MATH 8120 - HOMEWORK ASSIGNMENT 2

DUE WEDNESDAY, JANUARY 22 BY 4:00PM

Let Γ be a discrete abelian group. A character on Γ is a homomorphism $\chi : \Gamma \to \mathbb{T}$. We let $\hat{\Gamma}$ denote the set of characters, and we endow $\hat{\Gamma}$ with a group structure by defining $(\chi\psi)(s) = \chi(s)\psi(s)$.

Exercise 0.1. Show that $\hat{\Gamma}$ can naturally be identified with the spectrum $\sigma(C^*\Gamma)$ and that with the induced topology we have that $\hat{\Gamma}$ is a compact group.

The Gelfand transform gives an isomorphism $C^*\Gamma \cong C(\hat{\Gamma})$ and we have that $\ell^1\Gamma$ is a dense subspace in $C^*\Gamma$, thus the Gelfand transform gives a map from $\ell^1\Gamma$ onto a dense subspace of $C(\hat{\Gamma})$ which maps $\xi \in \ell^1\Gamma$ to the function $\hat{\Gamma} \ni \chi \mapsto \sum_{s\in\Gamma} \xi(s)\chi(s)$. This is closely related to the Fourier transform $\mathcal{F} : \ell^1\Gamma \to C(\hat{\Gamma})$ given by $\mathcal{F}(\xi)(\chi) = \sum_{s\in\Gamma} \xi(s)\overline{\chi(s)}$.

Exercise 0.2 (Plancherel's theorem)**.** Let λ denote the Haar measure on $\hat{\Gamma}$ such that $\lambda(\hat{\Gamma}) = 1$. Show that the Fourier transform extends to a unitary operator between $\ell^2\Gamma$ and $L^2(\hat{\Gamma}, \lambda)$.

We denote by $\mathcal{F}_0 : \ell^2\Gamma \to L^2(\hat{\Gamma}, \lambda)$ the unitary from the previous exercise. (Which is also called the Fourier transform).

The left regular representation of a group Γ is $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ defined by $(\lambda(s)\xi)(t) = \xi(s^{-1}t)$. The reduced group C^*-algebra of Γ is the C^*-algebra generated by $\lambda(\Gamma)$ and is denoted by $C^*_r(\Gamma)$.

We now consider the natural inclusions $C(\hat{\Gamma}) \subset L^\infty(\hat{\Gamma}) \subset \mathcal{B}(L^2(\hat{\Gamma}))$. Where the latter inclusion associates each function $f \in L^\infty(\hat{\Gamma})$ to the operator $L^2(\hat{\Gamma}) \ni \xi \mapsto f\xi \in L^2(\hat{\Gamma})$.

Exercise 0.3. Let Γ be a discrete abelian group. Show that $\mathcal{F}(C^*_r\Gamma)\mathcal{F}^* = C(\hat{\Gamma})$ and $\mathcal{F}(L\Gamma)\mathcal{F}^* = L^\infty(\hat{\Gamma})$. In particular, for abelian discrete groups, the canonical $*$-homomorphism from $C^*\Gamma$ onto $C^*_r\Gamma$ given by $\ell^1\Gamma \ni \xi \mapsto \lambda(\xi) \in C^*_r\Gamma$, is a $*$-isomorphism.

Exercise 0.4. Let Γ be a discrete group. Consider the following conditions:

1. The canonical $*$-homomorphism from $C^*\Gamma$ onto $C^*_r\Gamma$ is a $*$-isomorphism.
2. For any unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and any $\xi \in \ell^1\Gamma$, we have $\|\pi(\xi)\| \leq \|\lambda(\xi)\|$.
3. For any finite set $F \subset \Gamma$ we have $\|\sum_{s\in F} \lambda(s)\| = |F|$.
4. For any $\varepsilon > 0$ and $F \subset \Gamma$ finite, there exists a unit vector $\xi \in \ell^2\Gamma$ such that $\sum_{s\in F} \|\lambda(s)\xi - \xi\|^2 < \varepsilon$.

Prove the implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4).[1]

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[1]In fact, all four of these conditions are equivalent.

Hint: To prove (3) $\Leftrightarrow$ (4) you may want to use the fact that if $\{\xi_i\}_{i=1}^n$ is a family of unit vectors in a Hilbert space then we have $\sum_{i,j=1}^n \|\xi_i - \xi_j\|^2 = 2n^2 - 2\sum_{i,j=1}^n \langle \xi_i, \xi_j \rangle = 2n^2 - 2\|\sum_{i=1}^n \xi_i\|^2$.