Notes on operator algebras

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Chapter 1

Spectral theory

If A is a complex unital algebra then we denote by $G(A)$ the set of elements which have a two sided inverse. If $x \in A$, the **spectrum** of x is

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \notin G(A)\}.$$

The complement of the spectrum is called the **resolvent** and denoted $\rho_A(x)$.

Proposition 1.0.1. *Let A be a unital algebra over $\mathbb{C}$, and consider $x, y \in A$. Then $\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}$.*

Proof. If $1 - xy \in G(A)$ then we have

$$\begin{aligned}(1 - yx)(1 + y(1 - xy)^{-1}x) &= 1 - yx + y(1 - xy)^{-1}x - yxy(1 - xy)^{-1}x \\ &= 1 - yx + y(1 - xy)(1 - xy)^{-1}x = 1.\end{aligned}$$

Similarly, we have

$$(1 + y(1 - xy)^{-1}x)(1 - yx) = 1,$$

and hence $1 - yx \in G(A)$. ■

1.1 Banach and C^*-algebras

A **Banach algebra** is a Banach space A, which is also an algebra such that

$$\|xy\| \leq \|x\|\|y\|.$$

An **involution** $*$ on a Banach algebra is a conjugate linear period two anti-isomorphism such that $\|x^*\| = \|x\|$, for all $x \in A$. An **involutive** Banach algebra is a Banach algebra, together with a fixed involution.

If an involutive Banach algebra A additionally satisfies

$$\|x^*x\| = \|x\|^2,$$

for all $x \in A$, then we say that A is a C^***-algebra**. If a Banach or C^*-algebra is unital, then we further require $\|1\| = 1$.

Note that if A is a unital involutive Banach algebra, and $x \in G(A)$ then $(x^{-1})^* = (x^*)^{-1}$, and hence $\sigma_A(x^*) = \overline{\sigma_A(x)}$.

Lemma 1.1.1. *Let A be a unital Banach algebra and suppose $x \in A$ such that $\|1 - x\| < 1$, then $x \in G(A)$.*

Proof. Since $\|1 - x\| < 1$, the element $y = \sum_{k=0}^{\infty}(1 - x)^k$ is well defined, and it is easy to see that $xy = yx = 1$. ■

Proposition 1.1.2. *Let A be a unital Banach algebra, then $G(A)$ is open, and the map $x \mapsto x^{-1}$ is a continuous map on $G(A)$.*

Proof. If $y \in G(A)$ and $\|x - y\| < \frac{1}{\|y^{-1}\|}$ then $\|1 - xy^{-1}\| < 1$ and hence by the previous lemma $xy^{-1} \in G(A)$ (hence also $x = xy^{-1}y \in G(A)$) and

$$\begin{aligned} \|xy^{-1}\| &\leq \sum_{n=0}^{\infty} \|(1 - xy^{-1})\|^n \\ &\leq \sum_{n=0}^{\infty} \|y^{-1}\|^n \|y - x\|^n = \frac{1}{1 - \|y\|^{-1}\|y - x\|}. \end{aligned}$$

Hence,

$$\begin{aligned} \|x^{-1} - y^{-1}\| &= \|x^{-1}(y - x)y^{-1}\| \\ &\leq \|y^{-1}(xy^{-1})^{-1}\|\|y^{-1}\|\|y - x\| \leq \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\|\|y - x\|}\|y - x\|. \end{aligned}$$

Thus continuity follows from continuity of the map $t \mapsto \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\|t}t$, at $t = 0$. ■

Proposition 1.1.3. *Let A be a unital Banach algebra, and suppose $x \in A$, then $\sigma_A(x)$ is a non-empty compact set.*

Proof. If $\|x\| < |\lambda|$ then $\frac{x}{\lambda} - 1 \in G(A)$ by Lemma 1.1.1, also $\sigma_A(x)$ is closed by Proposition 1.1.2, thus $\sigma_A(x)$ is compact.

To see that $\sigma_A(x)$ is non-empty note that for any linear functional $\varphi \in A^*$, we have that $f(z) = \varphi((x - z)^{-1})$ is analytic on $\rho_A(x)$. Indeed, if $z, z_0 \in \rho_A(x)$ then we have

$$(x - z)^{-1} - (x - z_0)^{-1} = (x - z)^{-1}(z - z_0)(x - z_0)^{-1}.$$

Since inversion is continuous it then follows that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \varphi((x - z_0)^{-2}).$$

We also have $\lim_{z \to \infty} f(z) = 0$, and hence if $\sigma_A(x)$ were empty then f would be a bounded entire function and we would then have $f = 0$. Since $\varphi \in A^*$ were arbitrary this would then contradict the Hahn-Banach theorem. ■

Theorem 1.1.4 (Gelfand-Mazur)**.** *Suppose A is a unital Banach algebra such that every non-zero element is invertible, then $A \cong \mathbb{C}$.*

Proof. Fix $x \in A$, and take $\lambda \in \sigma(x)$. Since $x - \lambda$ is not invertible we have that $x - \lambda = 0$, and the result then follows. ∎

If $f(z) = \sum_{k=0}^n a_k z^k$ is a polynomial, and $x \in A$, a unital Banach algebra, then we define $f(x) = \sum_{k=0}^n a_k x^k \in A$.

Proposition 1.1.5 (The spectral mapping formula for polynomials)**.** *Let A be a unital Banach algebra, $x \in A$ and f a polynomial. then $\sigma_A(f(x)) = f(\sigma_A(x))$.*

Proof. If $\lambda \in \sigma_A(x)$, and $f(z) = \sum_{k=0}^n a_k z^k$ then

$$\begin{aligned} f(x) - f(\lambda) &= \sum_{k=1}^n a_k (x^k - \lambda^k) \\ &= (x - \lambda) \sum_{k=1} a_k \sum_{j=0}^{k-1} x^j \lambda^{k-j-1}, \end{aligned}$$

hence $f(\lambda) \in \sigma_A(x)$. conversely if $\mu \notin f(\sigma_A(x))$ and we factor $f - \mu$ as

$$f - \mu = \alpha_n (x - \lambda_1) \cdots (x - \lambda_n),$$

then since $f(\lambda) - \mu \neq 0$, for all $\lambda \in \sigma_A(x)$ it follows that $\lambda_i \notin \sigma_A(x)$, for $1 \leq i \leq n$, hence $f(x) - \mu \in G(A)$. ∎

If A is a unital Banach algebra and $x \in A$, the **spectral radius** of x is

$$r(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|.$$

Note that by Proposition 1.1.3 the spectral radius is finite, and the supremum is attained. Also note that by Proposition 1.0.1 we have the very useful equality $r(xy) = r(yx)$ for all x and y in a unital Banach algebra A. A priori the spectral radius depends on the Banach algebra in which x lives, but we will show now that this is not the case.

Proposition 1.1.6 (The spectral radius formula)**.** *Let A be a unital Banach algebra, and suppose $x \in A$. Then $\lim_{n\to\infty} \|x^n\|^{1/n}$ exists and we have*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

Proof. By Proposition 1.1.5 we have $r(x^n) = r(x)^n$, and hence

$$r(x)^n \leq \|x^n\|,$$

showing that $r(x) \leq \liminf_{n\to\infty} \|x^n\|^{1/n}$.

To show that $r(x) \geq \limsup_{n\to\infty} \|x^n\|^{1/n}$, consider the domain $\Omega = \{z \in \mathbb{C} \mid |z| > r(x)\}$, and fix a linear functional $\varphi \in A^*$. We showed in Proposition 1.1.3 that $z \mapsto \varphi((x-z)^{-1})$ is analytic in Ω and as such we have a Laurent expansion

$$\varphi((z-x)^{-1}) = \sum_{n=0}^{\infty} \frac{a_n}{z^n},$$

for $|z| > r(x)$. However, we also know that for $|z| > \|x\|$ we have

$$\varphi((z-x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n}.$$

By uniqueness of the Laurent expansion we then have that

$$\varphi((z-x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n},$$

for $|z| > r(x)$.

