Operator algebras

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Theorem: If $p, q \in \mathcal{P}(M)$

then p & q are centrally orthogonal iff $pMq = \{0\}$

iff $\nexists$ $p_0 \leq p$ $p_0 \neq 0$

$q_0 \leq q$ st $p_0 \sim q_0$

Proof:

$z(p)z(q) = 0$ iff

$\langle xp\xi, yq\eta \rangle = 0$ $\forall \xi, \eta \in \mathcal{H}$
$x, y \in M$

iff $qy^*xp = 0$ $\forall x, y \in M$

iff $pMq = \{0\}$

if $p_0 \leq p$ $p_0 \neq 0$ $p_0 \sim q_0 \leq q$

$\therefore$ $v^*v = p_0$ $vv^* = q_0$

$\therefore$ $pv^*q = v^* \neq 0$ $\Rightarrow pMq \neq \{0\}$

$$\boxed{z(p) = \bigwedge_{\substack{q \in \mathcal{Z}(M) \\ qp = pq = p}} q} = \bigvee_{q \sim p} q$$

$z(p)$: central support.

p & q are centrally orthogonal if $z(p)z(q) = 0$.

Lemma: If $p \in M$ is a projection

then $\boxed{z(p) = \bigvee_{x \in M} [xp\mathcal{H}]} = [Mp\mathcal{H}]$

If $pxq \neq 0$ for some $x \in M$

$\therefore$ $p_0 = [pxq\mathcal{H}] \neq 0$ $p_0 \leq p$

$p_0 \sim [qx^*p\mathcal{H}] \leq q$.

Theorem (The comparison theorem):

If M a von Neumann algebra and $p, q \in \mathcal{P}(M)$ then $\exists$ a central projection $z \in \mathcal{P}(Z(M))$ st.

$$pz \precsim qz \text{ and } q(1-z) \precsim p(1-z)$$

Proof:

By Zorn's lemma $\exists$ ~~a~~ maximal families $\{q_\alpha\}_\alpha$ & $\{p_\alpha\}_\alpha$ ~~of~~ of pairwise orthogonal projections st $\sum_\alpha q_\alpha \leq q$ $\sum_\alpha p_\alpha \leq p$

$$q_\alpha \sim p_\alpha \quad \forall \alpha.$$

set $q_0 = \sum_\alpha q_\alpha$ $p_0 = \sum_\alpha p_\alpha$

claim:

$$z(q - q_0) \; z(p - p_0) = 0$$

Ex: If $M = L^\infty[0,1]$

$p = 1_{[0,2/3]}$ $q = 1_{[1/3,1]}$

$p \not\precsim q$ $q \not\precsim p$

$z = p$

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cor: If M is a factor, i.e. $Z(M) = \mathbb{C}$. And $p, q \in \mathcal{P}(M)$ then exactly one of the following hold:

$$p < q \quad p \sim q \quad q < p$$

Proof:

$\mathcal{P}(Z(M)) = \{0, 1\}$

if $z = 0 \implies q \precsim p$

if $z = 1 \implies p \precsim q$

if both hold then $p \sim q$

Claim: $Z(q-q_0)Z(p-p_0)=0$

otherwise $\exists$ $\tilde{q} \le q-q_0$ $\tilde{q} \ne 0$
$\tilde{p} \le p-p_0$ st $\tilde{q} \sim \tilde{p}$

Then $\{q_\alpha\} \cup \{\tilde{q}\}$ $\{p_\alpha\} \cup \{\tilde{p}\}$

would contradict maximality.

$$z_1 = Z(p-p_0) \le 1 - z_2 = 1 - Z(q-q_0)$$
$\Rightarrow (p-p_0)z_0 = 0$

Also $p_0 \sim q_0 \Rightarrow p_0 z_2 \sim q_0 z_2$

$\therefore p z_2 = p_0 z_2 \sim q_0 z_2 \le q z_2$

$$q(1-z_2) = q_0(1-z_2) \sim p_0(1-z_2) \le p(1-z_2).$$

$\square$

Remark: If $p \in \mathcal{P}(M)$ then
$pMp \subset B(p\mathcal{H})$ is a vN alg.

Types of projections

M a vN alg $p \in \mathcal{P}(M)$

Then p is:

- minimal if $\dim(pMp) = 1$
  $\Downarrow$ $\Leftrightarrow$ $q \le p \Rightarrow q = 0$ or $q = p$. ($p \ne 0$)
- abelian if pMp is abelian.
  $\Downarrow$
- finite if $q \le p$ $q \sim p$ $\Longrightarrow q = p$
  $\Downarrow$
- semi-finite if there exist pairwise orthogonal finite projections $\{p_\alpha\}$ st $p = \sum p_\alpha$.
  Ex: $M = B(\ell^2 \mathbb{N})$ if $\{\xi_i\}$ an ONB then $1 = \sum P_{\mathbb{C}\xi_i}$ is semi-finite.
- Purely infinite if $p \ne 0$ & has no non-zero finite subprojections.
  $\Rightarrow$ these do exist

Lemma: If $\{P_\alpha\}$ a family of pairwise centrally orthogonal projections. If each P_α is abelian (resp. finite) then so is $P = \sum_\alpha P_\alpha$

Proof:
If $\alpha \neq \beta$ $\quad P_\alpha M P_\beta = \{0\}$

$$PxP = \sum_\alpha P_\alpha x P_\alpha \quad \forall \; x \in M$$

$$(PxP)(PyP) = \sum_\alpha P_\alpha x P_\alpha y P_\alpha$$

$P_\alpha M P_\alpha$ is abelian

$$= \sum_\alpha P_\alpha y P_\alpha x P_\alpha$$
$$= (PyP)(PxP)$$

If P_α is finite $\forall \alpha$.
& if $u \in M$ st $uu^* \leq u^*u = P$.

then $\forall \alpha$

$$Z(P_\alpha) u^* u Z(P_\alpha) = Z(P_\alpha) P_\alpha Z(P_\alpha) = P_\alpha$$

$$u Z(P_\alpha) u^* = Z(P_\alpha) uu^* \leq Z(P_\alpha) P = P_\alpha$$

P_α finite $\Longrightarrow$ $u Z(P_\alpha) u^* = P_\alpha$

$$uu^* = u Z(P) u^* = u\Big(\sum_\alpha Z(P_\alpha)\Big)u^* = \sum_\alpha P_\alpha = P$$

$\square$