.** We now turn to a systematic construction of further examples of effective spaces, and proceed by recalling the following two results of Kalinin:

Proposition 3.10. [18, Proposition 3.3] *A finite product of effective pairs is an effective pair.* □

Proposition 3.11. [18, Proposition 3.11] *Let (X, c) be a G-space, ξ a complex vector bundle on X endowed with an anti-linear involution covering the involution* c, *and Y an associated bundle of flags in ξ equipped with the induced involution. If X is effective then Y is effective.* □

3.2.1. *Effective blowing-up.* Let (Y, B, c) be a G-pair, where Y is a complex manifold equipped with an anti-holomorphic involution c, and $B \subseteq Y$ is a c-invariant complex submanifold of complex codimension at least two. Consider the blow-up $\pi : X \to Y$ of Y along B, and denote by E the exceptional divisor. Notice that (X, E) is a G-pair when equipped with the induced anti-holomorphic involution.

Proposition 3.12. *If Y is effective and $F(B) = \emptyset$, then X is effective.*

Proof. Since $F(B) = \emptyset$, the pull-back homomorphism $\pi^* : H^*(Y) \to H^*(X)$ induces an isomorphism between the second pages of Kalinin's spectral sequences for X and Y. Therefore, due to naturality of Kalinin's spectral sequences, it induces isomorphisms up to the limit. Since, in addition, π establishes a homeomorphism between $F(Y)$ and $F(X)$, by the naturality of Kalinin's filtration, $\pi^* : H^*(F(Y)) \to H^*(F(X))$ induces an isomorphism between Kalinin's filtrations in $F(Y)$ and $F(X)$. Hence, the effectivity of Y is equivalent to that for X. □

Proposition 3.13. *If B and (Y, B) are effective, then E, X and (X, E) are effective.*

Proof. By Proposition 3.11, we find that the effectivity of B implies the effectivity of E. On the other hand, from Proposition 2.9 we see that the effectivity of the pair (Y, B) is equivalent to effectivity of the pair (X, E). As a consequence of Proposition 2.11, we find that the boundary homomorphisms $\delta^i_{\mathbb{R}} : H^i(F(E)) \to H^{i+1}(F(X), F(E))$ are trivial for every $i \geq 0$. Hence, for each $i \geq 0$ we have a short exact sequence

$$0 \to \mathcal{F}_i(X, E) \to \mathcal{F}_i(X) \to \mathcal{F}_i(E) \to 0.$$

Therefore, we get a short exact sequence

$$0 \to \operatorname{Sq}^{-1} \mathcal{F}_i(X,E) \to \operatorname{Sq}^{-1} \mathcal{F}_i(X) \to \operatorname{Sq}^{-1} \mathcal{F}_i(E) \to 0.$$

From the effectivity of (X,E) and E, we have

$$\operatorname{Sq}^{-1} \mathcal{F}_i(X,E) = H^{\leq \frac{i}{2}}(F(X), F(E)) \quad \text{and} \quad \operatorname{Sq}^{-1} \mathcal{F}_i(E) = H^{\leq \frac{i}{2}}(F(E))$$

Since the inclusion and relative homomorphisms respect the dimension grading, we find that

$$\operatorname{Sq}^{-1} \mathcal{F}_i(X) = H^{\leq \frac{i}{2}}(F(X)),$$

and so X is effective. □

3.3. **Constructions of maximal spaces.** We discuss next the behavior of maximal and Galois maximality, respectively, for some general constructions, which parallels the discussion on Kalinin effectivity in the previous section.

As a direct application of Künneth's formula and Proposition 2.16 we have:

Proposition 3.14. *A finite product of maximal (Galois maximal) pairs is a maximal (Galois maximal) pair.* □

Proposition 3.15. *Let (X, c) be a G-space, ξ be a complex vector bundle on X endowed with an anti-linear involution covering the involution* c, *and Y an associated bundle of flags in ξ equipped with the induced involution.*

1) *X is maximal if and only if Y is maximal.*
2) *X is Galois maximal if and only if Y is Galois maximal.*

Proof. Let $\pi : Y \to X$ denote the projection. From the Leray-Hirsch theorem for the fibrations $\pi : Y \to X$ and $\pi_{|F(Y)} : F(Y) \to F(X)$, we obtain

$$\beta_*(X)\beta_*(\mathrm{Fl}) = \beta_*(Y) \quad \text{and} \quad \beta_*(F(X))\beta_*(F(\mathrm{Fl})) = \beta_*(F(Y)). \tag{3.1}$$

Since Fl is a maximal variety, we have $\beta_*(\mathrm{Fl}) = \beta_*(F(\mathrm{Fl}))$. Therefore, from (3.1) we infer that the maximality of X is equivalent to the maximality of Y. Finally, the proof of item 2) follows immediately from the Leray-Hirsch isomorphism and Propositions 2.4 and 2.14. □

3.3.1. *Maximal blowing-up.* As in Section 3.2.1, let (Y, B, c) be a G-pair, where Y is a complex manifold equipped with an anti-holomorphic involution c, and $B \subseteq Y$ is a c-invariant complex submanifold of complex codimension at least two. Consider the blow-up $\pi : X \to Y$ of Y along B, and denote by E the exceptional divisor.

Proposition 3.16. *X is maximal if and only if Y and B are maximal.*

Proof. Let $d = \operatorname{codim}^{\mathbb{C}}_Y B$. Due to the well-known isomorphisms

$$H^*(X) \oplus H^*(B) = H^*(Y) \oplus H^*(E),$$
$$H^*(F(X)) \oplus H^*(F(B)) = H^*(F(Y)) \oplus H^*(F(E)),$$

we find

$$\mathfrak{D}(X) + \mathfrak{D}(B) = \mathfrak{D}(Y) + \mathfrak{D}(E).$$

On the other hand, by Leray-Hirsch theorem, we have

$$\beta_*(E) = d\beta_*(B) \quad \text{and} \quad \beta_*(F(E)) = d\beta_*(F(B)),$$

and so

$$\mathfrak{D}(X) = \mathfrak{D}(Y) + (d-1)\mathfrak{D}(B). \tag{3.2}$$

Since $d \geq 2$ and the Smith-Thom deficiency is non-negative, the result follows from (3.2). □

We conclude this section by recalling the following result of Derval [9] which establishes the behavior of Galois maximality under blow-up. For the reader's convenience, we sketch a short proof, different from Derval's, based on Proposition 2.16.

Proposition 3.17. [9, Proposition 3.5] *If Y and B are Galois maximal, then X is Galois maximal.*

Proof. By the blow-up formula, for every $i \geq 0$, there exists a G-invariant isomorphism

$$H^i(Y) \oplus \bigoplus_{k\geq 1} H^{i-2k}(B) \to H^i(X),$$

given by

$$(y, b_1, b_2, \dots) = \pi^*(y) + \sum_{k\geq 1} j_*(\pi_E^*(b_k)\xi^k)$$

where $\pi_E : E \to B$ is the restriction of π, E is identified with $\mathbb{P}(N_{B|Y})$, and ξ stands for $c_1(\mathcal{O}_E(1)) \pmod 2 \in H^2(E)$, (see, for example, [29, Th. 7.31] and its proof). Thus, applying Proposition 2.16 to the Poincaré dual isomorphisms, we obtain the result. □

Remark 3.18. In general, the examples found in Propositions 3.7, 3.8, 3.9, and 3.12 while effective, are not conjugation spaces. This shows that the class of effective spaces is strictly larger than the class of conjugation spaces.

4. Kalinin effectivity and stretchedness

Let (Y, B, c) be a G-pair, where Y is a complex manifold equipped with an anti-holomorphic involution c, and $B \subseteq Y$ is a c-invariant complex submanifold of complex codimension at least two. As suggested by Proposition 3.13, stability of Kalinin effectivity under blowing-up requires both the effectivity of the G-space B and that of the G-pair (Y, B). For stability of Smith-Thom maximality and Galois maximality under blow-up, one typically assumes Smith-Thom maximality (respectively Galois maximality) of both B and Y. In this section we introduce a class of G-pairs (Y, B) for which Kalinin effectivity or maximality of B and (Y, B) is equivalent to Kalinin effectivity or maximality of B and Y. This will be useful for the applications in the next section.

Before doing so, we consider the case in which both Kalinin effectivity and maximality are assumed for B and (Y, B).