Hence for $|z| > r(x)$ we have that $\lim_{n\to\infty} \frac{\varphi(x^{n-1})}{|z|^n} = 0$, for all linear functionals $\varphi \in A^*$. By the uniform boundedness principle we then have $\lim_{n\to\infty} \frac{\|x^{n-1}\|}{|z|^n} = 0$, hence $|z| > \limsup_{n\to\infty} \|x^n\|^{1/n}$, and thus

$$r(x) \geq \limsup_{n\to\infty} \|x^n\|^{1/n}. \blacksquare$$

An element x of a involutive algebra A is **normal** if $x^*x = xx^*$. An element x is **self-adjoint** (resp. **skew-adjoint**) if $x^* = x$ (resp. $x^* = -x$). Note that self-adjoint and skew-adjoint elements are normal.

Corollary 1.1.7. *Let A be a unital involutive Banach algebra and $x \in A$ normal, then $r(x^*x) \leq r(x)^2$. Moreover, if A is a C^*-algebra, then we have equality $r(x^*x) = r(x)^2$.*

Proof. By the previous proposition we have

$$r(x^*x) = \lim_{n\to\infty} \|(x^*x)^n\|^{1/n} = \lim_{n\to\infty} \|(x^*)^n x^n\|^{1/n} \leq \lim_{n\to\infty} \|x^n\|^{2/n} = r(x)^2.$$

By the C^*-identity, the inequality above becomes equality in a C^*-algebra. $\blacksquare$

Proposition 1.1.8. *Let A be a C^*-algebra and $x \in A$ normal, then $\|x\| = r(x)$.*

Proof. We first show this if x is self-adjoint, in which case we have $\|x^2\| = \|x\|^2$, and by induction we have $\|x^{2^n}\| = \|x\|^{2^n}$ for all $n \in \mathbb{N}$. Therefore, $\|x\| = \lim_{n\to\infty} \|x^{2^n}\|^{2^n} = r(x)$.

If x is normal then by Corollary 1.1.7 we have

$$\|x\|^2 = \|x^*x\| = r(x^*x) = r(x)^2. \blacksquare$$

Corollary 1.1.9. *Let A and B be two unital C^*-algebras and $\Phi : A \to B$ a unital $*$-homomorphism, then Φ is contractive. If Φ is a $*$-isomorphism, then Φ is isometric.*

Proof. Since Φ is a unital $*$-homomorphism we clearly have $\Phi(G(A)) \subset G(B)$, from which it follows that $\sigma_B(\Phi(x)) \subset \sigma_A(x)$, and hence $r(\Phi(x)) \leq r(x)$, for all $x \in A$. By Proposition 1.1.8 we then have

$$\|\Phi(x)\|^2 = \|\Phi(x^*x)\| = r(\Phi(x^*x)) \leq r(x^*x) = \|x^*x\| = \|x\|^2.$$

If Φ is a $*$-isomorphism then so is Φ^{-1} which then shows that Φ is isometric. ∎

Corollary 1.1.10. *Let A be a unital complex involutive algebra, then there is at most one norm on A which makes A into a C^*-algebra.*

Proof. If there were two norms which gave a C^*-algebra structure to A then by the previous corollary the identity map would be an isometry. ∎

Lemma 1.1.11. *Let A be a unital C^*-algebra, if $x \in A$ is self-adjoint then $\sigma_A(x) \subset \mathbb{R}$.*

Proof. Suppose $\lambda = \alpha + i\beta \in \sigma_A(x)$ where $\alpha, \beta \in \mathbb{R}$. If we consider $y = x - \alpha + it$ where $t \in \mathbb{R}$, then we have $i(\beta + t) \in \sigma_A(y)$ and y is normal. Hence,

$$\begin{aligned}(\beta + t)^2 \leq r(y)^2 &= \|y\|^2 = \|y^*y\| \\ &= \|(x - \alpha)^2 + t^2\| \leq \|x - \alpha\|^2 + t^2,\end{aligned}$$

and since $t \in \mathbb{R}$ was arbitrary it then follows that $\beta = 0$. ∎

Lemma 1.1.12. *Let A be a unital Banach algebra and suppose $x \notin G(A)$. If $x_n \in G(A)$ such that $\|x_n - x\| \to 0$, then $\|x_n^{-1}\| \to \infty$.*

Proof. If $\|x_n^{-1}\|$ were bounded then we would have that $\|1 - xx_n^{-1}\| < 1$ for some n. Thus, we would have that $xx_n^{-1} \in G(A)$ and hence also $x \in G(A)$. ∎

Proposition 1.1.13. *Let B be a unital C^*-algebra and $A \subset B$ a unital C^*-subalgebra. If $x \in A$ then $\sigma_A(x) = \sigma_B(x)$.*

Proof. Note that we always have $G(A) \subset G(B)$. If $x \in A$ is self-adjoint such that $x \notin G(A)$, then by Lemma 1.1.11 we have $it \in \rho_A(x)$ for $t > 0$. By the previous lemma we then have

$$\lim_{t \to 0} \|(x - it)^{-1}\| = \infty,$$

and thus $x \notin G(B)$ since inversion is continuous in $G(B)$.

For general $x \in A$, if $x \in G(B)$ then $x^*x \in G(B)$ and hence $x^*x \in G(A)$ by the argument above, so that $(x^*x)^{-1} \in A$. We then have $x^{-1} = (x^*x)^{-1}x^* \in A$, so that $x \in G(A)$. ∎

Because of the previous result we will henceforth write simply $\sigma(x)$ for the spectrum of an element in a C^*-algebra.

Exercise 1.1.14. Suppose A is a unital Banach algebra, and $I \subset A$ is a closed two sided ideal, then A/I is again a unital Banach algebra, when given the norm $\|a + I\| = \inf_{y \in I} \|a + y\|$, and $(a + I)(b + I) = (ab + I)$.

1.1.1 Examples

If K is a topological space, then the set $C_b(K)$ of all complex valued bounded continuous functions is a C^*-algebra with the supremum norm $\|f\|_\infty = \sup_{x \in K} |f(x)|$. Of course, this C^*-algebra will be most interesting for completely Hausdorff spaces, i.e., those spaces for which any two distinct points can be separated by a continuous function into $\mathbb{R}$. We may similarly consider the C^*-subalgebra $C_0(K)$ which consists of those functions f which vanish at infinity, i.e., if $\{x_\alpha\}_\alpha \subset K$ is a net which eventually escapes every compact set, then $\lim_{\alpha \to \infty} f(x_\alpha) = 0$.

Another commutative example is given by considering (X, μ) a measure space. Then we let $L^\infty(X, \mu)$ be the space of complex valued essentially bounded functions with two functions identified if they agree almost everywhere. This is a C^*-algebra with the norm being given by the essential supremum.

For a noncommutative example consider a Hilbert space $\mathcal{H}$. Then the space of all bounded operators $\mathcal{B}(\mathcal{H})$ is a C^*-algebra when endowed with the operator norm $\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|$.

Proposition 1.1.15. *Let $\mathcal{H}$ be a Hilbert space and suppose $x \in \mathcal{B}(\mathcal{H})$, then $\|x^*x\| = \|x\|^2$.*

Proof. We clearly have $\|x^*x\| \leq \|x^*\|\|x\|$. Also, $\|x^*\| = \|x\|$ since

$$\|x\| = \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x\xi, \eta \rangle| = \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle \xi, x^*\eta \rangle| = \|x^*\|.$$

To see the reverse inequality just note that

$$\begin{aligned} \|x\|^2 &= \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \langle x\xi, x\xi \rangle \\ &\leq \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x\xi, x\eta \rangle| \\ &= \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x^*x\xi, \eta \rangle| = \|x^*x\|. \end{aligned}$$ ■

Let Γ be a group. If $\xi, \eta \in \ell^1\Gamma$, then the convolution $\xi * \eta \in \ell^1\Gamma$ is defined by $(\xi * \eta)(s) = \sum_{t \in \Gamma} \xi(st)\eta(t^{-1})$. Under convolution $\ell^1\Gamma$ becomes a Banach algebra. This is also an involutive Banach algebra with involution given by $\xi^*(s) = \overline{\xi(s^{-1})}$.

