Math 3100 Exam 1 - Due Friday February 26th by 6:00pm central time

February 24, 2021

1 Instructions

Give a complete solution to each of the problems below. You are welcome to type your solutions in LaTeX and then submit the tex file, or you can write your solutions out on paper and submit a scanned pdf copy of work. In either case, you should submit your solutions by placing it in our shared folder in Vanderbilt's Box. Also, in either case you should write complete solutions, giving a professional presentation, as we've come to expect from the homework.

For the problems you are allowed to use without proof any result that we have proved in class or any theorem from the book that appears before Section 3.3. You are welcome to use other resources as well, but you should justify with a proof any results. If you significantly use an external resource then you should cite your source.

The solutions should be your own and you should not use any resource that involves active participation from another person. You should avoid discussing the exam with other people in any way, even a comment like "number 2 was tricky" or "number 3 wasn't too bad" conveys a significant amount of information and it would be improper to make or hear such comments.

Any questions regarding the exam should be asked directly to the instructor via email.

2 Problems

Problem 1 (20 points)**.** Prove that there exists a function $f : \mathbb{R} \to (\mathbb{R} \setminus \mathbb{Q})$ that is both 1-1 and onto.

Problem 2 (20 points)**.** Suppose $f : [-1, 1] \to [-1, 1]$ is continuous and monotone increasing. Fix $t \in [-1, 1]$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ by setting $a_1 = t$ and $a_n = f(a_{n-1})$ for $n \geq 2$. Prove that this sequence converges and that the limit $t_0 = \lim_{n\to\infty} a_n$ satisfies $f(t_0) = t_0$.

Problem 3 (20 points)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(s) - f(t)| < \frac{1}{2}|s - t|$$

for all $s, t \in \mathbb{R}$. Fix $t \in [-1, 1]$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ by setting $a_1 = t$ and $a_n = f(a_{n-1})$ for $n \geq 2$. Prove that this sequence converges and that the limit $t_0 = \lim_{n\to\infty} a_n$ satisfies $f(t_0) = t_0$.

Problem 4 (5 points)**.** Give an example of a continuous function $f : \mathbb{Q} \to \mathbb{R}$ such that there is no continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ satisfying $\tilde{f}(r) = f(r)$ for all $r \in \mathbb{Q}$, or else prove that no such function f exists.

Problem 5 (15 points)**.** Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous such that $f(r) = g(r)$ for every rational number $r \in \mathbb{Q}$, then $f(t) = g(t)$ for all real numbers $t \in \mathbb{R}$.

To set up this next problem we first give some definitions. If $f : E \to \mathbb{R}$ and $D \subset E$, then the **restriction of** f **to** E is the function $f_{|D} : D \to \mathbb{R}$ given by $f_{|D}(t) = f(t)$ for $t \in D$.

If $\mathcal{P}$ is some property of functions (e.g., $\mathcal{P}$ could be continuity, monotone, being constant, etc.) then we say that a function $f : E \to \mathbb{R}$ satisfies $\mathcal{P}$ **locally** (or is **locally** $\mathcal{P}$) if for each point $t_0 \in E$ there is some $\gamma > 0$ such that the restriction of f to $(t_0 - \gamma, t_0 + \gamma) \cap E$ satisfies the property $\mathcal{P}$. For instance, if we define a function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$f(t) = \begin{cases} -1 & \text{if } t < 0; \\ 1 & \text{if } t > 0, \end{cases}$$

then f is not constant, but it is locally constant.

A property $\mathcal{P}$ of functions is said to be a **local property** if any function that is locally $\mathcal{P}$ must also satisfy $\mathcal{P}$. Thus, from the example above we see that being constant is not a local property.

Problem 6 (20 points)**.** Is being continuous a local property? Either prove this or else give a counter-example.

Problem 7 (Extra Credit: 2 points for a correct guess or 10 points for a complete solution)**.** If $f : [-1, 1] \to [-1, 1]$ is monotone increasing, does there always exists some point $t_0 \in [-1, 1]$ such that $f(t_0) = t_0$? Either prove this or else give a counter-example.