Proposition 4.1. *If B and (Y, B) are maximal and effective, then Y is a conjugation space.*

Proof. From the exact sequence of the pair (Y, B)

$$\cdots \xrightarrow{\delta^{i-1}} H^i(Y, B) \to H^i(Y) \to H^i(B) \xrightarrow{\delta^i} \cdots,$$

we find that the total Betti numbers satisfy

$$\beta_*(Y) = \beta_*(B) + \beta_*(Y, B) - 2\sum_{i\geq 0} \dim \operatorname{Im} \delta^i. \tag{4.1}$$

Since B and the G-pair (Y, B) are effective, by Proposition 2.11, for every $i \geq 0$ the boundary operators $\delta^i_{\mathbb{R}} : H^i(F(B)) \to H^{i+1}(F(Y), F(B))$ vanish, and so we have short exact sequences:

$$0 \to H^i(F(Y), F(B)) \to H^i(F(Y)) \to H^i(F(B)) \to 0, \tag{4.2}$$

and so the total Betti numbers satisfy

$$\beta_*(F(Y)) = \beta_*(F(B)) + \beta_*(F(Y), F(B)). \tag{4.3}$$

Since B and the G-pair (Y, B) are maximal, we have

$$\beta_*(F(B)) = \beta_*(B) \quad \text{and} \quad \beta_*(F(Y), F(B)) = \beta_*(Y, B),$$

while Smith inequality gives $\beta_*(F(Y)) \leq \beta_*(Y)$. From (4.1) and (4.3) we obtain

$$\begin{aligned} \beta_*(B) + \beta_*(Y, B) - 2\sum_{i\geq 0} \dim \operatorname{Im} \delta^i &= \beta_*(Y) \\ &\geq \beta_*(F(Y)) \\ &= \beta_*(F(B)) + \beta_*(F(Y), F(B)) \\ &= \beta_*(B) + \beta_*(Y, B). \end{aligned}$$

Therefore, Y is maximal and the boundary maps δ^i are trivial for all $i \geq 0$. To prove that Y is effective, it remains to observe that from the exact sequence (4.2), we obtain an induced short exact sequence on filtrations:

$$0 \to \mathcal{F}_i(Y, B) \to \mathcal{F}_i(Y) \to \mathcal{F}_i(B) \to 0,$$

and so

$$0 \to \operatorname{Sq}^{-1} \mathcal{F}_i(Y, B) \to \operatorname{Sq}^{-1} \mathcal{F}_i(Y) \to \operatorname{Sq}^{-1} \mathcal{F}_i(B) \to 0.$$

From the effectivity of (Y, B) and B, we have

$$\operatorname{Sq}^{-1} \mathcal{F}_i(Y, B) = H^{\leq \frac{i}{2}}(F(Y), F(B)) \quad \text{and} \quad \operatorname{Sq}^{-1} \mathcal{F}_i(B) = H^{\leq \frac{i}{2}}(F(B)).$$

Since the inclusion and relative homomorphisms respect the dimension grading, we find that

$$\operatorname{Sq}^{-1} \mathcal{F}_i(Y) = H^{\leq \frac{i}{2}}(F(Y)),$$

and so Y is effective. Thus, by Proposition 2.22, being both effective and maximal, Y is a conjugation space. $\square$

Corollary 4.2. *If B and (Y, B) are maximal and effective, X is the blow-up of Y along B, and E is the exceptional divisor, then X, E and (X, E) are maximal and effective.*

Proof. The result follows from Proposition 4.1, and Propositions 3.11, 3.13 and 3.16. $\square$

Remark 4.3. The proof of Proposition 4.1 suggests that Kalinin effectivity and maximality of G-pairs have the same good behavior when the boundary maps δ^i and $\delta^i_{\mathbb{R}}$ vanish for every $i \geq 0$.

4.1. **Stretched G-pairs.** Let (X, c) be a connected compact complex manifold equipped with an anti-holomorphic involution c.

Definition 4.4. *A G-pair (A, B) of* c*-invariant subspaces of X is called a* stretched G-pair *if for every $p \geq 0$, the inclusion homomorphisms*

$$\begin{aligned} H^p(A) &\to H^p(B), \\ H^p(F(A)) &\to H^p(F(B)) \end{aligned} \tag{4.4}$$

are epimorphisms.

Lemma 4.5. *Let (A, B) and (B, C) be two G-pairs of* c*-invariant subspaces of X.*

1) *If the G-pair (A, C) is stretched, then the G-pair (B, C) is stretched.*
2) *If the G-pairs (A, B) and (B, C) are stretched, then the G-pair (A, C) is stretched.*

Proof. Let $i : B \to A$, $j : C \to B$ and $k : C \to A$ denote the inclusions, and $F(i) : F(B) \to F(A)$, $F(j) : F(C) \to F(B)$ and $F(k) : F(C) \to F(A)$, are the corresponding inclusions of fixed loci. By functoriality, for every $p \geq 0$ we have the following commutative diagrams:

$$\begin{tikzcd} H^p(A) \arrow[r, "i^*"] \arrow[dr, "k^*"'] & H^p(B) \arrow[d, "j^*"] \\ & H^p(C) \end{tikzcd} \qquad \begin{tikzcd} H^p(F(A)) \arrow[r, "F(i)^*"] \arrow[dr, "F(k)^*"'] & H^p(F(B)) \arrow[d, "F(j)^*"] \\ & H^p(F(C)). \end{tikzcd}$$

FIGURE 1. Stretchedness for triples.

Since $j^* \circ i^* = k^*$ and $F(j)^* \circ F(i)^* = F(k)^*$, the conclusions of the lemma follow. $\square$

Proposition 4.6. *Let (Y, B) be a stretched G-pair. Then*

1) *If Y and B are effective then (Y, B) is effective.*
2) *If B and (Y, B) are effective then Y is effective.*

3) *If Y and (Y, B) are effective then B is effective.*

Proof. Since the G-pair (Y, B) is stretched, for every integer $i \geq 0$ we have a short exact sequence

$$0 \to H^i(F(Y), F(B)) \to H^i(F(Y)) \to H^i(F(B)) \to 0. \tag{4.5}$$

By applying Kalinin's spectral sequence, for each $i \geq 0$ we get a short exact sequence

$$0 \to \mathcal{F}_i(Y, B) \to \mathcal{F}_i(Y) \to \mathcal{F}_i(B) \to 0.$$

Therefore, we have a short exact sequence

$$0 \to \operatorname{Sq}^{-1} \mathcal{F}_i(Y, B) \to \operatorname{Sq}^{-1} \mathcal{F}_i(Y) \to \operatorname{Sq}^{-1} \mathcal{F}_i(B) \to 0. \tag{4.6}$$

Since the inclusion and relative homomorphisms respect the dimension grading, from the short exact sequences (4.5) and (4.6) we infer that if two of the following relations

$$\begin{aligned} \operatorname{Sq}^{-1} \mathcal{F}_i(Y, B) &= H^{\leq i/2}(F(Y, B)), \\ \operatorname{Sq}^{-1} \mathcal{F}_i(Y) &= H^{\leq i/2}(F(Y)), \\ \operatorname{Sq}^{-1} \mathcal{F}_i(B) &= H^{\leq i/2}(F(B)) \end{aligned}$$

hold, then so does the third. □

An analogous result holds for Smith-Thom maximality:

Proposition 4.7. *Let (Y, B) be a stretched G-pair. Then*

1) *If Y and B are maximal then (Y, B) is maximal.*
2) *If B and (Y, B) are maximal then Y is maximal.*
3) *If Y and (Y, B) are maximal then B is maximal.*

Proof. By assumption, for every $i \geq 0$ we have short exact sequences

$$\begin{gathered} 0 \to H^i(Y, B) \to H^i(Y) \to H^i(B) \to 0 \\ 0 \to H^i(F(Y), F(B)) \to H^i(F(Y)) \to H^i(F(B)) \to 0. \end{gathered} \tag{4.7}$$

As a consequence, we find

$$\beta_*(Y) = \beta_*(B) + \beta_*(Y, B) \text{ and } \beta_*(F(Y)) = \beta_*(F(B)) + \beta_*(F(Y, B)),$$

and the claims follow immediately. □

The following examples show that no direct analogue of Propositions 4.6 and 4.7 holds for Galois maximality.

Example 4.8. Let (Y, c) be the *quadric ellipsoid,* i.e. $Y = \mathbb{P}^1 \times \mathbb{P}^1$ equipped with an anti-holomorphic involution whose fixed locus is homeomorphic to S^2 and let B be a hyperplane section with no real points. In this case, we can immediately see that the pair (Y, B) is stretched. By Proposition 3.8, Y is Galois maximal. In addition, the Galois maximality of (Y, B) is equivalent to the Galois maximality of $Y \setminus B$. The latter admits a G-equivariant deformation retractionto $(F(Y), \mathrm{c} = \mathrm{Id})$ which is Galois maximal since c acts as the identity. Hence, (Y, B) is Galois maximal. However, B is not Galois maximal since $F(B) = \emptyset$.

Example 4.9. Let (B, c_B) be the quadric ellipsoid and consider the interval $[-1,1]$ equipped with the involution $\mathrm{c}_{[-1,1]}(t) = -t$. Let Y be obtained from B and $[-1,1]$ by identifying the pair $\{\pm 1\}$ with a pair of complex conjugate points on a purely imaginary hyperplane section H of B. We equip Y with the involution c induced from c_B and $\mathrm{c}_{[-1,1]}$. Then (Y, B) is a Galois maximal G-pair, since the involution $\mathrm{c}_{[-1,1]}$ restricted to $Y \setminus B = (-1,1)$ is maximal. Notice now that while B is also Galois maximal, Y is not Galois maximal. Indeed, if $\xi \in H_1(Y)$ is the 1-cycle given by the interval $[-1,1]$ completed by a meridian in H, then the Kalinin differential on the second page is

$$d^2(\xi) = [H] \neq 0 \in {}^2H_2 = H^1(G, H_2(Y)) = \mathbb{F}_2[H].$$

It should also be noted that in this example the G-pair (Y, B) is stretched.