1.2 The Gelfand transform

Let A be a abelian Banach algebra, the **spectrum** of A, denoted by $\sigma(A)$, is the set of non-zero continuous homomorphsims $\varphi : A \to \mathbb{C}$, which we endow with the weak*-topology as a subset of A^*.

Note that if A is unital, and $\varphi : A \to \mathbb{C}$ is a homomorphism, then it follows easily that $\ker(\varphi) \cap G(A) = \emptyset$. In particular, this shows that $\varphi(x) \in \sigma(x)$, since $x - \varphi(x) \in \ker(\varphi)$. Hence, for all $x \in A$ we have $|\varphi(x)| \leq r(x) \leq \|x\|$. Since, $\varphi(1) = 1$ this shows that $\|\varphi\| = 1$. We also have $\|\varphi\| = 1$ in the non-unital case, this will follow from Theorem 2.1.1.

It is easy to see that when A is unital $\sigma(A)$ is closed and bounded, by the Banach-Alaoglu theorem it is then a compact Hausdorff space.

Proposition 1.2.1. *Let A be a unital abelian Banach algebra. Then the association $\varphi \mapsto \ker(\varphi)$ gives a bijection between the spectrum of A and the space of maximal ideals.*

Proof. If $\varphi \in \sigma(A)$ then $\ker(\varphi)$ is clearly an ideal, and if we have a larger ideal I, then there exists $x \in I$ such that $\varphi(x) \neq 0$, hence $1 - x/\varphi(x) \in \ker(\varphi) \subset I$ and so $1 = (1 - x/\varphi(x)) + x/\varphi(x) \in I$ which implies $I = A$.

Conversely, if $I \subset A$ is a maximal ideal, then $I \cap G(A) = \emptyset$ and hence $\|1 - y\| \geq 1$ for all $y \in I$. Thus, $\overline{I}$ is also an ideal and $1 \notin I$ which shows that $I = \overline{I}$ by maximality. We then have that A/I is a unital Banach algebra, and since I is maximal we have that all non-zero elements of A/I are invertible. Thus, by the Gelfand-Mazur theorem we have $A/I \cong \mathbb{C}$ and hence the projection map $\pi : A \to A/I \cong \mathbb{C}$ gives a continuous homomorphism with kernel I. ∎

Corollary 1.2.2. *Let A be a unital abelian Banach algebra, and $x \in A \setminus G(A)$. Then there exists $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$. In particular, $\sigma(A)$ is a non-empty compact set.*

Proof. If $x \notin G(A)$ then the ideal generated by x is proper, hence by Zorn's lemma we see that x is contained in a maximal ideal $I \subset A$, and from Proposition 1.2.1 there exists $\varphi \in \sigma(A)$ such that $\Gamma(x)(\varphi) = \varphi(x) = 0$.

Considering $x = 0$ shows that $\sigma(A) \neq \emptyset$. We leave it as an exercise to see that $\sigma(A)$ is a weak*-closed subset of A^*, which is then compact by the Banach-Alaoglu theorem. ∎

Suppose A is a unital abelian C^*-algebra which is generated (as a unital C^*-algebra) by a single (normal) element x, if $\lambda \in \sigma(x)$ then we can consider the closed ideal generated by $x - \lambda$ which is maximal since x generates A. This therefore induces a map from $\sigma(x)$ to $\sigma(A)$. We leave it to the reader to check that this map is actually a homeomorphism.

Theorem 1.2.3 (Stone)**.** *Let X be a topological space. For each $x \in X$ denote by $\beta_x : C_b(X) \to \mathbb{C}$ the homomorphism given by $\beta_x(f) = f(x)$, then $X \ni x \mapsto \beta_x \in \sigma(C(X))$ is a continuous map with dense image which satisfies the universal property that if $\pi : X \to K$ is any continuous map into a compact*

Hausdorff space K, then there exists a unique continuous map $\beta_\pi : \sigma(C(X)) \to K$, such that for $x \in X$ we have $\pi(x) = \beta_\pi(\beta_x)$. In particular, if X is a compact Hausdorff space then β is a homeomorphism.

Proof. If $\{x_i\} \subset X$ is a net such that $x_i \to x$, then for any $f \in C_b(X)$ we have $\beta_{x_i}(f) = f(x_i) \to f(x) = \beta_x(f)$, hence $\beta_{x_i} \to \beta_x$ in the weak*-topology. Thus, $x \mapsto \beta_x$ is continuous. To show that this map has dense image we suppose by way of contradiction that $\varphi \in \sigma(C_b(X))$ is not in the closure of $\beta(X)$, and set $I = \ker(\varphi)$.

If $\psi \in \overline{\beta(X)}$, then there exists $f_\psi \in I$ such that $f_\psi \notin \ker(\psi)$. Hence, for some $c_\psi > 0$, and an open neighborhood O_ψ of ψ, we have that $|\psi'(f)| > c_\psi$ for all $\psi' \in O_\psi$. As $\overline{\beta(X)}$ is compact we may take a finite subcover of the cover $\{O_\psi\}_{\psi \in \overline{\beta(X)}}$. Thus, we obtain $f_1, \ldots, f_n \in I$, and $c > 0$ such that $\sum_{i=1}^n \psi(|f|^2) > c$ for all $\psi \in \overline{\beta(X)}$. In particular we have $\sum_{i=1}^n |f|^2(x) = \beta_x(\sum_{i=1}^n |f|^2) > c$, for all $x \in X$. Thus $\sum_{i=1}^n |f|^2$ is an invertible element in I contradicting the fact that I is a proper ideal. Thus, we must have that $\overline{\beta(X)} = \sigma(C_b(X))$.

If X is a compact Hausdorff space then β is surjective since the image is dense and compact. Moreover, β is injective since $C_b(X)$ separates points. Hence, β is a homoeomorphism, being a continuous bijection between compact Hausdorff spaces.

In general, to see that $\beta : X \to \sigma(C_b(X))$ satisfies the above universal property, suppose that K is a compact Hausdorff space and $\pi : X \to K$ is continuous. We then obtain a continuous map $\pi^* : C(K) \to C_b(X)$ given by $\pi^*(f)(x) = f(\pi(x))$. Thus, we obtain the continuous map $\tilde{\pi} : \sigma(C_b(X)) \to \sigma(C(K))$ by $\tilde{\pi}(\varphi)(g) = \varphi(\pi^*(g))$. Since K is compact and Hausdorff we have established above that $\beta^K : K \to \sigma(C_b(K))$ is a homeomorphism. Thus, we obtain a continuous map $\beta_\pi : \sigma(C_b(X)) \to K$ by setting $\beta_\pi = {\beta^K}^{-1} \circ \tilde{\pi}$. If $x \in X$, and $g \in C(K)$ then we compute directly

$$\tilde{\pi}(\beta_x)(\varphi)(g) = \beta_x(\pi^*(g)) = \pi^*(g)(x) = g(\pi(x)) = \beta^K_{\pi(x)}(g).$$

Hence, $\beta_\pi(\beta_x) = \pi(x)$. ■

If X is a topological space, then the **Stone-Čech compactification** of X consists of a compact Hausdorff space βX, together with a continuous map $\beta : X \to \beta X$, which satisfies the universal property given in the previous theorem. If follows easily that, up to homeomorphism, this is uniquely defined by its universal property. The previous theorem shows that βX exists and may be identified with $\sigma(C_b(X))$. The following easy consequence (implicit already in Tychonoff's work) was obtained independently by Čech using different methods:

Corollary 1.2.4 (Stone, Čech)**.** *Let X be a topological space, then $\beta : X \to \beta X$ is a homeomorphism onto its image if and only if X is a Tychonoff space, i.e., X is Hausdorff and given any closed set $F \subset X$, and a point $x \in X \setminus F$, there exists a continuous function $f : X \to [0,1]$, such that $f(x) = 1$, and $f(y) = 0$, for all $y \in F$.*

Let A be a unital abelian Banach algebra, the **Gelfand transform** is the map $\Gamma : A \to C(\sigma(A))$ defined by

$$\Gamma(x)(\varphi) = \varphi(x).$$

Theorem 1.2.5. *Let A be a unital abelian Banach algebra, then the Gelfand transform is a contractive homomorphism, and $\Gamma(x)$ is invertible in $C(\sigma(A))$ if and only if x is invertible in A.*

