Math 3100 Exam 2 - Solutions

March 30, 2021

1 Instructions

Give a complete solution to each of the problems below. You are welcome to type your solutions in LaTeX and then submit the tex file, or you can write your solutions out on paper and submit a scanned pdf copy of work. In either case, you should submit your solutions by placing it in our shared folder in Vanderbilt's Box. Also, in either case you should write complete solutions, giving a professional presentation, as we've come to expect from the homework.

For the problems you are allowed to use without proof any result that we have proved in class or any theorem from the book that appears in or before Section 5.4. You are welcome to use other resources as well, but you should justify with a proof any results. If you significantly use an external resource then you should cite your source. You must justify any claims you make even if it is not specifically requested by the problem.

The solutions should be your own and you should not use any resource that involves active participation from another person. You should avoid discussing the exam with other people in any way, even a comment like "number 2 was tricky" or "number 3 wasn't too bad" conveys a significant amount of information and it would be improper to make or hear such comments.

Any questions regarding the exam should be asked directly to the instructor via email.

2 Problems

Problem 1 (20 points)**.** A subset $A \subset \mathbb{R}$ is said to be complete if every Cauchy sequence in A converges to a point that is contained in A. Show that a subset of $\mathbb{R}$ is complete if and only if it is closed.

First suppose that every Cauchy sequence in A converges to a point in A. Let $x \in \mathbb{R}$ be any accumulation point of A. We may then construct a sequence $\{x_n\}_{n=1}^{\infty}$ by choosing, for each $n \geq 1$, some point $x_n \in A \cap (x - 1/n, x + 1/n)$. If $N \geq 1$ and $m, n \geq 2N$ then we have $x_n, x_m \in (x - 1/2N, x + 1/2N)$ and hence $|x_n - x_m| < 1/N$, hence $\{x_n\}_{n=1}^{\infty}$ is Cauchy and so, by hypothesis, converges to a point in A. But for $N \geq 1$ and $n \geq N$ we also have $|x_n - x| < 1/N$ and hence $\{x_n\}_{n=1}^{\infty}$ converges to x. Thus $x \in A$ and since A contains all of its accumulation points it it closed.

Conversely, suppose that A is closed, and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in A. Since the reals are complete we know that $\{x_n\}_{n=1}^{\infty}$ converges to some point $x \in \mathbb{R}$. If $x_n = x$ for some n then we have $x = x_n \in A$. Otherwise, if $x \neq x_n$ for all $n \geq 1$, then for each $\varepsilon > 0$ there exists $N \geq 1$ such that for $n \geq N$ we have $x_n \in (x - \varepsilon, x + \varepsilon) \setminus \{x\}$. This shows that $A \cap ((x - \varepsilon, x + \varepsilon) \setminus \{x\})$ is non-empty for all $\varepsilon > 0$ and hence x is an accumulation point of A. Since A is closed it follows that $x \in A$, and hence $\{x_n\}_{n=1}^{\infty}$ converges to a point in A.

Problem 2 (20 points)**.** Let $D = \mathbb{Q} \cap [0, 1]$ and suppose $f : D \to \mathbb{R}$. Show that the following conditions are equivalent:

1. f is uniformly continuous.
2. f is continuous and for each point $t \in [0, 1]$ the function f has a limit at t.
3. There exists a continuous function $g : [0, 1] \to \mathbb{R}$ such that $g(x) = f(x)$ for each $x \in D$.

We will first show $2 \implies 3$. Suppose that f is continuous and has a limit $g(t)$ at each point $t \in [0, 1]$. Note that since f is continuous we have that $g(t) = f(t)$ for each $t \in D$ and so g is an extension of f. It therefore remains to show that g is continuous.