4.2. **On the absence of stretchedness.** In general, in the absence of stretchedness, the behavior of Kalinin effectivity and maximality exhibited in Propositions 4.6 and 4.7, respectively, fails, as the following examples show. These examples show that stretchedness is not merely technical, but is genuinely needed in the cases discussed in Sections 5 and 6.

Example 4.10. Let $Y = \mathbb{CP}^2$ be equipped with the standard conjugation, and let $B \subseteq \mathbb{CP}^2$ be a nonempty real conic. Then, the G-pair (Y, B) is not stretched, while both Y and B are effective. However, since $F(Y \setminus B)$ is not connected, we see that the G-pair (Y, B) is not effective.

Example 4.11. Let (Y, c) be the quadric ellipsoid, and B a nonsingular hyperplane section with real points. Then, B and the pair (Y, B) are maximal, while Y is not. Notice that in this case the map $H^1(F(Y)) \to H^1(F(B))$ is the zero map, and so the pair (Y, B) is not stretched.

4.3. **Constructions preserving stretchedness.** We conclude this section with two results which discuss the behavior of stretchedness in two cases relevant for the applications discussed in the next section.

Proposition 4.12. *Let (Y, c) be a compact complex manifold equipped with an anti-holomorphic involution and $Z \subseteq Y$ a* c*-invariant submanifold. If $\mathcal{E}$ is a holomorphic vector bundle Y equipped with an anti-linear involution $\hat{\mathrm{c}}$ covering the involution* c*, and $\mathcal{F}$ a $\hat{\mathrm{c}}$-invariant subbundle of $\mathcal{E}$, then:*

1) *The G-pair $(\mathbb{P}(\mathcal{E}), \mathbb{P}(\mathcal{F}))$ is stretched.*
2) *If the G-pair (Y, Z) is stretched, then the G-pair $(\mathbb{P}(\mathcal{E}), \mathbb{P}(\mathcal{E})_{|Z})$ is stretched.*

Proof. Both items are direct consequences of the Leray-Hirsch theorem, since the tautological bundles of $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{E}_{|Z})$ are restrictions of the tautological bundle of $\mathbb{P}(\mathcal{E})$. In particular, the restriction maps on cohomology are surjective in each degree, both on the complex and real loci, so the pairs are stretched by Definition 4.4. ∎

Proposition 4.13. *Let* (Y, c) *be a compact connected complex manifold equipped with an anti-holomorphic involution and* Z *a* c*-invariant submanifold of* Y*. If the* G*-pair* (Y, Z) *is stretched, then the* G*-pair* $(\widetilde{Y}, E)$ *is stretched, where* $\widetilde{Y}$ *is the blow-up of* Y *along* Z *and* E *is the exceptional divisor of the blow-up.*

Proof. Let $\pi : \widetilde{Y} \to Y$ be blow-up map, and denote by $i : Z \to Y$ and $j : E \to \widetilde{Y}$ the inclusion maps. By the Leray-Hirsch theorem, for every $p \geq 0$ and $x \in H^p(E) = H^p(\mathbb{P}(N_{Z|Y}))$, we have

$$x = \sum_{i+2k=p} (\pi^* a_i)\xi^k,$$

where $\xi = c_1(\mathcal{O}_E(1)) \pmod 2 \in H^2(E)$ and $a_i \in H^i(Z)$. Notice now that $c_1(\mathcal{O}_E(1)) = j^*\eta$, where $\eta = c_1(\mathcal{O}_{\widetilde{Y}}(-E)) \in H^2(\widetilde{Y})$. Moreover, since (Y, Z) is stretched, for every $i \geq 0$ there exists $b_i \in H^i(Y)$ such that $b_i = i^*(a_i)$. Therefore

$$x = \sum_{i+2k=p} \pi^* a_i \xi^k = \sum_{i+2k=p} (\pi^* i^* b_i)(j^*\eta)^k = j^*(\sum_{i+2k=p} (\pi^* b_i)\eta^k) \in H^p(\widetilde{Y}),$$

and so the inclusion homomorphism $j^* : H^p(\widetilde{Y}) \to H^p(E)$ is surjective.

To prove the surjectivity of the inclusion homomorphism $H^p(F(\widetilde{Y})) \to H^p(F(E))$, we proceed in the same manner. By using again the Leray-Hirsch theorem, for every $p \geq 0$ and

$$x \in H^p(F(E)) = H^p(F(\mathbb{P}(N_{Z|Y}))) = H^p(\mathbb{P}_{\mathbb{R}}(N_{F(Z)|F(Y)})),$$

we find

$$x = \sum_{i+k=p} (\pi^* a_i)\xi^k,$$

where $a_i \in H^i(F(Z))$ and $\xi = w_1(\mathcal{O}_{F(E)}(1)) \in H^1(F(E))$. We notice next that $w_1(\mathcal{O}_E(1)) = j^*\eta$, where $\eta = w_1(\mathcal{O}_{\widetilde{Y}}(-E)) \in H^1(F(\widetilde{Y}))$. Moreover, since (Y, Z) is stretched, for every $i \geq 0$ there exists $b_i \in H^i(F(Y))$ such that $b_i = i^*(a_i)$. Therefore

$$x = \sum_{i+k=p} \pi^* a_i \xi^k = \sum_{i+k=p} (\pi^* i^* b_i)(j^*\eta)^k = j^*(\sum_{i+k=p} (\pi^* b_i)\eta^k) \in H^p(F(\widetilde{Y})).$$

Hence, the map $j^* : H^p(F(\widetilde{Y})) \to H^p(F(E))$ is surjective. $\square$

5. Wonderful Compactifications

Let X be a compact connected complex manifold. An *arrangement* $\mathcal{A}$ of submanifolds in X is a finite collection of submanifolds in X, not necessarily connected, such that for every $A, B \in \mathcal{A}$ we have $A \cap B \in \mathcal{A}$ and their tangent bundles satisfy $T(A \cap B) = TA \cap TB$, i.e., A and B intersect cleanly.

Definition 5.1. *Let* (X, c) *be a compact connected complex manifold equipped with an anti-holomorphic involution. An arrangement* $\mathcal{A}$ *of submanifolds in* X *is called a* G-arrangement *if* $\mathrm{c}(A) = A$ *for every* $A \in \mathcal{A}$.

We introduce next a G-equivariant analogue of Li's construction of wonderful compactifications of arrangements. The following definitions are adaptations of [25, Definition 2.2] to our G-equivariant setting.

Definition 5.2. *Let* (X, c) *be a compact connected complex manifold equipped with an anti-holomorphic involution. A subset* $\mathcal{B}$ *of a* G*-arrangement* $\mathcal{A}$ *of submanifolds in* X *is called a* G-building set of $\mathcal{A}$ *if for every* $A \in \mathcal{A}$*, the minimal elements of*$\{B \in \mathcal{B} : B \supseteq A\}$ *intersect transversally and their intersection is* A.

Definition 5.3. *Let* (X, c) *be a compact connected complex manifold equipped with an anti-holomorphic involution. A finite set* $\mathcal{B}$ *of* c-invariant *submanifolds of* X *is called a* G-building set *if the set of all possible intersections of collections of submanifolds from* $\mathcal{B}$ *forms a* G*-arrangement* $\mathcal{A}$ *and that* $\mathcal{B}$ *is a* G*-building set of* $\mathcal{A}$*. In this case,* $\mathcal{A}$ *is called the* arrangement induced by $\mathcal{B}$.

For any G-arrangement $\mathcal{A}$ of submanifolds in X, we define

$$X^\circ := X \setminus \bigcup_{A \in \mathcal{A}} A.$$

If $\mathcal{B}$ is a G-building set of a G-arrangement $\mathcal{A}$, we have

$$\bigcup_{B \in \mathcal{B}} B = \bigcup_{A \in \mathcal{A}} A \quad \text{and} \quad \bigcup_{B \in \mathcal{B}} F(B) = \bigcup_{A \in \mathcal{A}} F(A),$$

and so

$$X^\circ = X \setminus \bigcup_{B \in \mathcal{B}} B \quad \text{and} \quad F(X^\circ) = F(X) \setminus \bigcup_{B \in \mathcal{B}} F(B).$$

We define (*cf.* [25, Definition 1.1]) the wonderful compactification of X° induced by a G-building set $\mathcal{B}$ as follows:

Definition 5.4. *Let* (X, c) *be a compact connected complex manifold equipped with an anti-holomorphic involution, and* $\mathcal{B}$ *a* G*-building set of submanifolds in* X*. The closure of the image of the diagonal embedding*

$$X^\circ \hookrightarrow \prod_{B \in \mathcal{B}} \mathrm{Bl}_B X,$$

is called the wonderful compactification *of the arrangement induced by* $\mathcal{B}$*, and is denoted by* $X_{\mathcal{B}}$.

Definition 5.5. *Let* X *be a compact connected complex manifold and* $\pi : Bl_C X \to X$ *the blow-up of* X *along a submanifold* C*. For any analytic subspace* $V \subseteq X$*, the dominant transform of* V*, denoted by* $\widetilde{V}$*, is defined as the proper transform of* V *if* $V \nsubseteq C$*, and* $\pi^{-1}(V)$ *if* $V \subseteq C$.

