ON THE CLASSIFICATION OF ALE KÄHLER MANIFOLDS

HANS-JOACHIM HEIN, RAREŞ RĂSDEACONU, AND IOANA ŞUVAINA

Abstract. The underlying complex structure of ALE Kähler manifolds is exhibited as a resolution of a deformation of an isolated quotient singularity. As a consequence, there exist only finitely many diffeomorphism types of minimal ALE Kähler surfaces.

1. Introduction

The asymptotically locally Euclidean scalar-flat Kähler (ALE sfK) surfaces play a central rôle in the study of the Gromov-Hausdorff limit of constant scalar curvature Kähler surfaces. However, their classification is missing and, as Chen-LeBrun-Weber pointed out [6, page 1164], that "seems a daunting problem". Simply connected ALE Ricci-flat Kähler surfaces are constructed by Gibbons-Hawking [10], Hitchin [20], and Kronheimer [30], while the only non-simply connected ones are quotients of some of Hitchin's [20] (see also [44]). Kronheimer [31] and Şuvaina [44] show that these constructions yield the complete list of ALE Ricci-flat Kähler surfaces. More precisely, they classify the diffeomorphism type of the underlying smooth manifold. Moreover, given a complex structure, there exists a unique ALE Ricci-flat Kähler metric in a fixed cohomology class. In higher dimensions ALE Ricci-flat Kähler metrics were constructed by Calabi [4], Joyce [24] and Tian-Yau [46]. There are currently many known non-Ricci-flat ALE sfK metrics. In complex dimension two, such metrics were constructed by LeBrun [32, 33], Joyce [23], Calderbank-Singer [5], and recently by Honda [21, 22], Lock-Viaclovsky [36] and Han-Viaclovsky [15], while in higher dimensions, non-Ricci-flat ALE sfK metrics were found by Simanca [43] and recently by Apostolov-Rollin [1]. A common feature of all of the known examples of ALE sfK manifolds is that the underlying complex structure is a resolution of a deformation of a quotient singularity. We attack here the classification of ALE sfK manifolds by placing it in the broader context of ALE Kähler manifolds and show that this common feature is in fact a general phenomenon.

Theorem A. *Every ALE Kähler manifold asymptotic to $\mathbb{C}^n/G$ is isomorphic to a resolution of a deformation of the isolated quotient singularity $(\mathbb{C}^n/G, 0)$.*

Remark 1.1. *In his book, Joyce* [25, Sect. 8.9] *claims that such a result would follow from the results previously discussed in Sect. 8 of op.cit. However, a proof is not given.*

In dimension at least three, the deformation theory of isolated quotient singularities is very restrictive. In fact Schlessinger proved [42] that such singularities are

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rigid. In other words, if M is an ALE Kähler manifold of dimension at least three, then its complex structure near infinity is standard and M must be a resolution of the quotient singularity $\mathbb{C}^n/G$. Thus, as a consequence of Theorem A, we have:

Corollary B. *If $n \geq 3$, every ALE Kähler manifold asymptotic to $\mathbb{C}^n/G$ is biholomorphic to a resolution of the isolated singularity $\mathbb{C}^n/G$.*

Remark 1.2. *This recovers and refines the rigidity results of Hein-LeBrun in* [19, Section 2] *via a different method. In this direction, see also the very closely related results of Hamilton* [14, page 1–2] *and Tian* [45, Section 4].

The proof of Theorem A is based on the construction of an analytic divisorial compactification of ALE Kähler manifolds due to Hein and LeBrun [19] and Li [35]. The key property of the compactification is that the divisor at infinity is isomorphic to $\mathbb{P}^{n-1}/G$, where G is the group acting on the end of the given ALE manifold, and that the ring of sections of its normal bundle is in fact isomorphic to the ring of invariants of the isolated singularity $\mathbb{C}^n/G$. This is used to show that the starting ALE Kähler manifold is in fact a resolution of an affine algebraic variety. To identify this affine algebraic variety we appeal to the "sweeping out the cone with hyperplane sections" technique of Pinkham [40].

The deformation theory of isolated quotient singularities in complex dimension two is well-understood and it allows us to obtain the following consequence of Theorem A.

Theorem C. *For every finite subgroup $G \subset U(2)$, there exist only finitely many diffeomorphism types underlying minimal ALE Kähler surfaces which are asymptotic to $\mathbb{C}^2/G$.*

In contrast to dimension two, in higher dimensions it is a difficult task to impose a minimality condition on a resolution. For this reason we do not attempt here to explore a generalization of Theorem C in higher dimensions.

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2. Proof of Theorem A

2.1. **ALE Kähler manifolds.** There exist in the literature several definitions of an ALE Kähler manifold, each of them imposing various fall-off conditions of the metric at infinity. We are working in the following setting:

Definition 2.1. *A complete Kähler manifold (M, g) of complex dimension n is said to be asymptotically locally Euclidean (ALE) Kähler if there is a compact subset $K \subseteq M$ such that $M \setminus K$ has finitely many connected components, and for each component there exists a finite subgroup $G \subseteq U(n)$ acting freely on $\mathbb{C}^n \setminus B_R(0)$, and a diffeomorphism*

$$\pi : M \setminus K \to (\mathbb{C}^n \setminus B_R(0))/G,$$

where $B_R(0) = \{x \in \mathbb{C}^n : |x| \leq R\}$ is the ball of radius R for some fixed $R \geq 0$, centered at the origin, and, in these coordinates

$$g_{ij} - \delta_{ij} \in C^{2,\alpha}_{-\tau} \text{ for some } \tau > 0.$$

Here, the weighted Hölder spaces $C^{k,\alpha}_{-\tau}$ consist of $C^{k,\alpha}$ functions such that

$$\left(\sum_{j=0}^{k} |x|^j |\nabla^j f(x)|\right) + |x|^{k+\alpha} |\nabla^k f|_{C^{0,\alpha}(B_{|x|/10}(x))} = O(|x|^{-\tau}).$$

This definition can naturally be extended to tensor fields, and the resulting $C^{k,\alpha}_{-\tau}$ spaces then become Banach spaces when equipped with the obvious weighted analogs of the usual Hölder norms.

The connected components of $M \setminus K$ are called ends. Hein and LeBrun prove in [19, Proposition 2.5 and Proposition 4.2] that an ALE Kähler manifold has in fact only one end. (This is done under stronger fall-off assumptions at infinity than we have imposed here, but the alternative proofs mentioned in [19, Remark, page 195] actually only require the weaker fall-off $g_{ij} - \delta_{ij} \in C^1_{-\tau}$, $\tau > 0$). For the rest of the presentation, we will tacitly assume that M has only one end.

