Math 3100 Final Exam - Due Monday May 10th by 6:00pm central time

May 7, 2021

1 Instructions

Give a complete solution to each of the problems below. You are welcome to type your solutions in LaTeX and then submit the tex file, or you can write your solutions out on paper and submit a scanned pdf copy of work. In either case, you should submit your solutions by placing it in our shared folder in Vanderbilt's Box. Also, in either case you should write complete solutions, giving a professional presentation, as we've come to expect from the homework.

For the problems you are allowed to use without proof any result that we have proved in class or any theorem from the textbook. You are welcome to use other resources as well, but you should justify with a proof any results. If you significantly use an external resource then you should cite your source. You must justify any claims you make even if it is not specifically requested by the problem.

The solutions should be your own and you should not use any resource that involves active participation from another person. You should avoid discussing the exam with other people in any way, even a comment like "number 2 was tricky" or "number 3 wasn't too bad" conveys a significant amount of information and it would be improper to make or hear such comments.

In order to ensure prompt grading, **late submissions will not be accepted for the final examination**. Any questions regarding the exam should be asked directly to the instructor via email.

2 Problems

Problem 1 (12 points)**.** Fix $n \in \mathbb{N}$ and let S be a set. Suppose $\{A_s\}_{s\in S}$ is a family of distinct subsets of $\mathbb{Q}$ such that for all $s, t \in S$ with $s \neq t$ the set $A_s \cap A_t$ is finite and contains at most n elements. Prove that S is countable.

Problem 2 (13 points)**.** Fix a set $A \subset \mathbb{R}$ and let $\overline{A}$ denote the set of limit points of A, i.e., $x \in \overline{A}$ if and only if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in A$ such that $\{a_n\}_{n=1}^{\infty}$ converges to x.

1. Prove that $\overline{A}$ is closed, i.e., it contains all of its accumulation points. Also, show that if F is any closed set containing A, then F must also contain $\overline{A}$.
2. Show that for each $n \geq 1$ the set
$$G_n = \{x \in \mathbb{R} \mid \text{there exists } y \in A \text{ with } |x-y| < 1/n\}$$
is open.
3. Prove that $\cap_{n=1}^{\infty} G_n = \overline{A}$.

Problem 3 (12 points). Let $E \subset \mathbb{R}$ be a bounded set, and let $f : E \to \mathbb{R}$ be continuous. Prove that f is uniformly continuous if and only if f has a limit at each accumulation point of E. Give an example showing that this equivalence no longer holds if E is not assumed to be bounded.

Problem 4 (12 points). Let $\{t_n\}_{n=1}^{\infty}$ be any sequence of real numbers. Prove that $[0,1]$ is not a subset of $\bigcup_{t=1}^{\infty}(t_n - 2^{-n-1}, t_n + 2^{-n-1})$.

Problem 5 (13 points). Fix $x_0 \in (a,b)$. Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and is differentiable at all points in $[a,b] \setminus \{x_0\}$. Suppose also that f' has a limit at x_0. Show that f is differentiable at x_0 and that f' is continuous at x_0.

Problem 6 (12 points). Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that there exists a point $y \in [0,1]$ such that $\frac{1}{3} f(y) = \int_0^1 f(x) x^2 \, dx$.

Problem 7 (13 points). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $\mathbb{R}$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is such that $\{f_n\}_{n=1}^{\infty}$ converges to f point-wise, and $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly on $\mathbb{Q}$. Either prove that f must be continuous, or else give a counter-example showing that this need not be the case.

Problem 8 (13 points). Define the function $E(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Recall from class that we know this power series converges for all $x \in \mathbb{R}$ and E defines a differentiable function that satisfies $E'(x) = E(x)$.

1. Prove that for all $x, y \in \mathbb{R}$ we have $E(x+y) = E(x)E(y)$.
2. Prove that for all $q \in \mathbb{Q}$ we have $E(qx) = (E(x))^q$.
3. Prove that E is strictly increasing and has range $(0, \infty)$.
4. If $L : (0,\infty) \to \mathbb{R}$ is the inverse of E. Show that L is continuous, and that for all $a > 0$ and $x \in \mathbb{Q}$ we have $E(xL(a)) = a^x$.