Proposition 5.6. *Let (X, c) be a compact connected complex manifold equipped with an anti-holomorphic involution, $\mathcal{B}$ a G-building set with the induced G-arrangement $\mathcal{A}$ of submanifolds in X. Let F be a minimal element in $\mathcal{B}$ and let $\pi : \mathrm{Bl}_F X \to X$ be the blow-up of X along F. Denote the exceptional divisor by E.*

1) *The collection $\mathcal{A}_F$ of submanifolds in $\mathrm{Bl}_F X$ defined as*
$$\mathcal{A}_F := \{\widetilde{A}\}_{A\in\mathcal{A}} \cup \{\widetilde{A} \cap E\}_{A\in\mathcal{A},\, \emptyset\subsetneq A\cap F\subsetneq A}$$
is a G-arrangement of submanifolds in $\mathrm{Bl}_F X$.
2) *$\mathcal{B}_F := \{\widetilde{B}\}_{B\in\mathcal{B}}$ is a building set of $\mathcal{A}_F$.*

Proof. The proof follows from [25, Proposition 2.8], noting that since X is equipped with an anti-holomorphic involution c and F is a c-invariant submanifold of X, then the blow-up $\mathrm{Bl}_F X$ admits a naturally induced anti-holomorphic involution $\hat{\mathrm{c}}$ and the dominant transform of a c-invariant submanifold is $\hat{\mathrm{c}}$-invariant. □

We call $\mathcal{A}_F$ and $\mathcal{B}_F$ the *blow-up G-arrangement* and its *blow-up G-building set*, respectively.

Let (X, c) be a compact connected complex manifold equipped with an anti-holomorphic involution, $\mathcal{A}$ a G-arrangement of submanifolds in X, and $\mathcal{B} = \{B_1, \dots, B_N\}$ a G-building set of $\mathcal{A}$. We now describe the standard iterative construction of the wonderful compactification via successive blow-ups along the elements of the building set.

For every $k = 0, \dots, N$, we inductively define a triple
$$(X_k, \mathcal{A}^{(k)}, \mathcal{B}^{(k)})$$
consisting of a compact connected complex manifold (X_k, c_k) equipped with an anti-holomorphic involution, a G-arrangement $\mathcal{A}^{(k)}$ of submanifolds in X_k, and a G-building set $\mathcal{B}^{(k)}$, as follows:

(i) For $k = 0$, define $X_0 = X$, $\mathcal{A}^{(0)} = \mathcal{A}$, $\mathcal{B}^{(0)} = \mathcal{B}$.
(ii) Assume that $(X_{k-1}, \mathcal{A}^{(k-1)}, \mathcal{B}^{(k-1)})$ is constructed.

- Define X_k to be the blow-up of X_{k-1} along the submanifold $B_k^{(k-1)}$, and denote by $E^{(k)}$ the exceptional divisor.
- For each $B \in \mathcal{B}^{(k-1)}$, let $B^{(k)}$ denote its dominant transform in X_k, and define
$$\mathcal{B}^{(k)} := \{B^{(k)}\}_{B\in\mathcal{B}^{(k-1)}}.$$
- Define the collection of submanifolds of X_k given by
$$\mathcal{A}^{(k)} := \{\widetilde{A}\}_{A\in\mathcal{A}^{(k-1)}} \cup \{\widetilde{A} \cap E^{(k)}\}_{\emptyset\subsetneq A\cap B_k^{(k-1)}\subsetneq A}.$$

  Since $B_k^{(k-1)}$ is assumed c_{k-1}-invariant, by Proposition 5.6 and [25, Proposition 2.8], $\mathcal{A}^{(k)}$ is a G-arrangement of submanifolds in X_k equipped with the induced anti-holomorphic involution c_k and $\mathcal{B}^{(k)}$ is a G-building set of $\mathcal{A}^{(k)}$.

(iii) Continue the inductive construction until $k = N$. We obtain

$$(X_N, \mathcal{A}^{(N)}, \mathcal{B}^{(N)})$$

where all the subvarieties in the building set $\mathcal{B}^{(N)}$ are divisors.

We define

$$\mathrm{Bl}_{\mathcal{B}} X := X_N, \quad \mathcal{A}_{\mathcal{B}} := \mathcal{A}^{(N)}, \quad \widetilde{\mathcal{B}} := \mathcal{B}^{(N)}.$$

By construction, $\mathrm{Bl}_{\mathcal{B}} X$ is a compact complex manifold equipped with an anti-holomorphic involution, and $\mathcal{A}_{\mathcal{B}}$ is a G-arrangement of submanifolds in $\mathrm{Bl}_{\mathcal{B}} X$ induced by the G-building set $\widetilde{\mathcal{B}}$.

The following theorem summarizes the basic properties of the wonderful compactification, and is a direct consequence of [25, Proposition 2.13, Theorem 1.2 and Theorem 1.3]:

Theorem 5.7. *Let (X, c) be a compact complex manifold equipped with an anti-holomorphic involution and a nonempty G-building set $\mathcal{B} = \{B_1, \dots, B_N\}$, and let*

$$X^\circ = X \setminus \bigcup_{i=1}^{N} B_i.$$

Then:

1) $\mathrm{Bl}_{\mathcal{B}} X$ *is the wonderful compactification of the G-arrangement induced by the G-building set $\mathcal{B}$.*
2) *For each $B_i \in \mathcal{B}$ there is a nonsingular divisor $D_{B_i} \subset \mathrm{Bl}_{\mathcal{B}} X$, such that $D_{\mathcal{B}} = \bigcup_{i=1}^{N} D_{B_i} = \mathrm{Bl}_{\mathcal{B}} X \setminus X^\circ$ and $D_{\mathcal{B}}$ is a normal crossing divisor.*
3) *Let $\mathcal{I}_i$ be the ideal sheaf of $B_i \in \mathcal{B}$. The wonderful compactification $\mathrm{Bl}_{\mathcal{B}} X$ is isomorphic to the blow-up of X along the ideal sheaf $\mathcal{I}_1 \mathcal{I}_2 \cdots \mathcal{I}_N$.*
4) *If we arrange $\mathcal{B} = \{B_1, \dots, B_N\}$ in such an order that the first i terms $B_1, \dots, B_i$ form a G-building set for any $1 \le i \le N$, then*

$$\mathrm{Bl}_{\mathcal{B}} X = \mathrm{Bl}_{\widetilde{B}_N} \cdots \mathrm{Bl}_{\widetilde{B}_2} \mathrm{Bl}_{B_1} X, \tag{5.1}$$

where each blow-up is along a nonsingular subvariety. □

5.1. Stretched G-arrangements.

Definition 5.8. *Let (X, c) be a compact complex manifold equipped with an anti-holomorphic involution, and let $\mathcal{A}$ be a G-arrangement of submanifolds in X.*

1) *The G-arrangement $\mathcal{A}$ is called* stretched *if for any $A \in \mathcal{A}$ the inclusion homomorphisms*

$$\begin{gathered} H^p(X) \to H^p(A), \\ H^p(F(X)) \to H^p(F(A)) \end{gathered} \tag{5.2}$$

are surjective.

2) *The G-arrangement $\mathcal{A}$ is called effective, maximal or Galois maximal, if X and every $A \in \mathcal{A}$ are effective, maximal or Galois maximal, respectively.*

Remark 5.9. If a G-arrangement $\mathcal{A}$ is stretched, then by Lemma 4.5 one can see that for any $A, B \in \mathcal{A}$ such that $B \subseteq A$, the G-pair (A, B) is stretched.

Proposition 5.10. *Let (X, c) be a compact complex manifold equipped with an anti-holomorphic involution, $\mathcal{A}$ a G-arrangement of submanifolds in X, $\mathcal{B}$ a G-building set of $\mathcal{A}$, and let $B \in \mathcal{B}$ be a minimal element with respect to inclusion. Consider the blow-up G-arrangement $\mathcal{A}_B$ in the blow-up $\pi : \mathrm{Bl}_B X \to X$. Then*

1) *If the G-arrangement $\mathcal{A}$ is stretched, then the blow-up G-arrangement $\mathcal{A}_B$ is stretched.*
2) *If the G-arrangement $\mathcal{A}$ is stretched and effective, then the blow-up G-arrangement $\mathcal{A}_B$ is effective.*
3) *If the G-arrangement $\mathcal{A}$ is maximal or Galois maximal, then the blow-up G-arrangement $\mathcal{A}_B$ is maximal or Galois maximal, respectively.*

Proof. Let $i : A \to X$ and $j : B \to X$ denote the inclusion maps. We denote by $E = \mathbb{P}(N_{B|X})$ the exceptional divisor of the blow-up, and by $\widetilde{A}$ the dominant transform of A. Let $\widetilde{j} : E \to \mathrm{Bl}_B X$ and $\widetilde{i} : \widetilde{A} \to \mathrm{Bl}_B X$ be the inclusion maps, π_A, f_B and f_A the restriction of π to $\widetilde{A}$ and E, respectively. These maps give rise to the following commutative diagram:

$$\begin{tikzcd} \widetilde{A} \arrow[r, "\widetilde{i}"] \arrow[d, "\pi_A"'] & \mathrm{Bl}_B X \arrow[d, "\pi"] & E \arrow[l, "\widetilde{j}"'] \arrow[d, "f_B"] \\ A \arrow[r, "i"'] & X & B. \arrow[l, "j"] \end{tikzcd} \tag{5.3}$$

Recall that the blow-up arrangement $\mathcal{A}_B$ on $\mathrm{Bl}_B X$ is defined by

$$\mathcal{A}_B = \{\widetilde{A}\}_{A \in \mathcal{A}} \cup \{\widetilde{A} \cap E\}_{\emptyset \subsetneq A \cap B \subsetneq A}.$$

Before we proceed, notice that if B and X are effective, maximal or Galois maximal, then by Propositions 4.6, 3.13, 3.16, 3.17, we find that $\mathrm{Bl}_B X$ is effective, maximal or Galois maximal, respectively. We prove the three statements simultaneously. Since stretchedness concerns the surjectivity of restriction maps, we examine each submanifold appearing in the blow-up arrangement $\mathcal{A}_B$. This leads to four cases depending on the relative position of $A \in \mathcal{A}$ with respect to the blow-up center B.