Fix some point $t \in [0, 1]$ and choose a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0, 1]$ such that $\{t_n\}_{n=1}^{\infty}$ converges to to t. Since each $g(t_n)$ is defined as the limit of f at t_n we may then choose for each $n \geq 1$ a point $x_n \in D$ such that $|x_n - t_n| < 1/n$ and $|f(x_n) - g(t_n)| < 1/n$. Since $\{x_n - t_n\}_{n=1}^{\infty}$ converges to 0 we have that $\{x_n\}_{n=1}^{\infty}$ also converges to t. Moreover, since $\{x_n\}_{n=1}^{\infty}$ is a sequence in D, then we have that $\{f(x_n)\}_{n=1}^{\infty}$ converges to $g(t)$. Since $\{f(x_n) - g(t_n)\}_{n=1}^{\infty}$ converges to 0 it then follows that $\{g(t_n)\}_{n=1}^{\infty}$ also converges to $g(t)$. Since $\{t_n\}_{n=1}^{\infty}$ was an arbitrary sequence in $[0, 1]$ that converges to t it then follows that g is continuous at t, and since $t \in [0, 1]$ was arbitrary it follows that g is continuous on $[0, 1]$.

To see $3 \implies 1$ just note that since g is continuous and $[0, 1]$ is compact it follows that g is uniformly continuous, and hence so is f, since it is the restriction of g to D.

It is then left to show $1 \implies 2$. Suppose that f is uniformly continuous, $t \in [0, 1]$, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in D that converges to t. If $\varepsilon > 0$, then as f is uniformly continuous there exists $\delta > 0$ such that for $x, y \in D$, with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy there exists $N \geq 1$ such that for $n, m \geq N$ we have $|x_n - x_m| < \delta$ and hence $|f(x_n) - f(x_m)| < \varepsilon$. Thus, it follows that $\{f(x_n)\}_{n=1}^{\infty}$ is also Cauchy, and hence converges. Since

$\{x_n\}_{n=1}^{\infty}$ was an arbitrary sequence converging to t it then follows that f has a limit at t (this is Theorem 2.1 in the book).

Problem 3 (20 points)**.** Suppose $f : [0, \infty) \to [0, \infty)$ is differentiable and f' is uniformly continuous on $[0, \infty)$. Show that there exist constants $A, B, C \geq 0$ such that for all $x \in [0, \infty)$ we have $f(x) \leq Ax^2 + Bx + c$.

Since f' is uniformly continuous there exists $\delta > 0$ such that for all $x, y \in [0, \infty)$ with $|x - y| < 2\delta$ we have $|f'(x) - f'(y)| < 1$. Hence if $x \in [0, \infty)$, then by the triangle inequality, for $n \geq 1$ we have

$$\begin{aligned} |f'(x) - f'(0)| &\leq |f'(x) - f'(n\delta)| + \sum_{k=1}^{n} |f(k\delta) - f((k-1)\delta)| \\ &\leq |f'(x) - f'(n\delta)| + n. \end{aligned}$$

If we take n so that $(n+1)\delta \geq x > n\delta$ then we have $n < x/\delta$ and $|f'(x)-f'(0)| < 1 + x/\delta$. Hence for all $x \in [0, \infty)$ we have

$$|f'(x)| \leq |(1 + f'(0))| + \frac{1}{\delta}x = Ax + B,$$

where $A = \frac{1}{\delta} > 0$ and $B = |1 + f'(0)|$.

We now claim that for all $x \in \mathbb{R}$ we have $f(x) \leq f(0) + Bx + Ax^2$. Indeed, if this were not the case then for some $x \in (0, \infty)$ we would have $f(x) - f(0) > Bx + Ax^2$ and by the Mean Value Theorem there would exist some $c \in (0, x)$ such that

$$B + Ax \geq B + Ac \geq |f'(c)| \geq f'(c) = \frac{f(x) - f(0)}{x - 0} > B + Ax,$$

giving a contradiction.

Problem 4 (20 points)**.** Suppose $f : [a, b] \to \mathbb{R}$ is twice differentiable on $[a, b]$ and suppose that the equation $f(x) = x$ has at least three distinct solutions in $[a, b]$. Prove that there exists $c \in [a, b]$ such that $f''(c) = 0$.