Our definition for ALE Kähler manifolds is inspired by the work on compactifications of asymptotically conical manifolds of Li [35] (see also [18] and [8]). The stronger fall-off conditions at infinity in [19], where τ is required to be larger than $n-1$ are necessary for the definition of the ADM mass, but are not needed in the construction of the compactification.

2.2. **Divisorial compactification of ALE Kähler manifolds.** Let (M, g) be an ALE Kähler manifold of dimension $n \geq 2$. Let M_∞ be the end of M, and $\widetilde{M}_\infty$ the universal cover of M_∞. For $\tau > n-1$, Hein and LeBrun show [19, Lemma 2.1 and Lemma 4.1] that there exists a complex manifold $\mathscr{X}$ of dimension n containing an embedded complex hypersurface $\Sigma \simeq \mathbb{CP}^{n-1}$, with normal bundle of degree $+1$, and such that $\widetilde{M}_\infty$ is biholomorphic to $\mathscr{X} \setminus \Sigma$. For arbitrary $\tau > 0$, this follows using the method of Haskins-Hein-Nordström [18], which is also used in [19] to treat the case $n = 2$ and $1 < \tau \leq 3/2$. The method of Li [35] yields the same result assuming $C^{2n}_{-\tau}$ instead of $C^{2,\alpha}_{-\tau}$ fall-off of the metric, again for an arbitrary $\tau > 0$.

Since $H^1(\mathbb{CP}^{n-1}, \mathscr{O}_{\mathbb{CP}^{n-1}}(1)) = 0$, standard deformation theory arguments [26] imply that there is a complete analytic family of compact complex submanifolds of dimension $h^0(\mathbb{CP}^{n-1}, \mathscr{O}_{\mathbb{CP}^{n-1}}(1)) = n$, which represents all small deformations of $\Sigma \subset \mathscr{X}$ through compact complex submanifolds. Since $\mathbb{CP}^{n-1}$ is rigid, and since $h^{0,1}(\mathbb{CP}^{n-1}) = 0$, we may assume, by shrinking the size of the family if necessary, that every submanifold in the family is biholomorphic to $\mathbb{CP}^{n-1}$, and has normal bundle $\mathscr{O}_{\mathbb{CP}^{n-1}}(1)$. The union of these submanifolds fill out some open neighborhood $\mathscr{U}$ of Σ in $\mathscr{X}$. If $n \geq 3$, such a neighborhood is in fact biholomorphic to a neighborhood of $\mathbb{CP}^{n-1}$ of the hyperplane $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ [19, Lemma 2.2]. In dimension two, only a weaker result holds. Namely, the first infinitesimal neighborhood of $\Sigma \subset \mathscr{X}$ is isomorphic to the first infinitesimal neighborhood of $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Hein and LeBrun prove a stronger result [19, Lemma 4.4], but this isomorphism between the first infinitesimal neighborhoods suffices for our purpose in all dimensions.

The action of G extends continuously to $\mathscr{X}$ and since it is holomorphic on the complement of Σ, the induced action is actually holomorphic, and it sends Σ to itself. Moreover, we may assume that the open set $\mathscr{U} \subset \mathscr{X}$ is G-invariant. Since the automorphism group of the first infinitesimal neighborhood of Σ in $\mathscr{X}$ is isomorphic to the automorphism group of the first infinitesimal neighborhood of $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$, which is $GL(n;\mathbb{C})$, the action of the finite group G on $T\mathscr{X}|_\Sigma$ is realized by that of a finite subgroup of the maximal compact subgroup $U(n) \subset GL(n;\mathbb{C})$. Let $G_0 \subseteq G$

be the normal subgroup corresponding to the center $U(1) \subset U(n)$. The quotient $\mathscr{U}' = \mathscr{U}/G_0$ is then a smooth manifold, and $\mathscr{U} \to \mathscr{U}'$ is then a branched cyclic cover ramified along Σ. The induced action of G/G_0 on $\mathscr{U}'$ then has only cyclic isotropy groups with fixed locus in codimension 2, and let $\check{\mathscr{U}} = \mathscr{U}'/(G/G_0)$. Let $D \subset \check{\mathscr{U}}$ be the image of $\Sigma \subseteq \mathscr{U}$. We can now compactify M as a complex space X, by gluing $\check{\mathscr{U}}$ to M, and we have $X \setminus M = D = \mathbb{CP}^{n-1}/(G/G_0)$. The space X has at most quotient singularities in codimension 2, and its singular locus is contained in D.

In general, the sheaf $\mathscr{O}_X(D)$ is not locally free, but merely an orbi-line bundle [2], and the same is true for $N_{D|X} \simeq \mathscr{O}_D(D)$. However, this is not a serious obstacle, as will be seen later. For the moment, notice that since $N_{\Sigma|\mathscr{X}} \simeq N_{\mathbb{CP}^{n-1}|\mathbb{CP}^n} \simeq \mathscr{O}_{\mathbb{CP}^{n-1}}(1)$, we see that the orbi-line bundle $N_{D|X} \simeq \mathscr{O}_D(D)$ is ample, i.e. it admits a hermitian metric of positive curvature. When $n \geq 3$, this implies that X is a Kähler orbifold [19, Lemma 1.4]. If $n = 2$, since $D^2 > 0$ it follows [3, Chapter IV, Theorem 6.2] that X is projective.

The action of G on $T\mathscr{X}|_\Sigma$ induces an action on the rings of sections of normal bundles $R(\Sigma, \mathscr{O}_\Sigma(\Sigma)) \simeq R(\mathbb{CP}^{n-1}, \mathscr{O}_{\mathbb{CP}^{n-1}}(1))$, where

$$R(\Sigma, \mathscr{O}_\Sigma(\Sigma)) = \bigoplus_{k\geq 0} H^0(\Sigma, N_{\Sigma|\mathscr{X}}^{\otimes k})$$

and

$$R(\mathbb{CP}^{n-1}, \mathscr{O}_{\mathbb{CP}^{n-1}}(1)) = \bigoplus_{k\geq 0} H^0(\mathbb{CP}^{n-1}, \mathscr{O}_{\mathbb{CP}^{n-1}}(k)).$$

Therefore, there exists an isomorphism of invariant subrings

$$R(\Sigma, \mathscr{O}_\Sigma(\Sigma))^G \simeq R(\mathbb{CP}^{n-1}, \mathscr{O}_{\mathbb{CP}^{n-1}}(1))^G.$$

We define the section ring of $\mathscr{O}_D(D)$ as

$$R(D, \mathscr{O}_D(D)) := \bigoplus_{k\geq 0} H^0(D, \mathscr{O}_D(kD)).$$

Then $R(D, \mathscr{O}_D(D)) \simeq R(\Sigma, \mathscr{O}_\Sigma(\Sigma))^G$, and since $R(\mathbb{CP}^{n-1}, \mathscr{O}_{\mathbb{CP}^{n-1}}(1))$ is the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ in n variables, we have an isomorphism

$$R(D, \mathscr{O}_D(D)) \simeq \mathbb{C}[z_1, \ldots, z_n]^G, \tag{2.1}$$

the latter being the coordinate ring of the isolated quotient singularity $\mathbb{C}^n/G$.