Case 1: $A \cap B = \emptyset$.

Since $A \cap B = \emptyset$, the map π_A^* is the identity, and so $i^* = \widetilde{i}^* \circ \pi^*$. The stretchedness of the G-pair (X, A) implies that the map i^* is surjective, and so $\widetilde{i}^*$ is surjective, as well. Similarly, we find that $H^*(F(\mathrm{Bl}_B X)) \to H^*(F(\widetilde{A}))$ is surjective, which shows that the G-pair $(\mathrm{Bl}_B X, \widetilde{A})$ is stretched.

It remains to notice that, since the map π_A^* is the identity, if A is effective, maximal or Galois maximal, then $\widetilde{A}$ satisfies the corresponding property as well.

Case 2: $A \subseteq B$.

In this case $\widetilde{A} = \pi^{-1}(A)$ coincides with the projectivized normal bundle $\mathbb{P}(N_{B|X}|_A) \subseteq E$. By Lemmas 4.5 and 4.13, it suffices to show that the G-pair $(E, \widetilde{A})$ is stretched. However, since $E = \mathbb{P}(N_{B|X})$, this follows from Lemma 4.12.

If A is effective, from Proposition 3.11 we conclude that$\widetilde{A} = \mathbb{P}(N_{B|X_{|A}}) \subseteq E$ and, consequently, $\widetilde{A} \cap E = \widetilde{A}$ are effective. Similarly, if A is maximal or Galois maximal, then by Proposition 3.15, $\widetilde{A}$ and consequently, $\widetilde{A} \cap E = \widetilde{A}$ are maximal or Galois maximal, as well.

Case 3: $B \subseteq A$.

Let $\ell : B \to A$ denote the inclusion. In this case, $\widetilde{A}$ is the blow-up of A along B and $\widetilde{A} \cap E$ is the exceptional divisor $E_A = \mathbb{P}(N_{B|A})$. Notice we have a commutative diagram:

$$\begin{tikzcd}
E_A \arrow[r, "\widetilde{\ell}"] \arrow[d, "f_A"'] & \widetilde{A} \arrow[r, "\widetilde{i}"] \arrow[d, "\pi_A"] & \mathrm{Bl}_B X \arrow[d, "\pi"] \\
B \arrow[r, "\ell"'] & A \arrow[r, "i"'] & X,
\end{tikzcd} \tag{5.4}$$

where $\widetilde{\ell} : E_A \to \widetilde{A}$ denotes the inclusion, and $f_A : E_A \to B$ is the restriction of π to E_A.

We prove first that the G-pair $(\mathrm{Bl}_B X, \widetilde{A} \cap E)$ is stretched. Notice that $N_{B|A}$ is a holomorphic sub-bundle of $N_{B|X}$, and since A and B are c-invariant, it carries anti-linear involution covering the restriction of c to B. Therefore, by Lemma 4.12, the G-pair (E, E_A) is stretched. Since $\widetilde{A} \cap E = E_A$, using Lemmas 4.5 and 4.13, we can now see that G-pair $(\mathrm{Bl}_B X, \widetilde{A} \cap E)$ is stretched.

To prove that the G-pair $(\mathrm{Bl}_B X, \widetilde{A})$ is stretched, let R be the polynomial ring $\mathbb{F}_2[v]$, where v is of degree 2, and consider the diagram

$$\begin{tikzcd}
H^*(X) \otimes R \arrow[r, "\phi_X"] \arrow[d, "i^* \otimes Id"'] & H^*(\mathrm{Bl}_B X) \arrow[d, "\widetilde{i}^*"] \\
H^*(A) \otimes R \arrow[r, "\phi_A"'] & H^*(\widetilde{A}).
\end{tikzcd} \tag{5.5}$$

The horizontal maps are given by

$$\phi_X(x\otimes v^r) = \begin{cases} \pi^* x, \text{ if } r=0 \\ \widetilde{j}_*(f_B^* j^* x \cup \xi^r), \text{ if } r>0 \end{cases} \tag{5.6}$$

and

$$\phi_A(a\otimes v^r) = \begin{cases} \pi_A^* a, \text{ if } r=0 \\ \widetilde{\ell}_*(f_A^* \ell^* a \cup \xi_A^r), \text{ if } r>0, \end{cases} \tag{5.7}$$

where ξ and ξ_A stand for the reduction mod 2 of the first Chern class of the tautological line bundles of E and E_A, respectively.

Let us first show that the diagram (5.5) is commutative. Indeed, when $r = 0$, due to the commutativity of diagram (5.4), for every $x \in H^*(X)$, we have

$$\widetilde{i}^* \phi_X(x\otimes 1) = \widetilde{i}^* \pi^* x = \pi_A^* i^* x = \phi_A(i^* x\otimes 1).$$

For the case $r > 0$, we recall two well-known results:

1) If $[E]$ and $[E_A]$ denote the cohomology classes of E and E_A in $\mathrm{Bl}_B X$ and $\widetilde{A}$, respectively, then
$$\widetilde{i}^*[E] = [E_A], \tag{5.8}$$
2) For every $r \geq 0$, we have
$$\widetilde{j}_* \xi^r = [E]^{r+1} \quad \text{and} \quad \widetilde{\ell}_* \xi_A^r = [E_A]^{r+1}. \tag{5.9}$$

Using the commutativity of the diagrams (5.3) and (5.4), the formulas (5.8) and (5.9), and the projection formula, for every $r > 0$ and $x \in H^*(X)$, we have

$$\begin{aligned} \widetilde{i}^* \phi_X(x\otimes v^r) &= \widetilde{i}^*(\widetilde{j}_*(f_B^* j^* x \cup \xi^r)) \\ &= \widetilde{i}^*(\widetilde{j}_*(\widetilde{j}^* \pi^* x \cup \xi^r)) \\ &= \widetilde{i}^*(\pi^* x \cup [E]^{r+1}) \\ &= (\widetilde{i}^* \pi^*) x \cup [E_A]^{r+1} \\ &= (\pi_A{}^* i^*) x \cup [E_A]^{r+1}. \end{aligned}$$

Similarly, we compute

$$\begin{aligned} \phi_A(i^* x\otimes v^r) &= \widetilde{\ell}_*(f_A^* \ell^* i^* x \cup \xi_A^r) \\ &= \widetilde{\ell}_*(\widetilde{\ell}^* \pi_A^* i^* x \cup \xi_A^r) \\ &= (\pi_A^* i^*) x \cup [E_A]^{r+1} \\ &= \widetilde{i}^* \phi_X(x\otimes v^r), \end{aligned}$$

which shows that the diagram (5.5) is commutative[5].

By [14, Theorem 3.11], the cohomology of the blow-up is generated by the pull-back of classes from the base together with powers of the exceptional

[5] The same argument proves commutativity of the diagram (5.5) with coefficients in $\mathbb{Z}$.

divisor class, and so the maps ϕ_X and ϕ_A are surjective. Since the restriction map i^* is surjective, from the commutativity of diagram (5.5), we find that the map $\widetilde{i}^*$ is surjective, as well.

To prove that the restriction map $F(\widetilde{i}^*) : H^*(F(\mathrm{Bl}_B X)) \to H^*(F(\widetilde{A}))$ is surjective, we proceed in similar manner. Once again, the crucial ingredient is Theorem 3.11 in [14]. As pointed out in [14, Section 5], this result is also valid for the real blow-up, with the degree of the variable v reset to one and the first Chern class replaced by the first Stiefel-Whitney class. The other arguments are identical with those used for proving the surjectivity of $\widetilde{i}^*$ and will therefore not be repeated.

If the G-arrangement is effective, then A and B are effective. Hence, by Proposition 4.6, the G-pair (A, B) is effective. Since in our case, $\widetilde{A}$ is the blow-up of A along B and $\widetilde{A} \cap E$ is the exceptional divisor $E_A = \mathbb{P}(N_{B|A})$, by Proposition 3.13, $\widetilde{A}$ is effective. Furthermore, since $\widetilde{A} \cap E = \mathbb{P}(N_{B|A})$ and B is effective, from Proposition 3.11 we conclude that $\widetilde{A} \cap E$ is also effective.

Similarly, if the G-arrangement is maximal or Galois maximal, then A and B are maximal or Galois maximal, respectively. By applying Proposition 3.16 in the maximal case and Proposition 3.17 in the Galois maximal case, we see that $\widetilde{A}$ is maximal or Galois maximal, respectively. The maximality or Galois maximality of $\widetilde{A} \cap E = \mathbb{P}(N_{B|A})$ is a consequence of Proposition 3.15.

