Math 3100 Exam 1 - Solutions

March 10, 2021

1 Instructions

Give a complete solution to each of the problems below. You are welcome to type your solutions in LaTeX and then submit the tex file, or you can write your solutions out on paper and submit a scanned pdf copy of work. In either case, you should submit your solutions by placing it in our shared folder in Vanderbilt's Box. Also, in either case you should write complete solutions, giving a professional presentation, as we've come to expect from the homework.

For the problems you are allowed to use without proof any result that we have proved in class or any theorem from the book that appears before Section 3.3. You are welcome to use other resources as well, but you should justify with a proof any results. If you significantly use an external resource then you should cite your source.

The solutions should be your own and you should not use any resource that involves active participation from another person. You should avoid discussing the exam with other people in any way, even a comment like "number 2 was tricky" or "number 3 wasn't too bad" conveys a significant amount of information and it would be improper to make or hear such comments.

Any questions regarding the exam should be asked directly to the instructor via email.

2 Problems

Problem 1 (20 points)**.** Prove that there exists a function $f : \mathbb{R} \to (\mathbb{R} \setminus \mathbb{Q})$ that is both 1-1 and onto.

We know from class that $\sqrt{2}$ is irrational, and it then follows that $A = \{n\sqrt{2} \mid n \in \mathbb{Z}_+\}$ is a countably infinite set of irrational numbers. We also know from class that $\mathbb{Q}$ is countably infinite and so we may enumerate it in a sequence as $\{q_n\}_{n=1}^{\infty}$. We define $f : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Q}$ by setting $f(q_n) = 2n\sqrt{2}$, and $f(n\sqrt{2}) = (2n-1)\sqrt{2}$ for $n \geq 1$, and by setting $f(x) = x$ if $x \notin \mathbb{Q} \cup A$.

Since $\mathbb{Q}$, A, and $\mathbb{R} \setminus (\mathbb{Q} \cup A)$ are pairwise disjoint this function is well-defined. The sets $f(\mathbb{Q}), f(A)$ and $f(\mathbb{R} \setminus (\mathbb{Q} \cup A))$ are also pairwise disjoint, thus to see that f is 1-1 it suffices to check that the restriction of f to each of the sets $\mathbb{Q}$, A, and $\mathbb{R} \setminus (\mathbb{Q} \cup A)$ is 1-1, which is easily verified. If $x \in \mathbb{R} \setminus (\mathbb{Q} \cup A)$, then $f(x) = x$, and if $x \in A$, then either $x = 2n\sqrt{2} = f(q_n)$ for some $n \in \mathbb{Z}_+$ or else $x = (2n-1)\sqrt{2} = f(n\sqrt{2})$ for some $n \in \mathbb{Z}_+$, hence we see that f is onto.

Problem 2 (20 points)**.** Suppose $f : [-1, 1] \to [-1, 1]$ is continuous and monotone increasing. Fix $t \in [-1, 1]$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ by setting $a_1 = t$ and $a_n = f(a_{n-1})$ for $n \geq 2$. Prove that this sequence converges and that the limit $t_0 = \lim_{n\to\infty} a_n$ satisfies $f(t_0) = t_0$.

If $a_2 = f(a_1) = f(t) \leq t = a_1$, then as f is monotone increasing we see via induction we see that $a_n = f(a_{n-1}) \leq f(a_{n-2}) = a_{n-1}$, for all $n \geq 2$. If $a_2 \geq a_1$, then we similarly have that $a_n \geq a_{n-1}$ for all $n \geq 1$. Thus, in either case it follows that the sequence $\{a_n\}_{n=1}^{\infty}$ is monotone. Since the sequence is bounded in absolute value by 1 it then follows that the sequence converges to some number $t_0 \in [0, 1]$. Since f is continuous we have that $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(t_0)$, but for $n \geq 2$ we have $f(a_n) = a_{n+1}$ and hence $\{f(a_n)\}_{n=1}^{\infty}$ also converges to t_0, showing that $f(t_0) = t_0$.

Problem 3 (20 points)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(s) - f(t)| < \frac{1}{2}|s - t|$$

for all $s, t \in \mathbb{R}$. Fix $t \in [-1, 1]$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ by setting $a_1 = t$ and $a_n = f(a_{n-1})$ for $n \geq 2$. Prove that this sequence converges and that the limit $t_0 = \lim_{n\to\infty} a_n$ satisfies $f(t_0) = t_0$.

If $s = t$, then the above inequality is clearly false for all functions, and so no such function exists and the statement is therefore true since the assumption is vacuous.

If the strict inequality $<$ above is replaced with $\leq$, then the statement is still true (this is a special case of what is known as the Contraction Mapping Theorem) and we give of proof of this stronger result here. We first claim that $|a_{n+1} - a_n| \leq 2^{-n+1}|f(t) - t|$ for all $n \geq 1$. Indeed, if $n = 1$ this statement is clearly true, and if it hold for some $n \geq 1$, then we have

$$|a_{n+2} - a_{n+1}| = |f(a_{n+1}) - f(a_n)| \leq 2^{-1}|a_{n+1} - a_n| \leq 2^{-n}|f(t) - t|.$$

Hence the claim follows by induction.