The following theorem summarizes the discussion above:

Theorem 2.2. *Let M be an ALE Kähler manifold of complex dimension n, asymptotic at infinity to $\mathbb{C}^n/G$, where $G \subset U(n)$. Then M has only one end and there exists an n-dimensional complex Kähler orbifold X and a connected suborbifold divisor $D \subset X$ with the following properties:*

i) *$D \simeq \mathbb{P}^{n-1}/(G/G_0)$ as orbifolds.*
ii) *The normal orbi-bundle $N_{D|X} \simeq \mathscr{O}_D(D)$ is ample and its ring of sections $R(D, \mathscr{O}_D(D))$ is isomorphic to the coordinate ring of the isolated quotient singularity $\mathbb{C}^n/G$.*
iii) *$M = X \setminus D$ as complex manifolds.*

2.3. **Algebro-geometric properties of the compactification.** The compactification constructed above exhibits several features which will play a key role in the proof of Theorem A. Closely related results were proved by Conlon-Hein [7, Section 2.3], [8, Section 2] using different arguments.

For the convenience of the reader, we start by recalling some terminology.

Definition 2.3. *Let S be a complex variety and E a Weil divisor.*

i) *We say that E is $\mathbb{Q}$-Cartier if there exists an integer $m > 0$ such that mE is Cartier.*
ii) *We say that S is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier.*

Definition 2.4. *We say that a normal variety S has rational singularities if $R^i f_* \mathscr{O}_Y = 0$ for every resolution $f : S' \to S$ of S, and for every $i > 0$.*

Lemma 2.5 ([28], Proposition 5.15)**.** *Let S be a complex variety with quotient singularities only. Then S is $\mathbb{Q}$-factorial, and its singularities are rational.*

Definition 2.6. *Let S be a complex variety.*

i) *We say that a Cartier divisor L on S is big if there exist a positive constant $c = c(L)$ and a positive integer m_0 such that*
$$h^0(S, L^{\otimes m}) \geq cm^{dim S},$$
for every $m \geq m_0$. We say that a $\mathbb{Q}$-Cartier divisor E on S is big if there exists an positive integer k such that kE is a big Cartier divisor.
ii) *We say that a Cartier divisor L is pseudo-ample if there exists a positive integer m_0 such that $L^{\otimes m}$ is globally generated for $m \geq m_0$. We say that a $\mathbb{Q}$-Cartier divisor E on S is pseudo-ample if there exists an positive integer k such that kE is a pseudo-ample Cartier divisor.*

Definition 2.7. *A complex variety S is called Moishezon if the transcendence degree of its field of meromorphic function is $dim S$.*

The following criterion is a standard results in the smooth setting. For convenience of the reader, we include a proof in the orbifold setting.

Lemma 2.8. *An orbifold S is Moishezon if and only if it admits a big $\mathbb{Q}$-Cartier divisor.*

Proof. Let $\pi : T \to S$ be a resolution of singularities. Since the transcendence is a bimeromorphism invariant, we see that S is Moishezon if and only if T is Moishezon. Moreover, from the definition, we see that S admits a big $\mathbb{Q}$-Cartier divisor if and only if it admits a big Cartier divisor.

Let L_S be a big Cartier divisor on S. Since T is normal and π has connected fibers, by the projection formula
$$h^0(T, \pi^* L_S^m) = h^0(S, \pi_* \pi^* L_S^m) = h^0(S, L_S^m) \geq cm^{\dim S},$$
for some $c > 0$ and m sufficiently large. Therefore $\pi^* L_S$ is a big Cartier divisor on T. Hence T is Moishezon [37, Theorem 2.2.15], and so S is Moishezon as well.

Conversely, since S is Moishezon then T is Moishezon as well. As above, by [37, Theorem 2.2.15] there exists a big line bundle L_T on T. Replacing L_T by a large tensor power, we may assume $h^0(T, L_T) > 0$ and so there exists an effective divisor H such that $L_T = \mathscr{O}_T(H)$. Let $F = \pi(H)$, and discard all components of codimension strictly larger than one. Since L_T is big, the divisor H is not π-exceptional and so F is a non-trivial Weil divisor on S. Moreover, since S has only quotient singularities, there exists $k > 0$ such that kF is Cartier [28, Proposition 5.15]. Since π is birational, for every positive integer m there exists an effective π-exceptional divisor E_m such that $\pi^*\mathscr{O}_S(mkF) = \mathscr{O}_T(mkH + E_m)$. This yields an injection of sheaves $L_T^{mk} \to \pi^*\mathscr{O}_S(mkF)$. By passing to global sections,

$$h^0(S, \mathscr{O}_S(mkF)) = h^0(T, \pi^*\mathscr{O}_S(mkF)) > h^0(T, L_T^m) > c' m^{\dim S},$$

for some constant $c' > 0$ and m sufficiently large, and so $\mathscr{O}_S(F)$ is a big $\mathbb{Q}$-Cartier divisor on S. □

The following lemma is well-known, and it holds in larger generality. As the authors could not find a reference in the literature, we include a short proof for the reader's convenience, in the case of our compactification (X, D).

Lemma 2.9. *For every $m \geq 0$, the following sequence is exact:*

$$0 \to \mathscr{O}_X((m-1)D) \to \mathscr{O}_X(mD) \to \mathscr{O}_D(mD) \to 0. \tag{2.2}$$

Proof. In general, we have the exact sequence of the Weil divisor D:

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0, \tag{2.3}$$

which yields (2.2) in the case $m = 0$. If D is Cartier, then $\mathscr{O}_X(mD)$ is locally free (for example, if X is smooth, i.e., when $G = \{1\}$). Since tensoring by a line bundle preserves exactness, from (2.3) we would immediately see that (2.2) is true. However, D is merely $\mathbb{Q}$-Cartier, in which case such argument fails. We provide here an argument for the general case.

