Math 3100 Exam 3 - Due Friday April 23rd by 6:00pm central time

April 21, 2021

1 Instructions

Give a complete solution to each of the problems below. You are welcome to type your solutions in LaTeX and then submit the tex file, or you can write your solutions out on paper and submit a scanned pdf copy of work. In either case, you should submit your solutions by placing it in our shared folder in Vanderbilt's Box. Also, in either case you should write complete solutions, giving a professional presentation, as we've come to expect from the homework.

For the problems you are allowed to use without proof any result that we have proved in class or any theorem from the textbook that appears in Chapters 0-6. You are welcome to use other resources as well, but you should justify with a proof any results. If you significantly use an external resource then you should cite your source. You must justify any claims you make even if it is not specifically requested by the problem.

The solutions should be your own and you should not use any resource that involves active participation from another person. You should avoid discussing the exam with other people in any way, even a comment like “number 2 was tricky” or “number 3 wasn't too bad” conveys a significant amount of information and it would be improper to make or hear such comments.

Any questions regarding the exam should be asked directly to the instructor via email.

2 Problems

Problem 1 (20 points)**.** Let $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in $[a,b]$ and let $f : [a,b] \to \mathbb{R}$ be a bounded function that is continuous except at points in the sequence $\{x_n\}_{n=1}^{\infty}$. Prove that f is Riemann integrable.

Problem 2 (20 points)**.** Let $f : [a,b] \to \mathbb{R}$ be a bounded Riemann integrable function. Given $s < t$ define a function $g_{s,t} : [a, b] \to \mathbb{R}$ by

$$g_{s,t}(x) = \begin{cases} f(x) & \text{if } f(x) \in [s, t) \\ 0 & \text{if } f(x) \notin [s, t). \end{cases}$$

a) Prove that for all $s < t$ the function $g_{s,t}$ is Riemann integrable.

b) Prove that for all $\varepsilon > 0$ there exists a Riemann integrable function $g : [a, b] \to \mathbb{R}$ such that the range of g is a finite set and such that $|f(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$.

Problem 3 (20 points)**.** Let $f : [a, b] \to [0, \infty)$ be a continuous function. Prove that the sequence $\left\{ \left(\int_a^b (f(x))^n \, dx \right)^{1/n} \right\}_{n=1}^{\infty}$ converges and describe its limit.

Problem 4 (20 points)**.** For each $x \in \mathbb{R}$ consider the series $\sum_{n=1}^{\infty} \frac{x}{n^3(1+nx^2)}$. Show that this is absolutely convergent for each $x \in \mathbb{R}$, and that the resulting function $f(x) = \sum_{n=1}^{\infty} \frac{x}{n^3(1+nx^2)}$ is continuous.

Problem 5 (20 points)**.** For each of the series below, determine all $x \in \mathbb{R}$ for which the sires converges. Remember that you must justify your answers by stating the appropriate theorems that you use.

1. $\sum_{n=1}^{\infty} \frac{x^{n^2}}{2^n}$.
2. $\sum_{n=1}^{\infty} \frac{1}{nx^n}$.
3. $\sum_{n=1}^{\infty} (1 + 2\sqrt{x})x^{2n}$.
4. $\sum_{n=1}^{\infty} (2^n x^{2n} - 3^n x^{2n+1})$.
5. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{1+x^{2n}}$.