Case 4: $\emptyset \subsetneq A \cap B \subsetneq A$ and $B \not\subseteq A$.

In this case, $\widetilde{A}$ is the blow-up of A along $A \cap B$ and $E_A = \mathbb{P}(N_{A\cap B|A})$. Let $g : A \cap B \to A$ and $k : A \cap B \to X$ denote the inclusions.

Notice we have a commutative diagram

$$\begin{tikzcd} E_A \arrow[r, "\widetilde{g}"] \arrow[d, "f_A"'] & \widetilde{A} \arrow[r, "\widetilde{i}"] \arrow[d, "\pi_A"] & \mathrm{Bl}_B X \arrow[d, "\pi"] \\ A \cap B \arrow[r, "g"'] & A \arrow[r, "i"'] & X, \end{tikzcd} \tag{5.10}$$

where $\widetilde{g} : E_A \to \widetilde{A}$ denotes the inclusion, and, as before, $f_A : E_A \to B$ is the restriction of π to E_A.

As in the previous case, to show that the G-pair $(\mathrm{Bl}_B X, \widetilde{A} \cap E)$ is stretched, it suffices to argue that the G-pair (E, E_A) is stretched. However, since A and B meet cleanly, $N_{A\cap B|A}$ is a holomorphic vector sub-bundle of the restriction of $N_{B|X}$ to $A \cap B$. Hence, by applying Lemma 4.12, it follows that the G-pair (E, E_A) is indeed stretched.

To show that the pair $(\mathrm{Bl}_B X, \widetilde{A})$ is stretched we proceed as in the previous case and consider the diagram (5.5)

$$\begin{tikzcd} H^*(X)\otimes R \arrow[r, "\phi_X"] \arrow[d, "i^*\otimes Id"'] & H^*(\mathrm{Bl}_B X) \arrow[d, "\widetilde{i}^*"] \\ H^*(A)\otimes R \arrow[r, "\phi'_A"'] & H^*(\widetilde{A}), \end{tikzcd} \tag{5.11}$$

where the map ϕ_X is defined as in (5.6), while the map ϕ'_A is given by

$$\phi'_A(a\otimes v^r) = \begin{cases} \pi_A^* a, \text{ if } r=0 \\ \widetilde{g}_*(f_A^* g^* a \cup \xi_A^r), \text{ if } r>0, \end{cases} \tag{5.12}$$

where ξ_A the reduction mod 2 of the first Chern class of the tautological line bundle of E_A, respectively. Using the commutativity of the diagram (5.10) and the projection formula, for every $r>0$ and $x\in H^*(X)$, we have

$$\begin{aligned} \phi'_A(i^*x\otimes v^r) &= \widetilde{g}_*(f_A^* g^* i^* x \cup \xi_A^r) \\ &= \widetilde{g}_*(\widetilde{g}^*\pi_A^* i^* x\cup \xi_A^r), \\ &= (\pi_A^* i^*)x \cup [E_A]^{r+1} \\ &= \widetilde{i}^*\phi_X(x\otimes v^r), \end{aligned}$$

while, as in the previous case,

$$\widetilde{i}^*\phi_X(x\otimes 1) = \widetilde{i}^*\pi^* x = \pi_A^* i^* x = \phi_A(i^*x\otimes 1),$$

proving that the diagram (5.11) is commutative. By [14, Theorem 3.11] the maps ϕ_X and ϕ'_A are surjections. Since the G-pair (X, A) is stretched, the map i^* is a surjection implying that the map $\widetilde{i}^*$ is a surjection, as well.

The argument proving that the map $F(\widetilde{i}^*) : H^*(F(\mathrm{Bl}_B X)) \to H^*(F(\widetilde{A}))$ is surjective coincides with the one in the previous case and will not be repeated.

Since the G-arrangement $\mathcal{A}$ is effective, then A and $A\cap B$, which both belong to $\mathcal{A}$, are effective. As a consequence of Proposition 4.6, the G-pair $(A, A\cap B)$ is effective, and so, by Proposition 3.13, we find that $\widetilde{A}$ is effective. In addition, since A and B meet cleanly, then $\widetilde{A}\cap E = \mathbb{P}(N_{A\cap B|A})$. It is an effective G-space due to the effectivity of $A\cap B$ and Lemma 4.12.

Similarly, if the G-arrangement is maximal or Galois maximal, then A, B and $A\cap B$ are maximal or Galois maximal, respectively. Since $\widetilde{A}$ is the blow-up of A along $A\cap B$, then $\widetilde{A}$ is maximal by Proposition 3.16 or Galois maximal according to Proposition 3.17, respectively. The maximality or Galois maximality of $\widetilde{A}\cap E = \mathbb{P}(N_{A\cap B|A})$ follows from Proposition 3.15. $\square$

Theorem 5.11. *Let $\mathcal{A}$ be a G-arrangement of submanifolds in a compact connected complex manifold X, $\mathcal{B}\subseteq\mathcal{A}$ a building set, and $(X_{\mathcal{B}}, \mathcal{A}_{\mathcal{B}})$ its wonderful compactification. Then:*

1) *If the arrangement $\mathcal{A}$ is stretched then the G-arrangement $(X_{\mathcal{B}}, \mathcal{A}_{\mathcal{B}})$ is stretched.*
2) *If the G-arrangement $\mathcal{A}$ is maximal or Galois maximal then the G-arrangement $(X_{\mathcal{B}}, \mathcal{A}_{\mathcal{B}})$ is maximal or Galois maximal, respectively.*
3) *If the G-arrangement $\mathcal{A}$ is stretched and effective then the G-arrangement $(X_{\mathcal{B}}, \mathcal{A}_{\mathcal{B}})$ is effective.*

Proof. Let $\mathcal{B} = \{B_1, \dots, B_N\}$ be a building set of $\mathcal{A}$. The proof of Theorem 5.11 is by induction on N. If the arrangement $\mathcal{A}$ is stretched, from Proposition 5.10 we see that each time when we blow-up along an element of the building set, the resulting G-arrangement is stretched. Iterating, the resulting G-arrangement $(X_{\mathcal{B}}, \mathcal{A}_{\mathcal{B}})$ is stretched, proving the first item.

The second item follows from Proposition 3.16 and Proposition 3.17, respectively, which show that the Smith-Thom maximality (Galois maximality) is preserved under blowing-up a maximal (Galois maximal) manifold along a smooth maximal (Galois maximal) center.

The third item follows by a repeated application of Propositions 5.10. □

As a direct consequence of Theorem 5.11, we find:

Corollary 5.12. *If the G-arrangement $\mathcal{A}$ is stretched, maximal and effective, then $X_{\mathcal{B}}$ is a conjugation space.*

6. Examples

In this section we illustrate the general results obtained above by applying them to several natural geometric constructions.

6.1. **Wonderful compactification of subspace arrangements - Proof of Theorem A.** Let $\mathbb{P}^n$ be the complex projective space equipped with the standard conjugation c, and let $\mathcal{A}$ be an arrangement generated by a finite collection $\{L_i\}_{i \in I}$ of c-invariant linear subspaces. Then for every $i \in I$ the G-pair $(\mathbb{P}^n, L_i)$ is stretched, while $\mathbb{P}^n$ and each L_i is a conjugation space by Theorem 3.4. In this case, as a direct application of Corollary 5.12, we obtain a proof of Theorem A.

6.2. **The moduli space of real stable rational curves with marked points - Proof of Theorem B.** The Deligne-Mumford moduli space $\overline{\mathcal{M}}_{0,n}$ consists of isomorphism classes of n-pointed stable curves of arithmetic genus 0,

$$((C, J); p_1, \dots, p_n),$$

where C is an underlying tree of smooth spheres, J is a complex structure on C, and $p_1, \dots, p_n \in C$ are marked points. For each element σ of the permutation group S_n we have a holomorphic bijection$g_\sigma : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$,

$$g_\sigma((C, J); p_1, \dots, p_n) = ((C, J); \sigma(p_1), \dots, \sigma(p_n)),$$

and a bijection $\mathrm{c}_\sigma : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$,

$$\mathrm{c}_\sigma((C, J); p_1, \dots, p_n) = ((C, -J); \sigma(p_1), \dots, \sigma(p_n)),$$

which is anti-holomorphic. According to Bruno and Mella [2], for every $n \geq 5$ the group of holomorphic automorphisms, $\mathrm{Aut}(\overline{\mathcal{M}}_{0,n})$, is composed of g_σ, $\sigma \in S_n$. Notice that $\mathrm{c}_{\mathrm{Id}} \circ g_\sigma = \mathrm{c}_\sigma$ is an involution if and only if $\sigma^2 = \mathrm{Id}$. This implies that the set of real structures on $\overline{\mathcal{M}}_{0,n}$ is exhausted by the anti-holomorphic involutions c_σ with $\sigma^2 = \mathrm{Id}$.

Examples [3]:

1) $\overline{\mathcal{M}}_{0,3}$ is a point.
2) $\overline{\mathcal{M}}_{0,4}$ is biholomorphic to $\mathbb{CP}^1$ with the standard real structure, independently of σ.
3) $\overline{\mathcal{M}}_{0,5}$ is biholomorphic to $\mathbb{P}^2$ blown-up at four points. These four points are real if $\sigma = \mathrm{Id}$, two of them are real and the other two are imaginary complex conjugate points, if σ is a transposition, and the four points form two pairs of imaginary complex conjugate points, if σ consists of two transpositions.