It is enough to show that (2.2) holds locally near D. Let $\pi : \mathscr{U} \to \mathscr{U}/G$ be the orbifold chart around D, and $\Sigma \subset \mathscr{U}$ the hyperplane at infinity. Since $\mathscr{U}$ is smooth, for every $m \geq 0$ we have an exact sequence:

$$0 \to \mathscr{O}_{\mathscr{U}}((m-1)\Sigma) \to \mathscr{O}_{\mathscr{U}}(m\Sigma) \to \mathscr{O}_\Sigma(m\Sigma) \to 0.$$

Pushing forward onto $\mathscr{U}/G$ we get

$$0 \to \pi_*\mathscr{O}_{\mathscr{U}}((m-1)\Sigma) \to \pi_*\mathscr{O}_{\mathscr{U}}(mH) \to \pi_*\mathscr{O}_\Sigma(m\Sigma) \to R^1\pi_*\mathscr{O}_{\mathscr{U}}((m-1)\Sigma).$$

Moreover, since π is finite, $R^1\pi_*\mathscr{O}_{\mathscr{U}}((m-1)\Sigma) = 0$. Restriction to G-invariant subsheaves induces the exact sequence

$$0 \to (\pi_*\mathscr{O}_{\mathscr{U}}((m-1)\Sigma))^G \to (\pi_*\mathscr{O}_{\mathscr{U}}(m\Sigma)))^G \to (\pi_*\mathscr{O}_\Sigma(m\Sigma))^G \to 0. \tag{2.4}$$

Since Σ is G-invariant, and $D = \pi(\Sigma)$, then $(\pi_*\mathscr{O}_{\mathscr{U}})^G \simeq \mathscr{O}_{\mathscr{U}/G}$ and $(\pi_*\mathscr{O}_\Sigma)^G \simeq \mathscr{O}_D$. We immediately see that $(\pi_*\mathscr{O}_{\mathscr{U}}(k\Sigma))^G \simeq \mathscr{O}_{\mathscr{U}/G}(kD)$, and $(\pi_*\mathscr{O}_\Sigma(k\Sigma))^G \simeq \mathscr{O}_D(kD)$ for every $k \geq 0$ [38, Proposition 2, Chapter II, Sect. 7]. Therefore (2.4) yields (2.2). □

Lemma 2.10. *Let (X, D) be the compactification constructed in Theorem 2.2. Then X is Moishezon.*

Proof. Since $\Sigma \simeq \mathbb{P}^{n-1}$, we have $H^i(\Sigma, \mathscr{O}_\Sigma(m\Sigma)) = H^i(\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(m)) = 0$ for any $i > 0$ and $m \geq 0$. As the map $\pi : \Sigma \to D$ is finite, by Leray's spectral sequence we also have $H^i(D, \mathscr{O}_D(mD)) = 0$ for every $i > 0$ and $m \geq 0$.

Passing to cohomology, from the short exact sequence of sheaves

$$0 \to \mathscr{O}_X((m-1)D) \to \mathscr{O}_X(mD) \to \mathscr{O}_D(mD) \to 0$$

we see that $h^i(X, \mathscr{O}_X(mD)) = h^i(X, \mathscr{O}_X((m-1)D))$ for $m > 0$ and $i > 1$, while $h^1(X, \mathscr{O}_X(mD)) \leq h^1(X, \mathscr{O}_X((m-1)D))$ for $m > 0$. In particular, the functions

$$m \mapsto h^i(X, \mathscr{O}_X(mD)),\ i \geq 1$$

are bounded. Since $D^n > 0$, by Riemann-Roch, it follows that

$$h^0(X, \mathscr{O}_X(mD)) \sim cm^n,$$

for some positive constant c. That means the $\mathbb{Q}$-Cartier divisor $\mathscr{O}_X(D)$ is big, and so, by Lemma 2.8, X is Moishezon. □

Proposition 2.11. *Let (X, D) be the compactification of an n-dimensional ALE Kähler manifold constructed in Theorem 2.2. We have*

i) $H^i(X, \mathscr{O}_X) = 0,\ i > 0.$
ii) $H^i(X, \mathscr{O}_X(mD)) = 0$, $m > 0$, $i > 0$.
iii) *The $\mathbb{Q}$-Cartier divisor $\mathscr{O}_X(D)$ is pseudo-ample.*
iv) *X is projective.*

Proof. i) The complex variety X is a Moishezon orbifold by Lemma 2.10. Therefore, as in [8, Proof of Theorem 1.2], the orbifold de Rham cohomology satisfies the Hodge decomposition and $H^{p,q}(X) \simeq \overline{H^{q,p}(X)}$. To prove the claim it is therefore sufficient to show that $H^0(X, \Omega^i_X) = 0$, for every $i \geq 1$, where Ω^i_X is the sheaf of holomorphic orbifold i-forms.

Let $\pi : \mathscr{U} \to \mathscr{U}/G$ be the standard orbifold chart near D. Since the hyperplane at infinity $\Sigma \simeq \mathbb{CP}^{n-1} \subset \mathscr{U}$ has normal bundle $\mathscr{O}_{\mathbb{CP}^{n-1}}(1)$, it follows that any line $\ell \in \Sigma$ has normal bundle $\mathscr{O}_{\mathbb{CP}^1}(1)^{\oplus(n-1)}$. Therefore, standard deformation theory shows that by deforming ℓ, one can sweep out a neighborhood of Σ in $\mathscr{U}$. That means, by eventually shrinking $\mathscr{U}$, for every point $p \in \mathscr{U}$ there exists a curve $C \simeq \mathbb{CP}^1$ with normal bundle $\mathscr{O}_{\mathbb{CP}^1}(1)^{\oplus(n-1)}$ passing through p.