Proof. It is easy to see that the Gelfand transform is a contractive homomorphism. Also, if $x \in G(A)$, then $\Gamma(a)\Gamma(a^{-1}) = \Gamma(aa^{-1}) = \Gamma(1) = 1$, hence $\Gamma(x)$ is invertible. Conversely, if $x \notin G(A)$ then by Corollary 1.2.2 there exists $\varphi \in \sigma(A)$ such that $\Gamma(x)(\varphi) = \varphi(x) = 0$. Hence, in this case $\Gamma(x)$ is not invertible. ∎

Corollary 1.2.6. *Let A be a unital abelian Banach algebra, then $\sigma(\Gamma(x)) = \sigma(x)$, and in particular $\|\Gamma(x)\| = r(\Gamma(x)) = r(x)$, for all $x \in A$.*

Theorem 1.2.7 (Gelfand)**.** *Let A be a unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C(\sigma(A))$ gives an isometric $*$-isomorphism between A and $C(\sigma(A))$.*

Proof. If x is self-adjoint then from Lemma 1.1.11 we have $\sigma(\Gamma(x)) = \sigma(x) \subset \mathbb{R}$, and hence $\overline{\Gamma(x)} = \Gamma(x^*)$. In general, if $x \in A$ we can write x as $x = a + ib$ where $a = \frac{x+x^*}{2}$ and $b = \frac{i(x^*-x)}{2}$ are self-adjoint. Hence, $\Gamma(x^*) = \Gamma(a - ib) = \Gamma(a) - i\Gamma(b) = \overline{\Gamma(a) + i\Gamma(b)} = \overline{\Gamma(x)}$ and so Γ is a $*$-homomorphism.

By Proposition 1.1.8 and the previous corollary, if $x \in A$ we have

$$\|x\| = r(x) = r(\Gamma(x)) = \|\Gamma(x)\|,$$

and so Γ is isometric, and in particular injective.

To show that Γ is surjective note that $\Gamma(A)$ is self-adjoint, and closed since Γ is isometric. Moreover, $\Gamma(A)$ contains the constants and clearly separates points, hence $\Gamma(A) = C(\sigma(A))$ by the Stone-Weierstrauss theorem. ∎

1.3 Continuous functional calculus

Let A be a C^*-algebra. An element $x \in A$ is:

- **positive** if $x = y^*y$ for some $y \in A$.
- a **projection** if $x^* = x^2 = x$.
- **unitary** if A is unital, and $x^*x = xx^* = 1$.
- **isometric** if A is unital, and $x^*x = 1$.
- **partially isometric** if x^*x is a projection.

We denote by A_+ the set of positive elements, and $a, b \in A$ are two self-adjoint elements then we write $a \leq b$ if $b - a \in A_+$. Note that if $x \in A$ then $x^* A_+ x \subset A_+$, in particular, if $a, b \in A$ are self-adjoint such that $a \leq b$, then $x^* a x \leq x^* b x$.

Since we have seen above that if A is generated as a unital C^*-algebra by a single normal element $x \in A$, then we have a natural homeomorphism $\sigma(x) \cong \sigma(A)$. Thus by considering the inverse Gelfand transform we obtain an isomorphism between $C(\sigma(x))$ and A which we denote by $f \mapsto f(x)$.

Theorem 1.3.1 (Continuous functional calculus)**.** *Let A and B be a unital C^*-algebras, with $x \in A$ normal, then this functional calculus satisfies the following properties:*

(i) The map $f \mapsto f(x)$ is a homomorphism from $C(\sigma(x))$ to A, and if $f(z, \overline{z}) = \sum_{j,k=0}^n a_{j,k} z^j \overline{z}^k$ is a polynomial, then $f(x) = \sum_{j,k=0}^n a_{j,k} x^j (x^*)^k$.

(ii) For $f \in C(\sigma(x))$ we have $\sigma(f(x)) = f(\sigma(x))$.

(iii) If $\Phi : A \to B$ is a C^*-homomorphism then $\Phi(f(x)) = f(\Phi(x))$.

(iv) If $x_n \in A$ is a sequence of normal elements such that $\|x_n - x\| \to 0$, Ω is a compact neighborhood of $\sigma(x)$, and $f \in C(\Omega)$, then for large enough n we have $\sigma(x_n) \subset \Omega$ and $\|f(x_n) - f(x)\| \to 0$.

Proof. Parts (i), and (ii) follow easily from Theorem 1.2.7. Part (iii) is obvious for polynomials and then follows for all continuous functions by approximation.

For part (iv), the fact that $\sigma(x_n) \subset \Omega$ for large n follows from continuity of inversion. If we write $C = \sup_n \|x_n\|$ and we have $\varepsilon > 0$ arbitrary, then we may take a polynomial $g : \Omega \to \mathbb{C}$ such that $\|f - g\|_\infty < \varepsilon$ and we have

$$\limsup_{n \to \infty} \|f(x_n) - f(x)\| \leq 2\|f - g\|_\infty C + \limsup_{n \to \infty} \|g(x_n) + g(x)\| < 2C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have $\lim_{n \to \infty} \|f(x_n) - f(x)\| = 0$. ∎

1.3.1 The non-unital case

If A is not a unital Banach-algebra then we may consider the space $\tilde{A} = A \oplus \mathbb{C}$ which is a algebra with multiplication

$$(x \oplus \alpha) \cdot (y \oplus \beta) = (xy + \alpha y + \beta x) \oplus \alpha\beta.$$

If, moreover, A has an involution $*$, then we may endow $\tilde{A}$ with an involution given by $(x \oplus \alpha)^* = x^* \oplus \overline{\alpha}$. We may place a norm on $\tilde{A}$ by setting $\|x \oplus \alpha\| = \|x\| + |\alpha|$, and in this way $\tilde{A}$ is a Banach algebra, the **unitization** of A, and the natural inclusion $A \subset \tilde{A}$ is an isometric inclusion.

If A is a C^*-algebra, then the norm defined above is not a C^*-norm. Instead, we may consider the norm given by

$$\|x \oplus \alpha\| = \sup_{y \in A, \|y\| \leq 1} \|xy + \alpha y\|.$$

In the setting of C^*-algebras we call $\tilde{A}$, with this norm, the **unitization** of A.

Proposition 1.3.2. *Let A be a non-unital C^*-algebra, then the unitization $\tilde{A}$ is again a C^*-algebra, and the map $x \mapsto x \oplus 0$ is an isometric $*$-isomorphism of A onto a maximal ideal in $\tilde{A}$.*

Proof. The map $x \mapsto x \oplus 0$ is indeed isometric since on one hand we have $\|x \oplus 0\| = \sup_{y \in A, \|y\| \leq 1} \|xy\| \leq \|x\|$, while on the other hand if $x \neq 0$, and we set $y = x^*/\|x^*\|$ then we have $\|x\| = \|xx^*\|/\|x^*\| = \|xy\| \leq \|x \oplus 0\|$.

The norm on $\tilde{A}$ is nothing but the operator norm when we view $\tilde{A}$ as acting on A by left multiplication and hence we have that this is at least a semi-norm such that $\|xy\| \leq \|x\|\|y\|$, for all $x, y \in \tilde{A}$. To see that this is actually a norm note that if $\alpha \neq 0$, but $\|x \oplus \alpha\| = 0$ then for all $y \in A$ we have $\|xy + \alpha y\| \leq \|x \oplus \alpha\|\|y\| = 0$, and hence $e = -x/\alpha$ is a left identity for A. Taking adjoints we see that e^* is a right identity for A, and then $e = ee^* = e^*$ is an identity for A which contradicts that A is non-unital. Thus, $\|\cdot\|$ is indeed a norm.