Proof of Theorem B. Kapranov's description [19] of the Deligne-Mumford moduli space of rational curves with n-marked points exhibits $\overline{\mathcal{M}}_{0,n}$ as the wonderful compactification of the arrangement $\mathcal{A}$ generated by the building set $\mathcal{B}$ consisting of all projective subspaces of $\mathbb{CP}^{n-3}$ spanned by any subset of fixed $n-1$ generic points $\Pi = \{p_1, p_2, \ldots, p_{n-1}\}$. This description coincides with the wonderful compactification of linear subspaces

$$A_{n-2} = \{(z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} \,|\, z_i = z_j \,,\, 1 \leq i < j \leq n-1\}$$

due to De Concini-Procesi [5, page 483] (see also [25]).

Kapranov's construction depends on the choice of one of the marked points, which we take to be the n-th marked point. The moduli space $\overline{\mathcal{M}}_{0,n}$ can be obtained from $\mathbb{CP}^{n-3}$ by an iterated blow-up

$$\kappa_n : \overline{\mathcal{M}}_{0,n} \to \mathbb{CP}^{n-3}$$

as follows: first, we blow up the points of Π, next the proper images of the lines through the points of Π, then the proper images of the planes spanned by points of Π, etc.

The Kapranov blow-up presentation of $\overline{\mathcal{M}}_{0,n}$ is compatible with the real structure c_σ induced by a permutation $\sigma \in S_n$ if and only if σ fixes the distinguished marked point. In this case $\mathbb{CP}^{n-3}$ carries the standard real structure

$$\mathrm{c}([z_1 : \cdots : z_{n-2}]) = [\bar{z}_1 : \cdots : \bar{z}_{n-2}]$$

and the points in Π are permuted accordingly, making the iterated blow-up sequence κ_n equivariant at each stage.

When $\sigma = \mathrm{Id}$, the $n-1$ points are chosen in $F(\mathbb{CP}^{n-3}) = \mathbb{RP}^{n-3}$. Hence, $\mathcal{B}$ is G-invariant, and $\mathcal{A}$ is a G-arrangement. Notice that since the elements of the buliding set $\mathcal{B}$ are real linear subspaces in $\mathbb{CP}^{n-3}$, equipped with the standard real structure, as in the proof of Theorem A we see that the G-arrangement $\mathcal{A}$ is stretched, maximal and effective. As a direct application of Corollary 5.12, we find that $(\overline{\mathcal{M}}_{0,n}, \mathrm{c}_{\mathrm{Id}})$ is a conjugation space, proving the first part of Theorem B.

To prove the second part, notice that if $\sigma \neq \mathrm{Id}$ and the fixed-point set $\mathrm{Fix}\,\sigma \neq \emptyset$, the set Π in the Kapranov model for $\overline{\mathcal{M}}_{0,n}$ can be chosen c-invariant and having $\mathrm{Card}\,|\,\mathrm{Fix}\,\sigma| - 1$ real points. The associated arrangement of all linear subspaces spanned by the subsets of Π is merely c-invariant. Note that at each step in the iterated blow-up κ_n the centers are pairwise disjoint varieties. As in the proof of Theorem A, the stretchedness of the blow-up centers is preserved. Blowing up along centers with nonempty real locus, the effectivity is preserved due to Theorem 5.11, while for blowing-up a pair of disjoint, complex conjugated centers the effectivity is preserved due to Proposition 3.12. The Galois maximality of $(\overline{\mathcal{M}}_{0,n}, \mathrm{c}_{\sigma})$ follows from Proposition 3.17 applied to Kapranov's iterated blowing-up description of the maximal G-space $(\mathbb{CP}^{n-3}, \mathrm{c})$ along Galois maximal centers with possible empty real locus. □

Remark 6.1. An alternative description of $\overline{\mathcal{M}}_{0,n}$ is due to Keel and it gives an alternative proof of the first part of Theorem B.

Keel's construction [20, 25] is the wonderful compactification of the triple $((\mathbb{P}^1)^{n-3}, \mathcal{A}, \mathcal{B})$, where $\mathcal{B}$ is the set of all diagonals and augmented diagonals

$$\Delta_{I,a} = \{(p_4, \dots, p_n) \in (\mathbb{P}^1)^{n-3},\ p_i = a \ \forall i \in I\}$$

for $I \subseteq \{4, \dots, n\}$, $|I| \geq 2$, and $a \in \{0, 1, \infty\}$. Here $\mathcal{A}$ is the set of all intersections of elements in $\mathcal{B}$. Again, when $\sigma = \mathrm{Id}$, using Lemma 6.3, one can see that $\mathcal{A}$ is a stretched, maximal G-arrangement, and we, as before, we find $(\overline{\mathcal{M}}_{0,n}, \mathrm{c}_{\mathrm{Id}}))$ is a conjugation space as a consequence of Corollary 5.12.

Remark 6.2. For the real structure $(\overline{\mathcal{M}}_{0,n}, \mathrm{c}_{\sigma})$ with $\mathrm{Fix}\,\sigma = \emptyset$, an equivariant Kapranov model is unavailable, and it will be treated elsewhere.

Proof of Corollary C. The proof is a straightforward consequence of Theorems B and 2.19. □

6.3. **Wonderful compactifications of configuration spaces - Proof of Theorem D.** Let (X, c) be a compact connected complex manifold equipped with an anti-holomorphic involution. For $n \geq 2$, a diagonal in X^n is the submanifold

$$\Delta_I = \{(p_1, \dots, p_n) \in X^n \,|\, p_i = p_j, \forall\, i, j \in I\}$$

for $I \subseteq \{1, \dots, n\}$, with $|I| \geq 2$. A polydiagonal is an intersection of diagonals

$$\Delta_{I_1 \dots I_k} := \Delta_{I_1} \cap \dots \cap \Delta_{I_k}$$

where $I_i \subseteq \{1, \dots, n\}, |I_i| \geq 2$ for each $1 \leq i \leq k$. The collection $\mathcal{A}$ of all diagonals and polydiagonals is an arrangement of submanifolds in X^n, and the configuration space of ordered n-tuples of distinct labeled points in X is defined as

$$\mathrm{Conf}_n(X) := X^n \setminus \bigcup_{|I|\geq 2} \Delta_I = \{(p_1,\dots,p_n)\in X^n \,|\, p_i\neq p_j, \forall\, i\neq j\}.$$

Since all diagonals Δ_I are c_n-invariant with respect to the induced anti-holomorphic involution $\mathrm{c}_n : X^n \to X^n$ given by

$$\mathrm{c}_n(p_1,\dots,p_n) = (\mathrm{c}(p_1),\dots,\mathrm{c}(p_n)),$$

$\mathcal{A}$ is a G-arrangement, and $\mathrm{Conf}_n(X)$ is a G-space.

Lemma 6.3. *For every $1\leq i\leq k$ and every $I_i\subseteq\{1,\dots,n\}$ with $|I_i|\geq 2$, the G-pair $(X^n,\Delta_{I_1\dots I_k})$ is stretched.*

Proof. The proof is a direct consequence of the Künneth formula. □

The space $\mathrm{Conf}_n(X)$ admits several wonderful G-compactifications[6].

1) An example is the Fulton-MacPherson compactification [11], denoted by $X[n]$. In this case, the building set $\mathcal{B}$ is the set of all diagonals Δ_I, for $I\subseteq\{1,\dots,n\}$, with $|I|\geq 2$.
2) Ulyanov's compactification [27], denoted by $X\langle n\rangle$, is another wonderful G-compactification of $\mathrm{Conf}_n(X)$. In this case, the building set $\mathcal{B}$ is the set of all polydiagonals.
3) The Kuperberg-Thurston compactification X^Γ, where Γ is a connected graph with n labeled vertices [25]. In this case, $\mathcal{B}$ is an appropriate set of polydiagonals.

The proof of Theorem D is now a direct consequence of Lemma 6.3 and Theorem 5.11.

7. Hilbert Squares

In this section we apply Kalinin effectivity to the study of Hilbert squares of real algebraic varieties and obtain a formula for the Smith-Thom deficiency of $X^{[2]}$.

Let (X,c) be a compact n-dimensional complex manifold equipped with a real structure. We denote by F the fixed locus of c, and let F_i, $i=1,\dots,r$ be the connected components of F. The involution $\tau : X\times X\to X\times X$ permuting the factors lifts to an involution $\mathrm{Bl}(\tau)$ on the blowup $\mathrm{Bl}_\Delta(X\times X)$ of $X\times X$ along the diagonal $\Delta\subset X\times X$. The quotient of $\mathrm{Bl}_\Delta(X\times X)$ by $\mathrm{Bl}(\tau)$ is then naturally isomorphic to the Hilbert square $X^{[2]}$. The real structure c on X naturally lifts to a real structure on $X^{[2]}$, denoted by $\mathrm{c}^{[2]}$.