Let $s \in H^0(X, \Omega^i_X)$. The restriction of s to $\mathscr{U}/G$ induces a holomorphic section $\pi^* s \in H^i(\mathscr{U}, \Omega^i_{\mathscr{U}})$. However, the restriction of any holomorphic i-form on U to a $C \simeq \mathbb{CP}^1$ with normal bundle $\mathscr{O}_{\mathbb{CP}^1}(1)^{\oplus(n-1)}$ is a section of exterior product

$$\bigwedge^i \left(\mathscr{O}_{\mathbb{CP}^1}(-2) \oplus \mathscr{O}_{\mathbb{CP}^1}(-1)^{\oplus(n-1)}\right),$$

and so it vanishes identically. Since such curves sweep out the entire $\mathscr{U}$, it follows that $\pi^* s$ vanishes identically on $\mathscr{U}$ which implies that $\pi^* s = 0$ by analytic continuation. This implies $s = 0$.

ii) As in the proof of Lemma 2.10 we can infer that

$$h^i(X, \mathscr{O}_X(mD)) = h^i(X, \mathscr{O}_X((m-1)D)),\ \text{for every } m, i > 0$$

and

$$h^1(X, \mathscr{O}_X(mD)) \leq h^1(X, \mathscr{O}_X((m-1)D))$$

for every $m \geq 1$. By induction, and using that $H^i(X, \mathscr{O}_X) = 0,\ i > 0$, it follows now that $H^i(X, \mathscr{O}_X(mD)) = 0$, for all $m \geq 0$, and $i > 0$.

iii) From Proposition 2.11 ii) above, we see that for every $m > 0$ there is a short exact sequence of vector spaces

$$0 \to H^0(X, \mathscr{O}_X((m-1)D)) \to H^0(X, \mathscr{O}_X(mD)) \to H^0(D, \mathscr{O}_D(mD)) \to 0. \quad (2.5)$$

Let k be an positive integer such that $\mathscr{O}_X(kD)$ is Cartier. Since in this case $\mathscr{O}_D(kD)$ is also Cartier and $\mathscr{O}_D(D)$ is ample, $\mathscr{O}_D(mkD)$ is globally generated for m large enough. Using a defining section of D and the fact that the restriction map on global sections is surjective (2.5), it follows that for all points $x \in X$ there is an $s \in H^0(X, \mathscr{O}_X(mkD))$ such that $s(x) \neq 0$. Thus, $\mathscr{O}_X(mkD)$ is globally generated for $m \gg 0$.

iv) If $n = 2$, this is [19, Proposition 4.2]. If $n \geq 3$, we proved in Lemma 2.10 that X is Moishezon, while by [19, Lemma 2.4] we know it is Kähler. Since X has at most quotient singularities, Namikawa's projectivity criterion [39, Corollary 1.7] implies that X is in fact projective. □

Remark 2.12. *Proposition 2.11 iv) allows us to work in the Zariski topology hereafter.*

We study next the structure of X via the linear system $|mkD|$ for $m \gg 0$, where k is a fixed positive integer such that kD is Cartier, e.g., $k = |G|$. In the proof of Proposition 2.11 we noticed that there exists a sufficiently large integer m such that $\mathscr{O}_D(mkD)$ is very ample and $\mathscr{O}_X(mkD)$ is globally generated. For such m, let $\phi : X \to \mathbb{P}^N$ be the morphism defined by the complete linear system $|mkD|$. Moreover, since D is big as observed in the proof of Lemma 2.10, by eventually increasing m, the map ϕ is birational onto its image [28, Lemma 2.60]. To simplify the notation, let $X' = \phi(X)$ and $D' = \phi(D)$.

Proposition 2.13. *The complex variety X' is normal and the map ϕ is an isomorphism in a neighborhood of D. In particular, the ϕ-exceptional set in X does not intersect D.*

Proof. We show first that X' is normal. This follows immediately from the Stein factorization theorem [16, III, Corollary 11.5]. Since ϕ is a birational morphism, by the uniqueness of the Stein factorization, we have $\phi_*\mathscr{O}_X = \mathscr{O}_{X'}$. Let u be a rational function on (some open affine subvariety of) X' which is integral over $\mathscr{O}_{X'}$. Then $u \circ \phi$ is integral over $\mathscr{O}_X$, and so it lies in $\mathscr{O}_X$, as X is smooth. In other words, u lies in $\phi_*\mathscr{O}_X = \mathscr{O}_{X'}$, which means X is normal.

The proof that the map ϕ is an isomorphism in a neighborhood of D follows as in [17, Theorem 4.2]. □

2.4. **Rings of sections.** The structure of X' is encoded in the Proj construction [16, Chapter II, Section 2] applied to the ring of sections of D. The basic results on graded rings and rings of sections needed here are discussed in Appendix A.

Definition 2.14. *Let Y be a projective variety, and E be a $\mathbb{Q}$-Cartier Weil divisor on Y. The ring of sections of E in Y is*

$$R(Y, E) = \bigoplus_{m \geq 0} H^0(E, \mathscr{O}_Y(mE)).$$

The multiplication of sections makes a section ring a $\mathbb{Z}_{\geq 0}$-graded ring where the homogeneous elements of degree i are the sections $s \in H^0(Y, \mathscr{O}(iE))$.

Lemma 2.15. *The ring of sections $R(X', D')$ is finitely generated and*

$$X' \simeq \operatorname{Proj} R(X', D').$$

Proof. By construction, the $\mathbb{Q}$-Cartier divisor D' is a hyperplane section of X'. The result now follows from Proposition A.7. □

Let $0 \neq S \in H^0(X', \mathscr{O}_{X'}(D'))$ be a degree 1 homogeneous element of $R(X', D')$ spanning the image of $H^0(X', \mathscr{O}_{X'})$ in $H^0(X', \mathscr{O}_{X'}(D'))$. More precisely, S is given by multiplication by a section vanishing along D'.

Lemma 2.16. *There exists an isomorphism of* $\mathbb{C}$*-algebras*

$$R(X', D') \simeq R(D', D'|_{D'})[S].$$

Proof. By Proposition 2.11 ii) and the projection formula, for every $m \geq 1$ we have an exact sequence of $\mathbb{C}$-vector spaces

$$0 \to H^0(X', \mathscr{O}_{X'}((m-1)D')) \to H^0(X', \mathscr{O}_{X'}(mD')) \to H^0(D', \mathscr{O}_{D'}(mD')) \to 0, \tag{2.6}$$

which shows that the restriction of graded $\mathbb{C}$-vector spaces

$$r : R(X', D') \to R(D', D'|_{D'})$$

is surjective. The map $\phi : X \to X'$ induces an isomorphism between the rings of sections $R(D', D'|_{D'}) \simeq R(D, D|_D)$. Let now $s_1, \dots, s_p$ be algebra generators of $R(D', D'|_{D'})$ induced from $R(D, D|_D)$. We can assume that these generators are homogeneous. There exists homogeneous elements $S_1, \dots, S_p \in R(X', D')$ such that $r(S_i) = s_i$, $i = 1, \dots, p$. Since $H^0(X', \mathscr{O}_{X'}(mD'))$ are finite dimensional vector spaces, by induction, from (2.6) we infer that $\{S, S_1, \dots, S_p\}$ are algebra generators of $R(X', D')$. Hence $R(X', D') \simeq R(D', D'|_{D'})[S]$ as $\mathbb{C}$-algebras, as S is a generator in degree 1. □

