Operator algebras

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Theorem: If $p, q \in \mathcal{P}(M)$ then p & q are centrally orthogonal iff $pMq = \{0\}$

iff $\nexists$ $p_0 \leq p$ $p_0 \neq 0$ $q_0 \leq q$ st $p_0 \sim q_0$

Proof:

$z(p)z(q) = 0$ iff

$\langle xp\xi, yq\eta \rangle = 0$ $\forall \xi, \eta \in \mathcal{H}$ $x, y \in M$

iff $qy^*xp = 0$ $\forall x, y \in M$

iff $pMq = \{0\}$

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if $p_0 \leq p$ $p_0 \neq 0$ $p_0 \sim q_0 \leq q$

$\therefore$ $v^*v = p_0$ $vv^* = q_0$ $\Rightarrow pMq \neq \{0\}$

$\therefore$ $pv^*q = v^* \neq 0$

$$\boxed{z(p) = \bigwedge_{\substack{q \in Z(M) \\ qp = pq = p}} q} = \bigvee_{q \sim p} q$$

$z(p)$: central support.

p & q are centrally orthogonal if $z(p)z(q) = 0$.

Lemma: If $p \in M$ is a projection

then $\boxed{z(p) = \bigvee_{x \in M} [xp\mathcal{H}]} = [Mp\mathcal{H}]$

If $pxq \neq 0$ for some $x \in M$

$\therefore p_0 = [pxq\mathcal{H}] \neq 0$ $p_0 \leq p$

$p_0 \sim [qx^*p\mathcal{H}] \leq q$.

Theorem (The comparison theorem):
If M a von Neumann algebra and $p, q \in \mathcal{P}(M)$ then $\exists$ a central projection $z \in \mathcal{P}(Z(M))$ st.

$$pz \precsim qz \text{ and } q(1-z) \precsim p(1-z)$$

Proof:
By Zorn's lemma $\exists$ ~~a~~ maximal families $\{q_\alpha\}_\alpha$ ~~of~~ of pairwise orthogonal projections & $\{p_\alpha\}_\alpha$ st $\sum_\alpha q_\alpha \leq q$ $\sum_\alpha p_\alpha \leq p$

$$q_\alpha \sim p_\alpha \quad \forall \alpha.$$

set $q_0 = \sum_\alpha q_\alpha \quad p_0 = \sum_\alpha p_\alpha$

claim: $z(q - q_0)\, z(p - p_0) = 0$

Ex: If $M = L^\infty[0,1]$
$p = 1_{[0,2/3]} \quad q = 1_{[1/3,1]}$
$p \not\precsim q \quad q \not\precsim p$

$z = p$

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cor: If M is a factor, i.e. $Z(M) = \mathbb{C}$. And $p, q \in \mathcal{P}(M)$ then exactly one of the following hold:

$$p < q \quad p \sim q \quad q < p$$

Proof:
$\mathcal{P}(Z(M)) = \{0, 1\}$
if $z = 0 \Rightarrow q \precsim p$
if $z = 1 \Rightarrow p \precsim q$
if both hold then $p \sim q$

Claim: $z(q-q_0)z(p-p_0)=0$

otherwise $\exists$ $\tilde{q}\le q-q_0$ $\tilde{q}\ne 0$
$\tilde{p}\le p-p_0$ st $\tilde{q}\sim\tilde{p}$

Then $\{q_\alpha\}\cup\{\tilde{q}\}$ $\{p_\alpha\}\cup\{\tilde{p}\}$

would contradict maximality.

$z_1=z(p-p_0)\le 1-z_2=1-z(q-q_0)$
$\hookrightarrow (p-p_0)z_0=0$

Also $p_0\sim q_0 \Rightarrow p_0z_2\sim q_0z_2$

$\therefore pz_2=p_0z_2\sim q_0z_2\le qz_2$

$$q(1-z_2)=q_0(1-z_2)\sim p_0(1-z_2)\le p(1-z_2).$$

$\square$

Remark: If $p\in\mathcal{P}(M)$ then
$pMp\subset B(p\mathcal{H})$ is a vN alg.

Types of projections

M a vN alg $p\in\mathcal{P}(M)$

Then p is:

- minimal if $\dim(pMp)=1$

$\Leftrightarrow$ $p\ne 0$ $q\le p\Rightarrow q=0$ or $q=p$.