It is easy to see then that $\tilde{A}$ is then complete, and hence all that remains to be seen is the C^*-identity. Since, each for each $y \in A$, $\|y\| \leq 1$ we have $(y \oplus 0)^*(x \oplus \alpha) \in A \oplus 0 \cong A$ it follows that the C^*-identity holds here, and so

$$\begin{aligned} \|(x \oplus \alpha)^*(x \oplus \alpha)\| &\geq \|(y \oplus 0)^*(x \oplus \alpha)^*(x \oplus \alpha)(y \oplus 0)\| \\ &= \|(x \oplus \alpha)(y \oplus 0)\|^2. \end{aligned}$$

Taking the supremum over all $y \in A$, $\|y\| \leq 1$ we then have

$$\|(x \oplus \alpha)^*(x \oplus \alpha)\| \geq \|x \oplus \alpha\|^2 \geq \|(x \oplus \alpha)^*(x \oplus \alpha)\|.$$

Note that the C^*-identity also entails that the adjoint is isometric. Indeed, for $x \oplus \alpha \in \tilde{A}$ we have $\|x \oplus \alpha\|^2 = \|(x \oplus \alpha)^*(x \oplus \alpha)\| \leq \|(x \oplus \alpha)^*\|\|x \oplus \alpha\|$, and hence $\|x \oplus \alpha\| \leq \|(x \oplus \alpha)^*\|$, and the reverse inequality then follows from symmetry. ■

Lemma 1.3.3. *If A is a non-unital abelian C^*-algebra, then any multiplicative linear functional $\varphi \in \sigma(A)$ has a unique extension $\tilde{\varphi} \in \sigma(\tilde{A})$.*

Proof. If we consider $\tilde{\varphi}(x \oplus \alpha) = \varphi(x) + \alpha$ then the result follows easily. ■

In particular, this shows that $\sigma(A)$ is homeomorphic to $\sigma(\tilde{A}) \setminus \{\varphi_0\}$ where φ_0 is defined by $\varphi(x, \alpha) = \alpha$. Thus, $\sigma(A)$ is locally compact.

If $x \in A$ then the **spectrum** $\sigma(x)$ of x is defined to be the spectrum of $x \oplus 0 \in \tilde{A}$. Note that for a non-unital C^*-algebra A, since $A \subset \tilde{A}$ is an ideal it follows that $0 \in \sigma(x)$ whenever $x \in A$.

By considering the embedding $A \subset \tilde{A}$ we are able to extend the spectral theorem and continuous functional calculus to the non-unital setting. We leave the details to the reader.

Theorem 1.3.4. *Let A be a non-unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C_0(\sigma(A))$ gives an isometric isomorphism between A and $C_0(\sigma(A))$.*

Theorem 1.3.5. *Let A be a C^*-algebra, and $x \in A$ a normal element, then if $f \in C(\sigma(x))$ such that $f(0) = 0$, then $f(x) \in A \subset \tilde{A}$.*

Exercise 1.3.6. Suppose K is a non-compact, locally compact Hausdorff space, and $K \cup \{\infty\}$ is the one point compactification. Show that we have a natural isomorphism $C(K \cup \{\infty\}) \cong \widetilde{C_0(K)}$.

1.4 Applications of functional calculus

Given any element x in a C^*-algebra A, we can decompose x uniquely as a sum of a self-adoint and skew-adjoint elements $\frac{x+x^*}{2}$ and $\frac{x-x^*}{2}$. We refer to the self-adjoint elements $\frac{x+x^*}{2}$ and $i\frac{x^*-x}{2}$ the **real** and **imaginary** parts of x, note that the real and imaginary parts of x have norms no grater than that of x.

Proposition 1.4.1. *Let A be a unital C^*-algebra, then every element is a linear combination of four unitaries.*

Proof. If $x \in A$ is self-adjoint and $\|x\| \leq 1$, then $u = x + i(1 - x^2)^{1/2}$ is a unitary and we have $x = \frac{1}{2}(u + u^*)$. In general, we can decompose x into its real and imaginary parts and then write each as a linear combination of two unitaries. ∎

Also, if $x \in A$ is self-adjoint then from above we know that $\sigma(x) \subset \mathbb{R}$, hence by considering $x_+ = (0 \vee t)(x)$ and $x_- = -(0 \wedge t)(x)$ it follows easily from functional calculus that $\sigma(x_+), \sigma(x_-) \subset [0, \infty)$, $x_+x_- = x_-x_+ = 0$, and $x = x_+ - x_-$. We call x_+ and x_- the **positive** and **negative** parts of x.

1.4.1 The positive cone

Lemma 1.4.2. *Suppose we have self-adjoint elements $x, y \in A$ such that $\sigma(x), \sigma(y) \subset [0, \infty)$ then $\sigma(x + y) \subset [0, \infty)$.*

Proof. Let $a = \|x\|$, and $b = \|y\|$. Since x is self-adjoint and $\sigma(x) \subset [0, \infty)$ we have $\|a - x\| = r(a - x) = a$. Similarly we have $\|b - y\| = b$ and since $\|x + y\| \leq a + b$ we have

$$\sup_{\lambda \in \sigma(x+y)} \{a + b - \lambda\} = r((a + b) - (x + y)) = \|(a + b) - (x + y)\|$$
$$\leq \|x - a\| + \|y - b\| = a + b.$$

Therefore, $\sigma(x + y) \subset [0, \infty)$. ∎

Proposition 1.4.3. *Let A be a C^*-algebra. A normal element $x \in A$ is*

(i) self-adjoint if and only if $\sigma(x) \subset \mathbb{R}$.

(ii) positive if and only if $\sigma(x) \subset [0, \infty)$.

(iii) unitary if and only if $\sigma(x) \subset \mathbb{T}$.

(iv) a projection if and only if $\sigma(x) \subset \{0,1\}$.

Proof. Parts (i), (iii), and (iv) all follow easily by applying continuous functional calculus. For part (ii) if x is normal and $\sigma(x) \subset [0,\infty)$ then $x = (\sqrt{x})^2 = (\sqrt{x})^* \sqrt{x}$ is positive. It also follows easily that if $x = y^*y$ where y is normal then $\sigma(x) \subset [0,\infty)$. Thus, the difficulty arises only when $x = y^*y$ where y is perhaps not normal.

Suppose $x = y^*y$ for some $y \in A$, to show that $\sigma(x) \subset [0,\infty)$, decompose x into its positive and negative parts $x = x_+ - x_-$ as described above. Set $z = yx_-$ and note that $z^*z = x_-(y^*y)x_- = -x_-^3$, and hence $\sigma(zz^*) \subset \sigma(z^*z) \subset (-\infty, 0]$.

If $z = a+ib$ where a and b are self-adjoint, then we have $zz^* + z^*z = 2a^2 + 2b^2$, hence we also have $\sigma(zz^* + z^*z) \subset [0,\infty)$ and so by Lemma 1.4.2 we have $\sigma(z^*z) = \sigma((2a^2+2b^2) - zz^*) \subset [0,\infty)$. Therefore $\sigma(-x_-^3) = \sigma(z^*z) \subset \{0\}$ and since x_- is normal this shows that $x_-^3 = 0$, and consequently $x_- = 0$. ∎

Corollary 1.4.4. *Let A be a C^*-algebra. An element $x \in A$ is a partial isometry if and only if x^* is a partial isometry.*

Proof. Since x^*x is normal, it follows from the previous proposition that x is a partial isometry if and only if $\sigma(x^*x) \subset \{0,1\}$. Since $\sigma(x^*x) \cup \{0\} = \sigma(xx^*) \cup \{0\}$ this gives the result. ∎

Corollary 1.4.5. *Let A be a C^*-algebra, then the set of positive elements forms a closed cone. Moreover, if $a \in A$ is self-adjoint, and A is unital, then we have $a \leq \|a\|$.*

Proposition 1.4.6. *Let A be a C^*-algebra, and suppose $x, y \in A_+$ such that $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$. Moreover, if A is unital and $x, y \in A$ are invertible, then $y^{-1} \leq x^{-1}$.*

Proof. First consider the case that A is unital and x and y are invertible, then we have
$$y^{-1/2} x y^{-1/2} \leq 1,$$
hence
$$\begin{aligned} x^{1/2} y^{-1} x^{1/2} &\leq \|x^{1/2} y^{-1} x^{1/2}\| = r(x^{1/2} y^{-1} x^{1/2}) \\ &= r(y^{-1/2} x y^{-1/2}) \leq 1. \end{aligned}$$
Conjugating by $x^{-1/2}$ gives $y^{-1} \leq x^{-1}$.