By Proposition A.7, there exist canonical isomorphisms

$$f : D' \to \operatorname{Proj} R(D', D'|_{D'}) \text{ and } g : X' \to \operatorname{Proj} R(X', D').$$

Lemma 2.17. *The following diagram is commutative*

$$\begin{tikzcd} D' \arrow[r, "f"] \arrow[d, hook] & \operatorname{Proj} R(D', D'|_{D'}) \arrow[d, hook] \\ X' \arrow[r, "g"] & \operatorname{Proj} R(X', D') \end{tikzcd} \tag{2.7}$$

Proof. Since $\mathscr{O}_{X'}(D')$ and $\mathscr{O}_{D'}(D')$ are ample Cartier divisors on X' and D' respectively, Proposition A.7 provides the existence of certain isomorphisms $f : X' \to \operatorname{Proj} R(X', D')$ and $g : D' \to \operatorname{Proj} R(D', D'|_{D'})$.

Let $s_1, \dots, s_p$ be homogeneous algebra generators of $R(D', D'|_{D'})$ of positive degrees $d_1, \dots, d_p > 0$, respectively. As in the proof of Lemma 2.16 we see that there exists homogeneous elements $S_1, \dots, S_p \in R(X', D')$ such that $r(S_i) = s_i$, $i = 1, \dots, p$. In particular $\deg S_i = \deg s_i = d_i$, $i = 1, \dots, p$. Let d be the least common multiple of $d_1, \dots, d_p$. Then, by the proof of Proposition A.4, it follows that the graded $\mathbb{C}$-algebra $R(D', D'|_{D'})^{(d)}$ is generated by sections in $H^0(D', \mathscr{O}_{D'}(dD'))$. Since $\deg S = 1$, the graded $\mathbb{C}$-algebra $R(X', D')^{(d)}$ is generated by sections in $H^0(X', \mathscr{O}_{X'}(dD'))$. From (2.6), this implies the restriction map yields a surjection of graded $\mathbb{C}$-algebras $R(X', D')^{(d)} \to R(D', D'|_{D'})^{(d)}$. By [13, Proposition (**2.9.2**)], such a surjection is equivalent the existence of an embedding

$$\operatorname{Proj} R(D', D'|_{D'})^{(d)} \hookrightarrow \operatorname{Proj} R(X', D')^{(d)}.$$

Furthermore, by Proposition A.6, this is equivalent to an embedding

$$\operatorname{Proj} R(D', D'|_{D'}) \hookrightarrow \operatorname{Proj} R(X', D').$$

By construction, this embedding is compatible with the embedding of D' in X' via the maps f and g defined above. □

2.5. **Deformation to the normal cone.** In this section we identify the structure of the affine variety $X' \setminus D'$ as a deformation of the quotient singularity $\mathbb{C}^n/G$, thus proving Theorem A. The specialization to the quotient singularity is accomplished by appealing to the classical "sweeping out the cone with hyperplane sections" construction of Pinkham [40, Sect. §7] (see also [29, Sect. 3.1]). This strategy was already used in [8, Sect. 3], although in the setting of [8] the required surjectivity of the restriction maps came from the fact that D' was anticanonical in X'.

Definition 2.18. *Let Y be a complex variety. A deformation of Y is a flat holomorphic map $\pi : V \to T$ of complex varieties, together with an isomorphism $\pi^{-1}(0) \simeq Y$, for some point $0 \in T$. We say that Y is deformed into the nearby fiber $\pi^{-1}(t)$, for $t \in T$.*

Definition 2.19. *Let Y be a projective variety equipped with an ample $\mathbb{Q}$-line bundle L. The affine cone over Y with conormal bundle L is* $\operatorname{Spec} R(Y, L)$, *while the projective cone is* $\operatorname{Proj} R(Y, L)[Z]$, *were the graded ring $R(Y, L)[Z]$ is defined as*

$$R(Y, L)[Z] = \bigoplus_{m \geq 0} \left(\bigoplus_{r=0}^{m} H^0(Y, L^{\otimes r}) \cdot Z^{m-r} \right).$$

The variable Z is considered of degree 1 in the graded ring $R(Y, L)[Z]$.

Proof of Theorem A. By Lemma 2.16 we have $R(X', D') \simeq R(D', D|_{D'})[S]$, where S is a degree 1 homogeneous element of $R(X', D')$. Hence, the projective cone of $\mathscr{O}_{X'}(D')$ over X' is $\operatorname{Proj} R(D', D'|_{D'})[S, Z]$,

Consider the deformation $\pi : \mathscr{Y} \to \mathbb{C}$, where

$$\mathscr{Y} := (S - tZ = 0) \subset \operatorname{Proj} R(X', D')[Z] \times \mathbb{C},\ t \in \mathbb{C}.$$

If $t \neq 0$, then we can use $Z = t^{-1}S$ to eliminate Z and find that the fiber $\mathscr{Y}_t$ is $\operatorname{Proj} R(D', D'|_{D'})[t^{-1}S] \simeq \operatorname{Proj} R(D', D'|_{D'})[S]$, i.e., $\mathscr{Y}_t = X'$. If $t = 0$, then the extra equation becomes $(S = 0)$, thus we get the fiber $\mathscr{Y}_0 = \operatorname{Proj} R(X', D')[Z] \cap (S = 0)$. However, since the restriction maps

$$r_m : H^0(X', \mathscr{O}_{X'}(mD')) \to H^0(D', \mathscr{O}_{D'}(mD')) \tag{2.8}$$

are surjective for every $m \geq 0$, as it immediately follows from Proposition 2.11, $\operatorname{Proj} R(D', D'|_{D'})[Z]$ is the subscheme of $\operatorname{Proj} R(X', D')[Z]$ cut out by the equation $S = 0$. In particular, it follows that the fiber $\mathscr{Y}_0$ is isomorphic to the cone $\operatorname{Proj} R(D', D'|_{D'})[Z]$.