We next note that if $m \geq 1$, then by the triangle inequality we have

$$|a_{n+m} - a_n| \leq \sum_{k=0}^{m-1} |a_{n+k+1} - a_{n+k}| \leq \sum_{k=0}^{m} 2^{-n-k+1}|f(t) - t| < 2^{-n+2}|f(t) - t|.$$

Hence, if $\varepsilon > 0$ is given and we take $N \geq 1$ such that $2^{-N+2}|f(t) - t| < \varepsilon$ then we see that for $n, m \geq N$ we have $|a_n - a_m| < \varepsilon$. Therefore we have that the sequence $\{a_n\}_{n=1}^{\infty}$ is Cauchy and hence converges to some number $t_0 \in \mathbb{R}$. Since $|f(s) - f(t)| \leq \frac{1}{2}|s - t|$ for all $s, t \in \mathbb{R}$ it follows that f is continuous and hence the same argument as in the previous problem shows that $f(t_0) = f_0$.

Note that, in fact, the limit t_0 is independent of the starting point $t \in \mathbb{R}$. Indeed, if $\tilde{t}_0$ is any point such that $f(\tilde{t}_0) = \tilde{t}_0$ then we have $|t_0 - \tilde{t}_0| = |f(t_0) - f(\tilde{t}_0)| \leq \frac{1}{2}|t_0 - \tilde{t}_0|$ and we then conclude that $|t_0 - \tilde{t}_0| = 0$ so that $t_0 = \tilde{t}_0$.

Problem 4 (5 points)**.** Give an example of a continuous function $f : \mathbb{Q} \to \mathbb{R}$ such that there is no continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ satisfying $\tilde{f}(r) = f(r)$ for all $r \in \mathbb{Q}$, or else prove that no such function f exists.

Consider $f : \mathbb{Q} \to \mathbb{R}$ defined by $f(t) = \frac{1}{t - \sqrt{2}}$. Since $\sqrt{2}$ is irrational this function is well-defined. Also if $r_n \in \mathbb{Q}$ is such that $\{r_n\}_{n=1}^{\infty}$ converges to $\sqrt{2}$ then we have that $\{f(r_n)\}_{n=1}^{\infty}$ is unbounded and hence does not converge. Therefore if $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is any extension of f then we see that $\tilde{f}$ is not continuous at $\sqrt{2}$.

Problem 5 (15 points)**.** Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous such that $f(r) = g(r)$ for every rational number $r \in \mathbb{Q}$, then $f(t) = g(t)$ for all real numbers $t \in \mathbb{R}$.

If $t \in \mathbb{R}$ then there exists a sequence of rational numbers $\{r_n\}_{n=1}^{\infty}$ that converges to t. Since f and g are both continuous, and since f and g agree on the rational numbers we have that $\{f(r_n)\}_{n=1}^{\infty} = \{g(r_n)\}_{n=1}^{\infty}$ converges to both $f(t)$ and $g(t)$. Hence, $f(t) = g(t)$.

To set up this next problem we first give some definitions. If $f : E \to \mathbb{R}$ and $D \subset E$, then the **restriction of f to E** is the function $f_{|D} : D \to \mathbb{R}$ given by $f_{|D}(t) = f(t)$ for $t \in D$.

If $\mathcal{P}$ is some property of functions (e.g., $\mathcal{P}$ could be continuity, monotone, being constant, etc.) then we say that a function $f : E \to \mathbb{R}$ satisfies $\mathcal{P}$ **locally** (or is **locally** $\mathcal{P}$) if for each point $t_0 \in E$ there is some $\gamma > 0$ such that the restriction of f to $(t_0 - \gamma, t_0 + \gamma) \cap E$ satisfies the property $\mathcal{P}$. For instance, if we define a function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$f(t) = \begin{cases} -1 & \text{if } t < 0; \\ 1 & \text{if } t > 0, \end{cases}$$

then f is not constant, but it is locally constant.

A property $\mathcal{P}$ of functions is said to be a **local property** if any function that is locally $\mathcal{P}$ must also satisfy $\mathcal{P}$. Thus, from the example above we see that being constant is not a local property.

Problem 6 (20 points)**.** Is being continuous a local property? Either prove this or else give a counter-example.

Let $f : E \to \mathbb{R}$ be locally continuous and fix $t \in E$ and $\varepsilon > 0$. Then there exists $\gamma > 0$ such that $f_{|E\cap(t-\delta,t+\delta)}$ is continuous at t and hence there exists $\delta > 0$ such that if $s \in E \cap (t-\delta, t+\delta)$ and $|t-s| < \delta$ then $|f(t) - f(s)| < \varepsilon$.

Let δ_0 be the minimum of δ and γ. Then if $s \in E$ with $|s-t| < \delta_0$ we have $s \in E \cap (t-\gamma, t+\gamma)$ and $|s-t| < \delta$. Hence $|f(s) - f(t)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary this then shows that f is continuous at t and since $t \in E$ was arbitrary this shows that f is continuous.

Problem 7 (Extra Credit: 2 points for a correct guess or 10 points for a complete solution)**.** If $f : [-1,1] \to [-1,1]$ is monotone increasing, does there always exists some point $t_0 \in [-1,1]$ such that $f(t_0) = t_0$? Either prove this or else give a counter-example.

Let $S = \{t \in [-1,1] \mid f(t) \geq t\}$. Then S is bounded above by 1, and S is non-empty since $-1 \in S$. Let $r = \sup S \in [-1,1]$. Since f is increasing for any $t \in S$ we have $f(r) \geq f(t) \geq t$. Thus, $f(r)$ is an upper bound for S and hence $f(r) \geq r$. Since f is increasing we have $f(f(r)) \geq f(r)$ showing that $f(r) \in S$ and since r is an upper bound for S we then have $f(r) \leq r \leq f(r)$. We therefore conclude that $f(r) = r$.