Math 3100 Homework Assignment Zero

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Exercise 1. Prove that there exist positive irrational numbers $a, b \in \mathbb{R}_+ \setminus \mathbb{Q}$ such that a^b is rational.

To prove this we first recall the following classical result, which we will prove below.

Theorem 1. *There is no rational number such that its square is* 2*.*

By the previous theorem we have that $\sqrt{2}$ is irrational. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we obtain a solution to the exercise. Otherwise, if $\sqrt{2}^{\sqrt{2}}$ is irrational, then as $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$ is rational, we obtain a solution to the exercise in this case also.

Proof of Theorem 1. Suppose the contrary, then we could find positive integers $a, b \in \mathbb{N} \setminus \{0\}$ such that

$$\left(\frac{a}{b}\right)^2 = 2. \tag{1}$$

By writing the fraction $\frac{a}{b}$ in reduced form we may further assume that a and b have no common factor other than 1. Multiplying Equation (1) by b^2 gives

$$a^2 = 2b^2 \tag{2}$$

and hence a^2 is an even number. Consequently we have that a is an even number and so we may write $a = 2c$ for some positive integer $c \in \mathbb{N} \setminus \{0\}$. Substituting this into Equation (2) gives

$$4c^2 = a^2 = 2b^2$$

and hence $2c^2 = b^2$, which means that b^2 is an even number. From this we deduce that b is also an even number. Since a and b are both even we then obtain a contradiction to the fact that a and b have no common factor other than 1. $\square$

Exercise 2. Give an uncountable family of infinite subsets of $\mathbb{N}$ such that any two of them are disjoint, or else prove that no such family exists.

No such family exists. We will prove this by contraposition. Suppose that $\{E_a\}_{a\in A}$ is a family of infinite subsets of $\mathbb{N}$, such that $E_a \cap E_b = \emptyset$ for all $a \neq b$. We will show that A must be a countable set.

Since the set $E_a \subset \mathbb{N}$ is infinite for each $a \in A$, it is nonempty. Thus, we may define a function $f : A \to \mathbb{N}$ by letting $f(a)$ denote the smallest element in the set E_a. Note that if $a, b \in A$ are such that $f(a) = f(b)$, then we have $f(a) \in E_a \cap E_b$, so that $E_a \cap E_b \neq \emptyset$. Since the family $\{E_a\}_{a\in A}$ consists of pairwise disjoint sets we must then have $a = b$. Thus, $f : A \to \mathbb{N}$ is injective and hence A is countable, since a set injects into a countable set if and only if that set itself is countable.

Exercise 3. Give an uncountable family of infinite subsets of $\mathbb{N}$ such that any two of them intersect in a finite set, or else prove that no such family exists.

Let X denote the space of infinite sequences with entries in $\{1, 2\}$. We have established in class that X is uncountable. For each infinite sequence $x \in X$ we let $E_x \subset \mathbb{N}$ denote the collection of all natural numbers obtained by concatenating the initial finite sequences of x. For example, if x is the sequence

$$1, 2, 2, 1, 1, 1, 2, 1, \ldots$$

then we have

$$E_x = \{1, 12, 122, 1221, 12211, 122111, 1221112, 12211121, \ldots\}.$$

Given $x, y \in X$ with $x \neq y$ we then have $|E_x \cap E_y| = n$ where $n < \infty$ is the first entry where the sequences x and y disagree. Thus, the family $\{E_x\}_{x\in X}$ is a family of infinite subsets of $\mathbb{N}$ such that any two distinct members of this family have finite intersection.

Miscellany

Definition 2. This is how to define a definition.

Equations get numbered in sequence with statements, as for example

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{3}$$

The numbering is done automatically with LaTeX. Labeling the equation (or theorem, etc.) can allow us to recall Equation (3) later, without having to refer to the specific numbering in our code. Although you may have to compile the tex file twice to get the numbers to match.

You can also typeset math directly in a paragraph by placing it within dollar signs. This is called "math mode". For example: $e^{\pi i} + 1 = 0$. This is useful, but remember that it is harder to read inline math than displayed math. For longer equations, or for equations that should be highlighted you can use:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Remember that letters get put in a different font in math mode, so whenever you are referencing a mathematical object you should always put it in dollar signs. For example, e and i are famous mathematical constants, but e and i are just two letters. For a string of inequalities you can use "align" or "align*". For example, if $u, v \in \mathbb{R}^n$ with $\|v\|^2 = v \cdot v \neq 0$, then setting $\lambda = \frac{u\cdot v}{\|v\|^2}$ we have

$$\begin{aligned} 0 &\leq \|u - \lambda v\|^2 \\ &= u \cdot u - \lambda u \cdot v - \lambda v \cdot u + \lambda^2 v \cdot v \\ &= \|u\|^2 - \frac{|u \cdot v|^2}{\|v\|^2} - \frac{|u \cdot v|^2}{\|v\|^2} + \frac{|u \cdot v|^2}{\|v\|^2} \\ &= \|u\|^2 - \frac{|u \cdot v|^2}{\|v\|^2}. \end{aligned}$$

This inequality, in turn, gives the Cauchy-Schwarz inequality:

$$|u \cdot v| \leq \|u\|\|v\|.$$

In writing mathematics, it is useful to have many symbols to represent various objects. These are often encoded into LaTeX, and it is not difficult to find many lists of symbols for LaTeXon the internet. These include greek letters $(\alpha, \beta, \Gamma, \Delta, \varepsilon)$, operators $(\otimes, +, \sum, \prod)$ and much more $(\leq, \geq, \diamond, \sim, \cong)$.

Arrays and diagrams

Matrices can be a bit tricky. Here's an example:

$$\det \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = ad - bc \tag{4}$$

Here's how you can define a function by cases:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Writing mathematics

Using LaTeXwill help to write mathematics that looks nice, but it will not help with the much more difficult task of writing nice mathematics. The books [1] and [2] are devoted to help students (and mathematicians) write elegant mathematics in ways that can effectively convey concepts and ideas to the reader.

The even more difficult task of producing nice mathematics will come through hard work, patience, and many hours of practice. Contrary to popular belief, mathematical intuition, needed to effectively attack difficult problems (and to know what problems are worth attacking), is not innate but is rather learned over many years of study.

References

[1] Paul R. Halmos How to write mathematics. https://sites.math.washington.edu/~lind/Resources/Halmos.pdf

[2] Donald E. Knuth, Tracy Larrabee and Paul M. Roberts Mathematical Writing. https://jmlr.csail.mit.edu/reviewing-papers/knuth_mathematical_writing.pdf