$\Downarrow$

- abelian if pMp is abelian.

$\Downarrow$

- finite if $q\le p$ $q\sim p$ $\Longrightarrow q=p$

$\Downarrow$

- semi-finite if there exist pairwise orthogonal finite projections $\{p_\alpha\}$ st $p=\sum p_\alpha$.

Ex: $M=B(\ell^2\mathbb{N})$ if $\{\xi_i\}$ an ONB then $1=\sum P_{\mathbb{C}\xi_i}$ is semi-finite.

- Purely infinite if $p\ne 0$ & has no non-zero finite subprojections.

$\hookrightarrow$ these do exist

Lemma: If $\{P_\alpha\}$ a family of pairwise centrally orthogonal projections. If each P_α is abelian (resp. finite) then so is $P = \sum_\alpha P_\alpha$

Proof:

If $\alpha \neq \beta$ $\quad P_\alpha M P_\beta = \{0\}$

$$PxP = \sum_\alpha P_\alpha x P_\alpha \quad \forall \; x \in M$$

$$(PxP)(PyP) = \sum_\alpha P_\alpha x P_\alpha y P_\alpha$$

$P_\alpha M P_\alpha$ is abelian

$$= \sum_\alpha P_\alpha y P_\alpha x P_\alpha$$
$$= (PyP)(PxP)$$

If P_α is finite $\forall \alpha$.

& If $u \in M$ st $uu^* \leq u^*u = P$.

then $\forall \alpha$

$$Z(P_\alpha) u^* u Z(P_\alpha) = Z(P_\alpha) P_\alpha Z(P_\alpha)$$
$$= P_\alpha$$

$$u Z(P_\alpha) u^* = Z(P_\alpha) uu^* \leq Z(P_\alpha) P$$
$$= P_\alpha$$

$$P_\alpha \text{ finite} \Longrightarrow u Z(P_\alpha) u^* = P_\alpha$$

$$uu^* = u Z(P) u^*$$
$$= u\left(\sum_\alpha Z(P_\alpha)\right) u^* = \sum_\alpha P_\alpha = P$$

$\square$

$p \in M$ is a projection. Then

- p is finite if whenever $u \in M$ st $u^*u \le p$ $uu^* = p$ then $u^*u = p$.
- p is semi-finite if it is a sum of orthogonal finite projections.
- p is purely infinite if it has no non-zero finite subprojection and $p \neq 0$.

Prop: If $p, q \in M$ are non-zero projections st $p \le q$ if q is finite then p is also finite.

Proof:

Case 1: $p \sim q$, ie $u^*u = p$ $uu^* = q$

suppose $\underline{v^*v \le p}$ $vv^* = p$.

Set $w = uvu^*$

$$w^*w = (uv^*\underbrace{u^*)(u}_{p}vu^*) = uv^*vu^* \le upu^* = uu^* = q$$

vv^*

$$ww^* = uv\underbrace{u^*u}_{p}v^*u^* = uvv^*u^* = upu^* = q$$

q finite $\Rightarrow w^*w = q$

ie $uv^*vu^* = upu^*$

$\Longrightarrow v^*v = p$

Case 2: $p \le q$

suppose $v^*v \le p$ $vv^* = p$

Set $w = v + (q-p)$

$$w^*w = ((q-p)+v^*)((q-p)+v) = (q-p) + v^*v \le q$$

$ww^* \le q$

q finite $\Rightarrow w^*w = q \Rightarrow v^*v = p$. $\square$

General: $p \sim q_0 \le q$

Cor: If $\overset{\neq 0}{p} \precsim q$ q is purely infinite then p is purely infinite.

Proof:

$p \precsim q \Rightarrow p \sim q_0 \leq q$

if $\exists$ $\overset{\neq 0}{p_0} \leq p$ p_0 finite

then $\exists$ $q_{00} \overset{\leq q_0}{\sim} p_0$ finite

$\Rightarrow$ q not purely infinite. $\square$

Prop: p is semi-finite iff p is the supremum of finite projections.

Proof:

($\Leftarrow$): suppose $p = \bigvee_\alpha p_\alpha$ p_α is finite

Take a maximal family of orthogonal finite subprojections of p. $\{q_\beta\}_\beta$.

set $q_0 = p - \sum_\beta q_\beta$.