Let $\mathscr{D} \subset \mathscr{Y}$ be the divisor at infinity $(Z = 0)$, and $\pi_{\mathscr{D}} : \mathscr{D} \to \mathbb{C}$ denote the restriction map. As above, we see that the fiber $\mathscr{D}_t$ of $\pi_{\mathscr{D}}$ is $\operatorname{Proj} R(D', D'|_{D'})$, for any $t \in \mathbb{C}$.

Let now $\phi : \mathscr{Y} \setminus \mathscr{D} \to \mathbb{C}$ denote the restriction of π to $\mathscr{Y} \setminus \mathscr{D}$. The map ϕ is flat and, by [13, Proposition (**8.3.2**)], the central fiber $\phi^{-1}(0)$ is the affine cone $\operatorname{Spec} R(D', D'|_{D'})$. Since $R(D', D'|_{D'}) \simeq R(D, D|_D)$, by Theorem 2.2 ii) we obtain $\phi^{-1}(0) = \mathbb{C}^n/G$. Furthermore, for $t \neq 0$ we have

$$\phi^{-1}(t) = \operatorname{Proj} R(X', D') \setminus \operatorname{Proj} R(D', D'|_{D'}) = X' \setminus D'.$$

Since, by Proposition 2.13, M is a resolution of $X' \setminus D'$, the proof of Theorem A is now completed. $\square$

3. Proof of Corollary B and Theorem C

Corollary B and Theorem C now follow from Theorem A by appealing to some known facts from the deformation theory of isolated quotient singularities.

Proof of Corollary B. This is a consequence of Theorem A and a result of Schlessinger saying that every isolated quotient singularity $\mathbb{C}^n/G$, where $n \geq 3$, is rigid [42]. See [8, Footnote 1] for some additional details on this type of argument, in particular, the use of $\mathbb{C}^*$-equivariance in order to conclude that the affine variety $X' \setminus D'$ of Section 2 is actually globally isomorphic to $\mathbb{C}^n/G$. $\square$

An interesting immediate consequence of our construction shows that in dimension two, the singularities of $X' \setminus D'$ are rather restricted.

Corollary 3.1. *If G is a finite subgroup of $U(2)$, then $X' \setminus D'$ has at most quotient singularities. Moreover, if G is a finite cyclic subgroup of $U(2)$, then $X' \setminus D'$ has at most cyclic quotient singularities.*

Proof. In the case of an arbitrary finite subgroup of $U(2)$, the result is the positive answer given by Esnault and Viehweg [9] to a conjecture of Riemenschneider. The case of a finite cyclic subgroup is Corollary 7.15 in [27]. $\square$

Before we proceed to the proof of Theorem C, we recall the following definition:

Definition 3.2. *Let (X, x) be a germ of a complex analytic space. A deformation $\mathscr{V} \to \mathscr{S}$ of X is called versal if for every other deformation $\mathscr{W} \to \mathscr{T}$ there exists a map $\psi : \mathscr{T} \to \mathscr{S}$, such that $\psi^*(\mathscr{V}) \simeq \mathscr{W}$. A versal deformation $\mathscr{V} \to \mathscr{S}$ is called miniversal if the induced map between the Zariski tangent spaces of $\mathscr{T}$ and $\mathscr{S}$ is uniquely determined by the isomorphism class of $\mathscr{W}$.*

Let (X, x) be an isolated singularity. In [12] Grauert constructs a miniversal deformation $\mathscr{V} \to \mathscr{S}$ of the germ (X, x) (see also [41]). In particular, a miniversal deformation space exists for isolated finite quotient singularities $\mathbb{C}^n/G$, where G is a finite subgroup of $U(n)$. Notice that given a germ of analytic space, if a versal deformation $\mathscr{V} \to \mathscr{S}$ exists, then the versal base space $\mathscr{S}$ is in general reducible, with components of varying dimensions, possibly with embedded components.

Proof of Theorem C. Let M be a minimal ALE Kähler surface asymptotic to $\mathbb{C}^2/G$ at infinity, were G is a finite subgroup of $U(2)$, and let $\phi : \mathscr{Y} \setminus \mathscr{D} \to \mathbb{C}$ be the deformation constructed in the proof of Theorem A, where the central fiber is $\phi^{-1}(0) = \mathbb{C}^2/G$, while $\phi^{-1}(t) = X' \setminus D'$ for $t \neq 0$. In particular, for $t \neq 0$, the fibers $\phi^{-1}(t)$ have the same singularities and the same diffeomorphism type.

Let $\Phi : \mathscr{V} \to \mathscr{S}$ be the miniversal deformation of the quotient singularity $\mathbb{C}^2/G$. Classical results due to Whitney and Thom (see [11, Chapter 1, Sect. §1.7]) applied to the map Φ show that the analytic variety $\mathscr{V}$ admits a complex analytic conelike stratification with respect to the map $\Phi : \mathscr{V} \to \mathscr{S}$. That means the set of points $p \in \mathscr{V}$ for which Φ restricted to the stratum through p has a critical point at p is closed and Φ is topologically locally trivial relative to the stratification. As it follows from [27, Theorem 3.9], the miniversal deformation space of a two-dimensional quotient singularity has finitely many components, each of them being quasi-projective, and so the stratification above has finitely many strata. Hence the fibers of Φ can have only finitely many diffeomorphism types, and by miniversality there exist only finitely many diffeomorphism types for the variety $X' \setminus D'$. The minimality condition on M implies that M is minimal resolution of $X' \setminus D'$. However, the minimal resolution of a quotient singularity is unique. Therefore, there are only finitely many deformation types for M. $\square$

Appendix A. Graded Rings

In this section we collect some results on graded rings which are used in this paper. These are known results to the experts in the field, see for example [13], but we include them for convenience.

Definition A.1. *A graded ring R is a ring together with a direct sum decomposition*

$$R = \bigoplus_{n \geq 0} R_n,$$

where each R_n is an abelian subgroup of R, such that $1 \in R_0$ and $R_m R_n \subseteq R_{m+n}$ for all $m, n \in \mathbb{Z}_{\geq 0}$. If $d \in \mathbb{Z}_{>0}$, we set $R^{(d)} = \bigoplus_{n \geq 0} R_{dn}$.

If R is a graded ring, then R_0 is also a ring, each R_n is an R_0-module, and R is an R_0-algebra. We assume for convenience that R_0 is Noetherian.

Example A.2. *If S is a irreducible, reduced algebraic variety, and L a line bundle on S we can form the section ring*

$$R(S, L) = \bigoplus_{n \geq 0} H^0(S, L^{\otimes m}),$$

in which case $R_0 = H^0(S, \mathscr{O}_S) \simeq \mathbb{C}$. Notice that if $0 \neq s \in H^0(S, L^{\otimes m})$ and $0 \neq t \in H^0(S, L^{\otimes n})$, then $s \otimes t \in H^0(S, L^{\otimes (m+n)})$ is non-zero. Hence the section ring R is a domain.