We also have
$$\|y^{-1/2} x^{1/2}\|^2 = \|y^{-1/2} x y^{-1/2}\| \leq 1,$$
therefore
$$\begin{aligned} y^{-1/4} x^{1/2} y^{-1/4} &\leq \|y^{-1/4} x^{1/2} y^{-1/4}\| = r(y^{-1/4} x^{1/2} y^{-1/4}) \\ &= r(y^{-1/2} x^{1/2}) \leq \|y^{-1/2} x^{1/2}\| \leq 1. \end{aligned}$$
Conjugating by $y^{1/4}$ gives $x^{1/2} \leq y^{1/2}$.

In the general case we may consider the unitization of A, and note that if $\varepsilon > 0$, then we have $0 \leq x + \varepsilon \leq y + \varepsilon$, where $x + \varepsilon$, and $y + \varepsilon$ are invertible, hence from above we have

$$(x + \varepsilon)^{1/2} \leq (y + \varepsilon)^{1/2}.$$

Taking the limit as $\varepsilon \to 0$ we obtain the result. ■

In general, a continuous real valued function f defined on an interval I is said to be **operator monotone** if $f(a) \leq f(b)$ whenever $\sigma(a), \sigma(b) \subset I$, and $a \leq b$. The previous proposition shows that the functions $f(t) = \sqrt{t}$, and $f(t) = -1/t$, $t > 0$ are operator monotone.

Note that if $x \in A$ is an arbitrary element of a C^*-algebra A, then x^*x is positive and hence we may define the **absolute value** of x as the unique element $|x| \in A_+$ such that $|x|^2 = x^*x$.

Corollary 1.4.7. *Let A be a C^*-algebra, then for $x, y \in A$ we have $|xy| \leq \|x\||y|$.*

Proof. Since $|xy|^2 = y^*x^*xy \leq \|x\|^2 y^*y$, this follows from the previous proposition. ■

1.4.2 Extreme points

Given a involutive normed algebra A, we denote by $(A)_1$ the unit ball of A, and $A_{\text{s.a.}}$ the subspace of self-adjoint elements.

Proposition 1.4.8. *Let A be a C^*-algebra.*

(i) The extreme points of $(A_+)_1$ are the projections of A.

(ii) The extreme points of $(A_{\text{s.a.}})_1$ are the self-adjoint unitaries in A.

(iii) Every extreme point of $(A)_1$ is a partial isometry in A.

Proof. (i) If $x \in (A_+)_1$, then we have $x^2 \leq 2x$, and $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$. Hence if x is an extreme point then we have $x = x^2$ and so x is a projection. For the converse we first consider the case when A is abelian, and so we may assume $A = C_0(K)$ for some locally compact Hausdorff space K. If x is a projection then $x = 1_E$ is the characteristic function on some open and closed set $E \subset K$, hence the result follows easily from the fact that 0 and 1 are extreme points of $[0, 1]$.

For the general case, suppose $p \in A$ is a projection, if $p = \frac{1}{2}(a + b)$ then $\frac{1}{2}a = p - b \leq p$, and hence $0 \leq (1 - p)a(1 - p) \leq 0$, thus $a = ap = pa$. We therefore have that a, b, and p commute and hence the result follows from the abelian case.

(ii) First note that if A is unital then 1 is an extreme point in the unit ball. Indeed, if $1 = \frac{1}{2}(a + b)$ where $a, b \in (A)_1$, then we have the same equation when replacing a and b by their real parts. Thus, assuming a and b are self-adjoint we have $\frac{1}{2}a = 1 - \frac{1}{2}b$ and hence a and b commute. By considering the unital C^*-subalgebra generated by a and b we may assume $A = C(K)$ for some compact Hausdorff space K, and then it is an easy exercise to conclude that $a = b = 1$.

If u is a unitary in A, then the map $x \mapsto ux$ is a linear isometry of A, thus since 1 is an extreme point of $(A)_1$ it follows that u is also an extreme point. In particular, if u is self-adjoint then it is an extreme point of $(A_{\text{s.a.}})_1$.

Conversely, if $x \in (A_{\text{s.a.}})_1$ is an extreme point then if $x_+ = \frac{1}{2}(a+b)$ for $a, b \in (A_+)_1$, then $0 = x_- x_+ x_- = \frac{1}{2}(x_- a x_- + x_- b x_-) \geq 0$, hence we have $(a^{1/2}x_-)^*(a^{1/2}x_-) = x_- a x_- = 0$. We conclude that $ax_- = x_- a = 0$, and similarly $bx_- = x_- b = 0$. Thus, $a - x_-$ and $b - x_-$ are in $(A_{\text{s.a.}})_1$ and $x = \frac{1}{2}((a - x_-) + (b - x_-))$. Since x is an extreme point we conclude that $x = a - x_- = b - x_-$ and hence $a = b = x_+$.

We have shown now that x_+ is an extreme point in $(A_+)_1$ and thus by part (i) we conclude that x_+ is a projection. The same argument shows that x_- is also a projection, and thus x is a self-adjoint unitary.

(iii) If $x \in (A)_1$ such that x^*x is not a projection then by applying functional calculus to x^*x we can find an element $y \in A_+$ such that $x^*xy = yx^*x \neq 0$, and $\|x(1 \pm y)\|^2 = \|x^*x(1 \pm y)^2\| \leq 1$. Since $xy \neq 0$ we conclude that $x = \frac{1}{2}((x + xy) + (x - xy))$ is not an extreme point of $(A)_1$. ■

Chapter 2

Representations and states

2.1 Approximate identities

If A is a Banach algebra, then a **left (resp. right) approximate identity** consists of a uniformly bounded net $\{a_\lambda\}_\lambda$ such that $\|a_\lambda x - x\| \to 0$, for all $x \in A$. An **approximate identity** is a net which is both a left and right approximate identity. If A is a Banach algebra, then the **opposite algebra** $A^{\rm op}$ is the Banach algebra which has the same Banach space structure as A, but with a new multiplication given by $x \cdot y = yx$. Then A has a left approximate identity if and only if $A^{\rm op}$ has a right approximate identity.

Theorem 2.1.1. *Let A be a C^*-algebra, and let $I \subset A$ be a left ideal, then there exists an increasing net $\{a_\lambda\}_\lambda \subset I$ of positive elements such that for all $x \in I$ we have*

$$\|xa_\lambda - x\| \to 0.$$

In particular, I has a right approximate identity. Moreover, if A is separable then the net can be taken to be a sequence.

Proof. Consider Λ to be the set of all finite subsets of $I \subset A \subset \tilde{A}$, ordered by inclusion. If $\lambda \in \Lambda$ we consider

$$h_\lambda = \sum_{x \in \lambda} x^* x, \qquad a_\lambda = |\lambda| h_\lambda (1 + |\lambda| h_\lambda)^{-1}.$$

Then we have $a_\lambda \in I$ and $0 \leq a_\lambda \leq 1$. If $\lambda \leq \lambda'$ then we clearly have $h_\lambda \leq h_{\lambda'}$ and hence by Proposition 1.4.6 we have that

$$\frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_{\lambda}\right)^{-1}.$$

Therefore

$$a_\lambda = 1 - \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_\lambda\right)^{-1} \leq 1 - \frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} = a_{\lambda'}.$$

If $y \in \lambda$ then we have

$$(y(1-a_\lambda))^*(y(1-a_\lambda)) \leq \sum_{x \in \lambda} (x(1-a_\lambda))^*(x(1-a_\lambda)) = (1-a_\lambda)h_\lambda(1-a_\lambda).$$

But $\|(1-a_\lambda)h_\lambda(1-a_\lambda)\| = \|h_\lambda(1+|\lambda|h_\lambda)^{-2}\| \leq \frac{1}{4|\lambda|}$, from which it follows easily that $\|y - ya_\lambda\| \to 0$, for all $y \in I$.

If A is separable then so is $\overline{I}$, hence there exists a countable subset $\{x_n\}_{n\in\mathbb{N}} \subset I$ which is dense in I. If we take $\lambda_n = \{x_1, \ldots, x_n\}$, then clearly $a_n = a_{\lambda_n}$ also satisfies

$$\|y - ya_n\| \to 0. \qquad \blacksquare$$

If I is self-adjoint then we also have $\|a_\lambda x - x\| = \|x^* a_\lambda - x^*\| \to 0$ and in this case $\{a_\lambda\}$ is an approximate identity. Taking $I = A$ we obtain the following corollary.