If $q_0 \neq 0$ $\therefore$ $q_0 p_\alpha \neq 0$ for some α.

$\therefore$ $\exists$ $q_{00} \neq 0$ $q_{00} \leq q$ $p_0 \leq p_\alpha$ st $q_{00} \sim p_0$

p_0 is finite $\therefore$ $\{q_\beta\}_\beta \cup \{p_0\}$ is a larger family of orthogonal subprojections, contradicting maximality. $\square$

Cor: If p is semi-finite then so is $Z(p)$. If p is purely ∞ then so is $Z(p)$.

Proof:

- p semifinite

$$Z(p) = \bigvee_{q \sim p} q \quad \text{semi-finite.}$$

- p is purely infinite iff p is centrally orthogonal to every finite subprojection.

$\Rightarrow$ $Z(p)$ is orthogonal to every finite subprojection, hence is purely infinite

Cor: If $p \precsim q$ & q is semi-finite then so is p.

Proof:

we may assume $q \in Z(M)$.

$\therefore p \leq q$.

let p_0 be a maximal semi-finite subprojection of p

$Z(p-p_0) \leq z(q) = q = \bigvee_{r \leq q \, r \text{ finite}} r$

if $p-p_0 \neq 0$ then $p-p_0$ and r have non-zero equivalent subprojections for some $r \leq q$ r finite.

giving a contradiction $\square$.

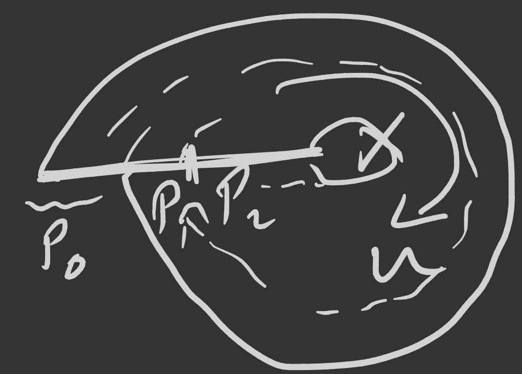
Lemma: Suppose M has no non-zero finite central projection, then $\exists\ p \in \mathcal{P}(M)$ st $p \sim 1-p \sim 1$

Proof:

$\exists\ u \in M$ st $uu^* < u^*u = 1$.

set $p_0 = 1 - uu^*$

set $p_n = u^n p_0 (u^n)^*$



$\{p_n\}$ pairwise orthogonal and $p_n \sim p_0$ $\forall$ n.

Let $\{q_\alpha\}$ be a maximal family of pairwise orthogonal & pairwise equivalent projections st $\{p_n\} \subset \{q_\alpha\}$.

Set $q_0 = 1 - \sum_\alpha q_\alpha$

By the comparison thm $\exists$ a central projection $z \in M$ st

$$q_0 z \precsim p_0 z \ \underline{\text{and}} \ p_0(1-z) \precsim q_0(1-z)$$

claim $z \neq 0$: otherwise $\exists$ $r_0 \leq q_0$ st $r_0 \sim p_0$. Then $\{q_\alpha\} \cup \{r_0\}$ would contradict maximality.

$\therefore$ $z \neq 0$

$$z = q_0 z + \sum q_\alpha z \precsim p_0 z + \sum_{q_\alpha \neq p_0} q_\alpha z$$

$$= \sum_{q_\alpha} q_\alpha z \leq z$$

$$\therefore z \sim \sum_{\alpha \in I} q_\alpha z$$

write $I = I_1 \sqcup I_2$ st $\underline{|I_1| = |I_2| = |I|}$

set $r = \sum_{\alpha \in I_1} q_\alpha z$ then

$$rz \sim \sum_{\alpha \in I_2} q_\alpha z \ \underline{\sim z \sim (1-r) z}$$

let $\{r_i\}_i$ be a maximal family of pairwise centrally orthogonal projections st.

$$r_i \sim z(r_i) \sim z(r_i) - r_i.$$

consider $p_0 = \sum z(r_i) \in Z(M)$

If $p_0 \neq 1$ then the above argument would produce a new centrally orthogonal r.

$\therefore$ $p_0 = 1$

set $\tilde{r} = \sum_i r_i$

then $\tilde{r} \sim z(\tilde{r}) = 1 \sim 1 - \tilde{r}$. $\square$

Thm: If p, q are finite then so is $p \vee q$.