Lemma A.3. *If R is a finitely generated R_0-algebra, then $R^{(d)}$ has the same property. Furthermore, the converse holds if R is a domain.*

Proof. Suppose that R is a finitely generated R_0-algebra, with generators $r_1, \ldots, r_p$, which are assumed to be nonzero and homogeneous with positive degrees $d_1, \ldots, d_p$. Let ℓ be the least common multiple of d and d_j. Consider the set of elements $s_i = r_i^{\ell/d_i} \in R_d$. The R_0-algebra $R' = R_0[s_1 \ldots, s_p]$ is finitely generated over R_0, hence it is Noetherian. We have the ring extensions $R' \hookrightarrow R^{(d)} \hookrightarrow R$ and since R is finite over R_0, it follows that also $R^{(d)}$ is finite over R_0. Since R_0 is a finitely generated R_0-algebra, we deduce that $R^{(d)}$ has the same property.

Conversely, suppose R is a domain and that $R^{(d)}$ is finitely generated over R_0, hence it is Noetherian. Let $m \in \mathbb{Z}_{\geq 0}$ and set $M^m = \bigoplus_{n \geq 0} R_{m+nd}$. Notice that M^m is an $R^{(d)}$-submodule of R. Moreover, it is finitely generated. This is trivial if $M^m = 0$. Otherwise, let $0 \neq h \in M^m$. Since R is a domain, multiplication by h^{d-1} induces an injective $R^{(d)}$-linear map $M^m \hookrightarrow R^{(d)}$. Since $R^{(d)}$ is Noetherian, we conclude that M^m is a finitely generated $R^{(d)}$-module, as claimed.

Notice now that $R = \sum_{m=0}^{d-1} M^m$. This implies that R is a finitely generated $R^{(d)}$-module. Since $R^{(d)}$ is a finitely generated R_0-algebra, we conclude that R has the same property. $\square$

We would like R to be generated by R_1. This is needed in Proposition A.7 below which implies Lemma 2.15 (see also [13, Sect. §2]). The graded ring R being genereated by degree one homogeneous elements is equivalent to having R a graded quotient of a polynomial algebra over R_0 (with the usual grading), i.e., $\operatorname{Proj} R$ is contained as a closed subset of some projective space over R_0. The following standard procedure shows how to overcome this difficulty and assure the generation by homogeneous elements of degree one.

From now on, we suppose $R = \bigoplus_{n\geq 0} R_n$ is a graded ring, finitely generated as R_0-algebra, where R_0 is Noetherian. Let $r_1, \dots, r_p$ be algebra generators of R over R_0. We can assume that these are homogeneous with positive degrees $d_1, \dots, d_p > 0$.

Proposition A.4. *Suppose R is a graded ring and a finitely generated R_0-algebra. There exists a positive integer d such that $R_d R_n = R_{n+d}$ for $n \geq d$. In particular, $R^{(d)}$ is generated over R_0 by R_d.*

Proof. Let d be the least common multiple of the d_j. Consider the set of elements $s_i = r_i^{d/d_i} \in R_d$, and let $s \in R_n$, where $n \geq d$. We have that s is a sum of products of powers of the r_j's, where each term is of degree n. Since in each term, at least one of the r_j must occur to power at least d/d_j, we can write each term in the sum as some s_j times something in R_{n-d}, wherefrom $R_n = R_d R_{n-d}$. In particular, $R_{nd} = (R_d)^n$ for every $n \geq 1$, which implies the last statement. □

Remark A.5. *By declaring the elements in R_d as being homogeneous of degree* 1, *the R_0-algebra $R^{(d)} = \bigoplus_{n\geq 0} R_{nd}$ constructed in the above proposition is generated as an R_0-algebra by elements of degree* 1.

Proposition A.6 ([13], Proposition (**2.4.7**))**.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a graded ring. For every integer d, there exists a canonical isomorphism* $\operatorname{Proj} R \simeq \operatorname{Proj} R^{(d)}$.

Proof. See *loc.cit.* □

Proposition A.7. *Let S be a normal, connected projective variety, and H be an ample $\mathbb{Q}$-Cartier divisor on S. Then the sections ring $R(S,H)$ is finitely generated and there exists a canonical isomorphism*

$$S \simeq \operatorname{Proj} R(S,H).$$

Proof. Let d be a large enough positive integer such that the line bundle $H^{\otimes d}$ is very ample and gives a projectively normal embedding. For the existence of such an integer d, see [16, Chapter II, Exercise 5.14]. Let $\iota : S \to \mathbb{P}(H^0(S, H^{\otimes d})^*) = \mathbb{P}^N$ be the embedding of S given by a basis of sections of $H^{\otimes d}$, and let $S' = \iota(S)$. Since $H^{\otimes d} \simeq \iota^* \mathscr{O}_{\mathbb{P}^N}(1)$, we have an isomorphism of $\mathbb{C}$-algebras $R(S, H^{\otimes d}) \simeq R(S', \mathscr{O}_{S'}(1))$. Since S' is projectively normal in $\mathbb{P}^N$, using again [16, Chapter II, Exercise 5.14], we see that the section ring $R(S', \mathscr{O}_{S'}(1))$ is a quotient of the homogeneous coordinate ring of $\mathbb{P}^N$ and we have an isomorphism $S' \simeq \operatorname{Proj} R(S', \mathscr{O}_{S'}(1)) \simeq \operatorname{Proj} R(S, H^{\otimes d})$. In particular, $R(S', \mathscr{O}_{S'}(1))$ is finitely generated and so is $R(S, H^{\otimes d})$. Hence, by Lemma A.3, we can conclude that $R(S,H)$ is finitely generated. Furthermore, by Proposition A.6, we have $S \simeq \operatorname{Proj} R(S,H)$. □

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Department of Mathematics, Fordham University, Bronx, NY 10458, USA
E-mail address: hhein@fordham.edu

Department of Mathematics, 1326 Stevenson Center, Vanderbilt University, Nashville, TN, 37240, USA
E-mail address: rares.rasdeaconu@vanderbilt.edu

Department of Mathematics, 1326 Stevenson Center, Vanderbilt University, Nashville, TN, 37240, USA
E-mail address: ioana.suvaina@vanderbilt.edu