Math 3100 Exam 2 - Due Friday March 26th by 6:00pm central time

March 25, 2021

1 Instructions

Give a complete solution to each of the problems below. You are welcome to type your solutions in LaTeX and then submit the tex file, or you can write your solutions out on paper and submit a scanned pdf copy of work. In either case, you should submit your solutions by placing it in our shared folder in Vanderbilt's Box. Also, in either case you should write complete solutions, giving a professional presentation, as we've come to expect from the homework.

For the problems you are allowed to use without proof any result that we have proved in class or any theorem from the book that appears in or before Section 5.4. You are welcome to use other resources as well, but you should justify with a proof any results. If you significantly use an external resource then you should cite your source. You must justify any claims you make even if it is not specifically requested by the problem.

The solutions should be your own and you should not use any resource that involves active participation from another person. You should avoid discussing the exam with other people in any way, even a comment like "number 2 was tricky" or "number 3 wasn't too bad" conveys a significant amount of information and it would be improper to make or hear such comments.

Any questions regarding the exam should be asked directly to the instructor via email.

2 Problems

Problem 1 (20 points)**.** A subset $A \subset \mathbb{R}$ is said to be complete if every Cauchy sequence in A converges to a point that is contained in A. Show that a subset of $\mathbb{R}$ is complete if and only if it is closed.

Problem 2 (20 points)**.** Let $D = \mathbb{Q} \cap [0, 1]$ and suppose $f : D \to \mathbb{R}$. Show that the following conditions are equivalent:

1. f is uniformly continuous.

2. f is continuous and for each point $t \in [0, 1]$ the function f has a limit at t.

3. There exists a continuous function $g : [0, 1] \to \mathbb{R}$ such that $g(x) = f(x)$ for each $x \in D$.

Problem 3 (20 points)**.** Suppose $f : [0, \infty) \to [0, \infty)$ is differentiable and f' is uniformly continuous on $[0, \infty)$. Show that there exist constants $A, B, C \geq 0$ such that for all $x \in [0, \infty)$ we have $f(x) \leq Ax^2 + Bx + c$.

Problem 4 (20 points)**.** Suppose $f : [a, b] \to \mathbb{R}$ is twice differentiable on $[a, b]$ and suppose that the equation $f(x) = x$ has at least three distinct solutions in $[a, b]$. Prove that there exists $c \in [a, b]$ such that $f''(c) = 0$.

Problem 5 (20 points)**.** Let $f : [0, 1] \to [0, \infty)$ be a bounded function that is Riemann integrable. Prove that for each $n \geq 1$ the function $[0, 1] \ni x \mapsto f(x)x^n$ is Riemann integrable and the sequence $\left\{\int_0^1 f(x)x^n \, dx\right\}_{n=1}^{\infty}$ converges to 0.

Hint: First try showing that the function $[0, 1] \ni x \mapsto f(x)x$ is Riemann integrable.