By construction, the branch locus $E\subset X^{[2]}$ of the double ramified covering $\mathrm{Bl}_\Delta(X\times X)\to X^{[2]}$ is naturally isomorphic to $\mathbb{P}(T^*X)$, it is $\mathrm{c}^{[2]}$-invariant and coincides with the exceptional divisor of the canonical projection $X^{[2]}\to X^{(2)}$ to the symmetric square $X^{(2)}$ of X. Notice (*cf.* [22]) that

[6] We refer the interested reader to [25] for details.

its fixed locus $F(E) = \mathbb{P}_{\mathbb{R}}(T^*F)$ is simultaneously the boundary for two real $2n$-manifolds, $\mathbb{H}_0$ and $\mathbb{H}$, whose interiors satisfy

$$\operatorname{Int}\mathbb{H}_0 \cong (X/\mathrm{c}) \setminus F, \quad \operatorname{Int}\mathbb{H} \cong F^{(2)} \setminus \Delta F,$$

where ΔF is the diagonal embedding of F in $F^{(2)}$. Let

$$\mathrm{in}_0 : F(E) \to \mathbb{H}_0 \quad \text{and} \quad \mathrm{in}_1 : F(E) \to \mathbb{H}$$

be the inclusion maps, and $\mu = (\mathrm{in}_0, \mathrm{in}_1) : F(E) \to \mathbb{H}_0 \sqcup \mathbb{H}$.

Recall a well known statement, dual to the one proved in [22, Lemma 2.10].

Lemma 7.1. *Let M be a compact n-dimensional manifold with boundary, and $\mathrm{in} : \partial M \to M$ the inclusion. Then*

$$\dim \operatorname{Coker}(\mathrm{in}^*) = \operatorname{rank}(\mathrm{in}^*) = \frac{1}{2} \dim H^*(\partial M).$$

□

Proposition 7.2. *If X is Galois maximal and effective, then* $\operatorname{Im}(\mathrm{in}_0^*) = \operatorname{Im}(\mathrm{in}_1^*)$.

Proof. By Lemma 7.1, it is sufficient to show that $\operatorname{Im}(\mathrm{in}_1^*) \cap \operatorname{Im}(\mathrm{in}_0^*)$ contains a subspace of dimension $\frac{n}{2}\beta_*(F)$.

Let us consider a decomposition $X = U \cup V$ where U is a closed equivariant tubular neighborhood of F and V is the closure of $X \setminus U$. Here, U can be identified with a disc subbundle $D(F)$ of T^*F and $U \cap V$ with a sphere subbundle $S(F) = \partial D(F) = \partial V$. By applying the Mayer-Vietoris sequence in equivariant cohomology to this decomposition, we get a long exact sequence

$$\cdots \to H^n_G(X) \to H^n_G(U) \oplus H^n_G(V) \to H^n_G(S) \to \ldots$$

Since U is homotopy equivalent to F and G acts freely on V and S, this sequence turns into

$$\cdots \to H^n_G(X) \to \left(\bigoplus_p H^p(F) \otimes H^{n-p}(\mathbb{RP}^\infty) \right) \oplus H^n(V/\mathrm{c}) \\ \to H^n(F(E)) = H^n(S/\mathrm{c}) \to \ldots$$

According to Proposition 2.17, due to effectivity of X, for every $z \in H^p(F), p \geq 0$, there exists $\hat{z} \in H^{2p}_G(X)$ such that the first component of the mapping

$$H^*_G(X) \to (H^*(F) \otimes H^*(\mathbb{RP}^\infty)) \oplus H^*(V/\mathrm{c})$$

sends $\hat{z}$ to $u^p z + u^{p-1} \operatorname{Sq}^1 z + \cdots + \operatorname{Sq}^p z$, where u is the characteristic class of the tautological line bundle over $BG = \mathbb{RP}^\infty$.

Let now $w \in H^*(V/\mathrm{c})$ denote the second summand of the image of $\hat{z}$. Due to exactness of the Mayer-Vietoris sequence, the image of

$$(u^p z + u^{p-1} \operatorname{Sq}^1 z + \cdots + \operatorname{Sq}^p z, w)$$

in $H^*(F(E)) = H^*(S/\mathrm{c})$ is equal to 0. This gives

$$\gamma^p z + \gamma^{p-1}\operatorname{Sq}^1 z + \cdots + \gamma \operatorname{Sq}^{p-1} z + \operatorname{Sq}^p z = \operatorname{in}_0^* w$$

where $\gamma \in H^1(F(E))$ is the characteristic class of the tautological line bundle over E.

Due to Proposition 3.9, $(F \times F, \tau)$ is also a Galois maximal effective G-space. Therefore, the above arguments apply literally to it and show that for every $z \in H^p(F)$ there exists $w' \in H^*(F^{(2)} \setminus \Delta F)$, such that

$$\gamma^p z + \gamma^{p-1}\operatorname{Sq}^1 z + \cdots + \gamma \operatorname{Sq}^{p-1} z + \operatorname{Sq}^p z = \operatorname{in}_1^* w'.$$

Finally, since, for any integer $l \geq 1$, we have

$$\operatorname{in}_0^* \gamma_0^l w = \gamma^l \operatorname{in}_0^* w \quad \text{and} \quad \operatorname{in}_1^* \gamma_1^l w' = \gamma^l \operatorname{in}_1^* w',$$

where γ_0 and γ_1 are characteristic classes of the corresponding 2-sheeted coverings, we conclude that $\operatorname{Im}(\operatorname{in}_1^*) \cap \operatorname{Im}(\operatorname{in}_0^*)$ contains a subspace generated by

$$\gamma^l \left(\gamma^p z + \gamma^{p-1}\operatorname{Sq}^1 z + \cdots + \gamma \operatorname{Sq}^{p-1} z + \operatorname{Sq}^p z\right), \; 0 \leq l \leq n-1-p, z \in H^p(F).$$

Thus, there remains to notice that these generators are linear independent, and therefore the dimension of this subspace is equal to

$$\sum_{p=0}^{n-1} (n-p)\beta_p(F) = \frac{n}{2}\beta_*(F).$$

□

Let $\delta_k = \operatorname{rank} \Delta_k$, $k \geq 0$ where Δ_k are the connecting homomorphisms in the Smith exact sequence (2.3).

Proposition 7.3. *Let X be a real nonsingular n-dimensional algebraic variety with $\operatorname{Tors}_2 H_*(X, \mathbb{Z}) = 0$ and Smith-Thom deficiency $\mathfrak{D}(X) = a$. Then the Smith-Thom deficiency of $X^{[2]}$ is given by*

$$\mathfrak{D}(X^{[2]}) = 2 \operatorname{rank} \mu_* + \sum_{k=1}^{2n} (2k-1)\delta_k + a\beta_* + \frac{a(a-1)}{2} - n\beta_*(F(X)) - \beta_{\mathrm{odd}}.$$

In particular, if X is maximal then $\mathfrak{D}(X^{[2]}) = 2 \operatorname{rank} \mu_* - n\beta_* - \beta_{\mathrm{odd}}$.

Proof. This computation repeats almost literally the proof of [22, Proposition 6.1]. The changes are caused by replacing the maximality assumption by Galois maximality, and are the following. The equality $\beta_*(F(E)) = n\beta_*$ is to be replaced by $\beta_*(F(E)) = n\beta_*(F(X))$. In particular, the equality $\beta_*(\mathbb{H}) = \frac{1}{2}\sum_{i=1}^{r} \beta_*^2(F_i) + \frac{n-1}{2}\beta_*$ is replaced by $\beta_*(\mathbb{H}) = \frac{1}{2}\sum_{i=1}^{r} \beta_*^2(F_i) + \frac{n-1}{2}\beta_*(F(X))$. By contrast, the equality $\beta_*(\mathbb{H}_0) = \frac{n}{2}\beta_*$, proved in [22, Lemma 3.1] as an application of the Smith exact sequence, is to be replaced by $\beta_*(\mathbb{H}_0) = n\beta_* - \dfrac{n}{2}\beta_*(F(X)) - \sum_{k=1}^{2n}(2k-1)\delta_k$. $\square$

Lemma 7.4. *If* $\mathrm{Im}(\mathrm{in}_0^*) = \mathrm{Im}(\mathrm{in}_1^*)$ *then* $\operatorname{rank}\mu_* = \frac{n}{2}\beta_*(F(X))$

Proof. Since $\beta_*(E) = n\beta_*(F(X))$, the relation stated follows from Lemma 7.1 applied to any of the halves U, V. $\square$

Proof of Theorem E. Due to Proposition 7.2 we have $\mathrm{Im}(\mathrm{in}_0^*) = \mathrm{Im}(\mathrm{in}_1^*)$. Therefore, $\operatorname{rank}\mu_* = \dim \mathrm{Im}(\mathrm{in}_0^*) = \dim \mathrm{Im}(\mathrm{in}_1^*)$ so that by Lemma 7.4 we get $\operatorname{rank}\mu_* = \frac{n}{2}\beta_*(F(X))$. This, together with Proposition 7.3, completes the proof. $\square$

Proof of Corollary F. Since X is a conjugation space, it is maximal and effective. In particular, to compute the Smith-Thom deficiency of $X^{[2]}$ one can apply Theorem E and we notice that for conjugation spaces we have $\beta_{\mathrm{odd}} = 0$, $\delta_k = 0$ and $\mathfrak{D}(X) = 0$. Therefore $\mathfrak{D}(X^{[2]}) = 0$. $\square$

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