Suppose we have $a \leq x_1 < x_2 < x_3 \leq b$ with $f(x_i) = x_i$. By the Mean Value Theorem there then exists $y_1 \in (x_1, x_2)$ and $y_2 \in (x_2, x_3)$ with

$$f'(y_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 1 = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = f'(y_2).$$

Since f' is also differentiable on $[a, b]$ and since $y_1 < y_2$, another application of the Mean Value Theorem then gives a point $c \in (y_1, y_2)$ such that

$$f''(c) = \frac{f'(y_2) - f'(y_1)}{y_2 - y_1} = 0.$$

Problem 5 (20 points)**.** Let $f : [0,1] \to [0,\infty)$ be a bounded function that is Riemann integrable. Prove that for each $n \geq 1$ the function $[0,1] \ni x \mapsto f(x)x^n$ is Riemann integrable and the sequence $\left\{\int_0^1 f(x)x^n\,dx\right\}_{n=1}^{\infty}$ converges to 0.

Hint: First try showing that the function $[0,1] \ni x \mapsto f(x)x$ is Riemann integrable.

Suppose $M > 0$ is such that $f(x) \leq M$ for all $x \in [0,1]$. Fix $\varepsilon > 0$. Since f is integrable there exists a partition P of $[0,1]$ such that $U(P,f) - L(P,f) < \varepsilon/2$. If we take $m \geq 1$ such that $1/m < \varepsilon/2M(b-a)$, then setting $P' = P \cup \{\frac{k}{m} \mid 0 \leq k \leq m\}$ we have that the mesh of P' is less than $\varepsilon/2M(b-a)$ and since P' is a refinement of P we have $U(P',f) - L(P',f) < \varepsilon/2$. We write P' as $P' = \{x_0, x_1, \ldots, x_n\}$.

Note that on each interval $[x_{i-1}, x_i]$ we have

$$\begin{aligned}
|M_i(f(x)x) - m_i(f(x)x)| &\leq x_i M_i(x) - x_{i-1} m_i(f) \\
&\leq (x_i - x_{i-1}) M_i(f) + x_{i-1}(M_i(f) - m_i(f)) \\
&\leq M\varepsilon/2M(b-a) + (M_i(f) - m_i(f)),
\end{aligned}$$

and hence

$$\begin{aligned}
U(P, f(x)x) - L(P, f(x)x) &= \sum_{i=1}^{n} (M_i(f(x)x) - m_i(f(x)x))(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} (x_i - x_{i-1}) M\varepsilon/2M(b-a) + (M_i(f) - m_i(f))(x_i - x_{i-1}) \\
&= \varepsilon/2 + (U(P',f) - L(P',f)) < \varepsilon.
\end{aligned}$$

Thus $f(x)x$ is integrable and by induction it then follows that $f(x)x^n$ is integrable for all $n \geq 1$.

Now fix $\varepsilon > 0$ and set $x_1 = 1 - \varepsilon/2M$. Since $x_1 < 1$ the sequence $\{x_1^n\}_{n=1}^{\infty}$ converges to 0. Hence there is some $N > 0$ such that for $n \geq N$ we have $x_1^{n+1} < \varepsilon/2M$. If $n \geq N$ and we consider the partition $P = \{0, x_1, 1\}$, then we have

$$\begin{aligned}
U(P, f(x)x^n) &= \left(\sup_{x \in [0,x_1]} f(x)x^n\right)(x_1 - 0) + \left(\sup_{x \in [x_1, x]} f(x)x^n\right)(1 - x_1) \\
&\leq M x_1^n x_1 + M(1 - x_1) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Since $L(P, f(x)x^n) \geq 0$ we then have $|\int_0^1 f(x)x^n\,dx| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary this then shows that the sequence $\left\{\int_0^1 f(x)x^n\,dx\right\}_{n=1}^{\infty}$ converges to 0.