MATH 8120 - HOMEWORK ASSIGNMENT 1

DUE FRIDAY, JANUARY 10 BY 4:00PM

Exercise 0.1. Let $\mathcal{H}$ be a Hilbert space. Show that $\mathcal{B}(\mathcal{H})$, with Banach space structure given by the operator norm, product structure given by composition, and involution given by adjoint is a C^*-algebra.

Let Γ be a discrete group. If ξ and η are complex functions on Γ then the convolution of ξ and η is given by $(\xi * \eta)(s) = \sum_{t \in \Gamma} \xi(st^{-1})\eta(t)$. This is defined whenever the series converges absolutely. We also define ξ^* by $\xi^*(s) = \overline{\xi(s^{-1})}$. Note that $\|\xi^*\|_p = \|\xi\|_p$ for any $1 \leq p \leq \infty$.

Exercise 0.2. Prove Young's inequality: If $1 \leq p, q \leq r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, then for $\xi \in \ell^p\Gamma$ and $\eta \in \ell^q\Gamma$ we have $\xi * \eta \in \ell^r\Gamma$ and

$$\|\xi * \eta\|_r \leq \|\xi\|_p \|\eta\|_q.$$

Exercise 0.3. Show that $\ell^1\Gamma$ is a Banach algebra under convolution.

Exercise 0.4. Let Γ be a discrete group. Find necessary and sufficient conditions on Γ so that the involutive Banach algebra $\ell^1(\Gamma)$ defined above is a C^*-algebra.

Exercise 0.5. Show that for each unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ there exists a unique Banach $*$-algebra representation $\tilde{\pi} : \ell^1\Gamma \to \mathcal{B}(\mathcal{H})$ such that $\tilde{\pi}(\delta_s) = \pi(s)$ for all $s \in \Gamma$.

Exercise 0.6. For each $\xi \in \ell^1\Gamma$ define $\|\xi\|_u = \sup_{\pi \in \mathrm{Rep}(\ell^1\Gamma)} \|\pi(\xi)\|$ where $\mathrm{Rep}(\ell^1\Gamma)$ denotes the space of all $*$-representations of $\ell^1\Gamma$ into bounded operators on a Hilbert space. Show that $\|\cdot\|_u$ defines a C^*-norm on $\ell^1\Gamma$, i.e., there exists a C^*-algebra $C^*\Gamma$ and a $*$-algebra homomorphism $\rho : \ell^1\Gamma \to C^*\Gamma$ such that $\|\xi\|_u = \|\rho(\xi)\|$ for all $\xi \in \ell^1\Gamma$. Also, show that ρ is injective.

The C^*-algebra $C^*\Gamma$ defined by the previous exercise is called the universal C^*-algebra associated to Γ. The previous two exercises show that $C^*\Gamma$ contains an isomorphic copy of Γ in its unitary group, and satisfies that universal property that any unitary representation of Γ has a unique extension to a unital $*$-representation of $C^*\Gamma$.

A vector $\xi \in \ell^2\Gamma$ is said to be a left (resp. right) convolver if $\xi * \eta \in \ell^2\Gamma$ (resp. $\eta * \xi \in \ell^2\Gamma$) for all $\eta \in \ell^2\Gamma$. Note that if $\xi \in \ell^1\Gamma$ then by Young's inequality it follows that ξ is a left and right convolver. If $\xi \in \ell^2\Gamma$ is a left convolver, then we denote by $L_\xi : \ell^2\Gamma \to \ell^2\Gamma$ the linear map given by $L_\xi(\eta) = \xi * \eta$. If ξ is a right convolver then we similarly define R_ξ.

Exercise 0.7. Show that the set of left convolvers forms a $*$-algebra under convolution, and that $\xi \mapsto L_\xi$ gives a unital injective $*$-representation of this algebra into $\mathcal{B}(\ell^2\Gamma)$.

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We let $L\Gamma \subset \mathcal{B}(\ell^2\Gamma)$ denote the image of the space of left convolvers under the map $\xi \mapsto L_\xi$. The next exercise shows that $L\Gamma$ is a von Neumann algebra.

Exercise 0.8. Suppose $T \in \mathcal{B}(\ell^2\Gamma)$ such that $[T, L_\xi] = 0$ for any left convolver $\xi \in \ell^2\Gamma$. Show that there exists a unique right convolver $\eta \in \ell^2\Gamma$ such that $T = R_\eta$.