Corollary 2.1.2. *Every C^*-algebra has an approximate identity consisting of an increasing net of positive elements.*

Using the fact that the adjoint is an isometry we also obtain the following corollary.

Corollary 2.1.3. *Let A be a C^*-algebra, and $I \subset A$ a closed two sided ideal. Then I is self-adjoint. In particular, I is a C^*-algebra.*

Exercise 2.1.4. Show that if A is a C^*-algebra such that $x \leq y \implies x^2 \leq y^2$, for all $x, y \in A_+$, then A is abelian.

Exercise 2.1.5. Let A be a C^*-algebra and $I \subset A$ a non-trivial closed two sided ideal. Show that A/I is again a C^*-algebra.

2.2 The Cohen-Hewitt factorization theorem

Let A be a Banach algebra with a left approximate identity. If X is a Banach space and $\pi : A \to \mathcal{B}(X)$ is a continuous representation, then a point $x \in X$ is a point of continuity if $\lim_{\lambda\to\infty} \|\pi(e_\lambda)x - x\| \to 0$, for some left approximate identity $\{e_\lambda\}$. Note that if $\{\tilde{e}_\alpha\}$ is another left approximate identity and $x \in X$ is a point of continuity, then we have $\lim_{\alpha\to\infty} \|\tilde{e}_\alpha e_\lambda x - e_\lambda x\| = 0$, for each λ, and hence it follows that x is a point of continuity with respect to any left approximate identity. We denote by X_c the set of points of continuity.

Theorem 2.2.1 (The Cohen-Hewitt factorization theorem)**.** *Let A be a Banach algebra with a left approximate identity, X a Banach space, and $\pi : A \to \mathcal{B}(X)$ a continuous representation. Then X_c is a closed invariant subspace, and we have $X_c = \pi(A)X$.*

Proof. It's easy to see that X_c is a closed invariant subspace, and it is also easy to see that $\pi(A)X \subset X_c$. Thus, it suffices to show $X_c \subset \pi(A)X$. To show this, we consider the Banach algebra unitization $\tilde{A}$, and extend π to a representation $\tilde{\pi} : \tilde{A} \to \mathcal{B}(X)$ by $\tilde{\pi}(x, \alpha) = \pi(x) + \alpha$.

Let $\{e_i\}_{i \in I}$ denote a left approximate unit, and set $M = \sup_i \|e_i\|$, so that $1 \leq M < \infty$. Set $\gamma = 1/4M$. We claim that $\gamma e_i + (1 - \gamma)$ is invertible and $\lim_{i \to \infty} \tilde{\pi}((\gamma e_i + (1-\gamma))^{-1})x = x$, for all $x \in X_c$. Indeed, we have

$$\|\gamma e_i - \gamma\| = \gamma(\|e_i\| + 1) \leq (M+1)/4M \leq 1/2. \tag{2.1}$$

Thus, $\gamma e_i + (1-\gamma)$ is invertible, and we have $(\gamma e_i + (1-\gamma))^{-1} = \sum_{k=0}^{\infty}(\gamma - \gamma e_i)^k$, hence

$$\|(\gamma e_i + (1-\gamma))^{-1}\| \leq \sum_{k=0}^{\infty} \gamma^k (1+M)^k \leq 2. \tag{2.2}$$

We then have

$$\lim_{i \to \infty} \tilde{\pi}((\gamma e_i + (1-\gamma))^{-1})x = \lim_{i \to \infty} \tilde{\pi}((\gamma e_i + (1-\gamma))^{-1}(\gamma e_i + (1-\gamma)))x = x.$$

Fix $x \in X_c$. We set $a_0 = 1$, and inductively define a sequence of invertible elements $\{a_n\} \subset \tilde{A}$, satisfying the following properties:

- $a_n - (1-\gamma)^n \in A$.
- $\|a_n - a_{n-1}\| < 2^{-n} + (1-\gamma)^{n-1}$.
- $\|a_n^{-1}\| \leq 2^n$.
- $\|\pi(a_n^{-1})x - \pi(a_{n-1}^{-1})x\| < 2^{-n}$.

Indeed, suppose that a_{n-1} has been constructed satisfying the above properties. Since $\{e_i\}_{i \in I}$ is a left approximate unit for A, and since $a_{n-1} - (1-\gamma)^{n-1} \in A$, there exists $i \in I$ such that $\|(e_i - 1)(a_{n-1} - (1-\gamma)^{n-1})\| < 2^{-n}$. Moreover from above we may choose i so that it also satisfies $\|\pi((\gamma e_i + (1-\gamma))^{-1})x - x\| < 2^{-2n}$.

If we set $a_n = (\gamma e_i + (1-\gamma))a_{n-1}$ then

$$\begin{aligned} a_n - (1-\gamma)^n &= (\gamma e_i + (1-\gamma))a_{n-1} - (1-\gamma)^n \\ &= \gamma e_i a_{n-1} + (1-\gamma)(a_{n-1} - (1-\gamma)^{n-1}) \in A, \end{aligned}$$

and from (2.1) we then have

$$\begin{aligned} \|a_n - a_{n-1}\| &= \|(\gamma e_i - \gamma)a_{n-1}\| \\ &\leq \|(\gamma e_i - \gamma)(a_{n-1} - (1-\gamma)^{n-1})\| + (1-\gamma)^{n-1}\|\gamma e_i - \gamma\| \\ &< 2^{-n} + (1-\gamma)^{n-1}. \end{aligned}$$

Moreover, from (2.2) we have

$$\|a_n^{-1}\| \leq \|a_{n-1}^{-1}\| \|(\gamma e_i + (1-\gamma))^{-1}\| \leq 2^n,$$

and hence

$$\|\pi(a_n^{-1})x - \pi(a_{n-1}^{-1})x\| \leq \|a_{n-1}^{-1}\|\|\pi((\gamma e_i + (1-\gamma))^{-1})x - x\| < 2^{-n}.$$

Thus, we have that $\{a_n\}$ and $\{\tilde{\pi}(a_n^{-1})x\}$ are Cauchy and hence converge to elements a and y respectively. Note that since $a_n - (1-\gamma)^n \in A$ it follows that $a \in A$. We then have $x = \lim_{n\to\infty} \tilde{\pi}(a_n)(\tilde{\pi}(a_n^{-1})x) = \pi(a)y \in \pi(A)X$. ∎

Corollary 2.2.2. *Let A be a Banach algebra with a left or right approximate identity. Then $A^2 = A$.*

Proof. If A has a left approximate identity then by considering left multiplication we obtain a representation of A into $\mathcal{B}(A)$ such that every point is a point of continuity. Hence the Cohen-Hewitt factorization theorem gives the result. If A has a right approximate identity then $A^{\rm op}$ has a left approximate identity the result again follows. ∎

2.3 States

If A is a C^*-algebra then A^* is a Banach space which is also an A-bimodule given by $(a \cdot \psi \cdot b)(x) = \psi(bxa)$. Moreover, the bimodule structure is continuous since

$$\|a \cdot \psi \cdot b\| = \sup_{x\in(A)_1} |\psi(bxa)| \leq \sup_{x\in(A)_1} \|\psi\|\|bxa\| \leq \|\psi\|\|b\|\|a\|.$$

A linear functional $\varphi : A \to \mathbb{C}$ on a C^*-algebra A is **positive** if $\varphi(x) \geq 0$, whenever $x \in A_+$. Note that if $\varphi : A \to \mathbb{C}$ is positive then so is $a^* \cdot \varphi \cdot a$ for all $a \in A$. A positive linear functional is **faithful** if $\varphi(x) \neq 0$ for every non-zero $x \in A_+$, and a **state** if φ is positive, and $\|\varphi\| = 1$. The state space $S(A)$ is a convex closed subspace of the unit ball of A^*, and as such it is a compact Hausdorff space when endowed with the weak*-topology.

Note that if $\varphi \in S(A)$ then for all $x \in A$, $x = x^*$, then $\varphi(x) = \varphi(x_+ - x_-) \in \mathbb{R}$. Hence, if $y \in A$ then writing $y = y_1 + iy_2$ where y_j are self-adjoint for $j = 1, 2$, we have $\varphi(y^*) = \varphi(y_1) - i\varphi(y_2) = \overline{\varphi(y)}$. In general, we say a functional is **Hermitian** if $\varphi(y^*) = \overline{\varphi(y)}$, for all $y \in A$. Note that by defining $\varphi^*(y) = \overline{\varphi(y^*)}$ then we have that $\varphi + \varphi^*$, and $i(\varphi - \varphi^*)$ are each Hermitian.

Also note that a positive linear functional $\varphi : A \to \mathbb{C}$ is bounded. Indeed, if $\{x_n\}_n$ is any sequence of positive elements in $(A)_1$ then for any $(a_n)_n \in \ell^1\mathbb{N}$ we have $\sum_n a_n\varphi(x_n) = \varphi(\sum_n a_n x_n) < \infty$. This shows that $(\varphi(x_n))_n \in \ell^\infty\mathbb{N}$ and since the sequence was arbitrary we have that φ is bounded on the set of positive elements in $(A)_1$. Writing an element x in the usual way as $x = x_1 - x_2 + ix_3 - ix_4$ then shows that φ is bounded on the whole unit ball.

Lemma 2.3.1. *Let $\varphi : A \to \mathbb{C}$ be a positive linear functional on a C^*-algebra A, then for all $x, y \in A$ we have $|\varphi(y^*x)|^2 \leq \varphi(y^*y)\varphi(x^*x)$.*

Proof. Since φ is positive, the sesquilinear form defined by $\langle x, y\rangle = \varphi(y^*x)$ is non-negative definite. Thus, the result follows from the Cauchy-Schwarz inequality. ■

Lemma 2.3.2. *Suppose A is a unital C^*-algebra. A linear functional $\varphi : A \to \mathbb{C}$ is positive if and only $\|\varphi\| = \varphi(1)$.*

Proof. First suppose φ is a positive linear functional, then for all $x \in A$ we have $\varphi(\|x + x^*\| \pm (x + x^*)) \geq 0$. Since φ is Hermitian we then have

$$|\varphi(x)| = |\varphi(\frac{x + x^*}{2})| \leq \|\frac{x + x^*}{2}\|\varphi(1) \leq \|x\|\varphi(1),$$

showing $\|\varphi\| \leq \varphi(1) \leq \|\varphi\|$.

Now suppose $\|\varphi\| = \varphi(1)$, and $x \in A$ is a positive element such that $\varphi(x) = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. For all $t \in \mathbb{R}$ we have

$$\begin{aligned} \alpha^2 + (\beta + t\|\varphi\|)^2 &= |\varphi(x + it)|^2 \\ &\leq \|x + it\|^2\|\varphi\|^2 = (\|x\|^2 + t^2)\|\varphi\|^2. \end{aligned}$$

Subtracting $t^2\|\varphi\|^2$ from both sides of this inequality shows $2\beta t\|\varphi\| \leq \|x\|^2\|\varphi\|$, thus $\beta = 0$.

Also, we have

$$\|x\|\|\varphi\| - \varphi(x) = \varphi(\|x\| - x) \leq \|\|x\| - x\|\|\varphi\| \leq \|x\|\|\varphi\|,$$

hence $\alpha > 0$. ■

Proposition 2.3.3. *If $\varphi : A \to \mathbb{C}$ is a positive linear functional on a C^*-algebra A, then φ has a unique extension to a positive linear functional $\tilde{\varphi}$ on the unitization $\tilde{A}$, such that $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. Suppose $\varphi : A \to \mathbb{C}$ is a positive linear functional. If $\{a_\lambda\}_\lambda$ is an approximate identity consisting of positive contractions as given by Theorem 2.1.1, then we have that $\varphi(a_\lambda^2)$ is a bounded net and hence has a cluster point $\beta > 0$. If $x \in A$, $\|x\| \leq 1$, then $|\varphi(x)| = \lim_{\lambda\to\infty} |\varphi(a_\lambda x)| \leq \liminf_{\lambda\to\infty} \varphi(a_\lambda^2)^{1/2}\varphi(x^*x)^{1/2} \leq \beta^{1/2}\|x\|\|\varphi\|^{1/2}$. Thus, we have $\|\varphi\| \leq \beta$ and hence $\beta = \|\varphi\|$, since we also have $\varphi(a_\lambda^2) \leq \|\varphi\|$, for all λ. Since β was an arbitrary cluster point we then have $\|\varphi\| = \lim_{\lambda\to\infty} \varphi(a_\lambda)$.

If we define $\tilde{\varphi}$ on $\tilde{A}$ by $\tilde{\varphi}(x, \alpha) = \varphi(x) + \alpha\|\varphi\|$, then for all $x \in A$, and $\alpha \in \mathbb{C}$ we then have $\tilde{\varphi}(x, \alpha) = \lim_{\lambda\to\infty} \varphi(a_\lambda x a_\lambda + \alpha a_\lambda^2)$. Thus, we have

$$\tilde{\varphi}((x, \alpha)^*(x, \alpha)) = \lim_{\lambda\to\infty} \varphi((xa_\lambda + a_\lambda)^*(xa_\lambda + a_\lambda)) \geq 0.$$

Uniqueness of such an extension follows from the previous lemma. ■

Proposition 2.3.4. *Let A be a C^*-algebra and $x \in A$. For each $\lambda \in \sigma(x)$ there exists a state $\varphi \in S(A)$ such that $\varphi(x) = \lambda$.*

Proof. By considering the unitization, we may assume that A is unital. Consider the subspace $\mathbb{C}x + \mathbb{C}1 \subset A$, with the linear functional φ_0 on this space defined by $\varphi_0(\alpha x + \beta) = \alpha\lambda + \beta$, for $\alpha, \beta \in \mathbb{C}$. Since $\varphi_0(\alpha x + \beta) \in \sigma(\alpha x + \beta)$ we have that $\|\varphi_0\| = 1$.

By the Hahn-Banach theorem there exists an extension $\varphi : A \to \mathbb{C}$ such that $\|\varphi\| = 1 = \varphi(1)$. By Lemma 2.3.2 $\varphi \in S(A)$, and we have $\varphi(x) = \lambda$. ■

Proposition 2.3.5. *Let A be a C^*-algebra, and $x \in A$.*

(i) $x = 0$ if and only if $\varphi(x) = 0$ for all $\varphi \in S(A)$.

(ii) x is self-adjoint if and only if $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(A)$.

(iii) x is positive if and only if $\varphi(x) \geq 0$ for all $\varphi \in S(A)$.

Proof. (i) If $\varphi(x) = 0$ for all $\varphi \in S(A)$ then writing $x = x_1 + ix_2$ where $x_j = x_j^*$, for $j = 1, 2$, we have $\varphi(x_j) = 0$ for all $\varphi \in S(A)$, $j = 1, 2$. Thus, $x_1 = x_2 = 0$ by Proposition 2.3.4

(ii) If $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(A)$ then $\varphi(x - x^*) = \varphi(x) - \overline{\varphi(x)} = 0$, for all $\varphi \in S(A)$. Hence $x - x^* = 0$.

(iii) If $\varphi(x) \geq 0$ for all $\varphi \in S(A)$ then $x = x^*$ and by Proposition 2.3.4 we have $\sigma(x) \subset [0, \infty)$. ■

2.3.1 The Gelfand-Naimark-Segal construction

A **representation** of a C^*-algebra A is a $*$-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$. If $\mathcal{K} \subset \mathcal{H}$ is a closed subspace such that $\pi(x)\mathcal{K} \subset \mathcal{K}$ for all $x \in A$ then the restriction to this subspace determines a **sub-representation**. If the only sub-representations are the restrictions to $\{0\}$ or $\mathcal{H}$ then π is **irreducible** , which by the double commutant theorem is equivalent to the von Neumann algebra generated by $\pi(A)$ being $\mathcal{B}(\mathcal{H})$. Two representations $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\rho : A \to \mathcal{B}(\mathcal{K})$ are **equivalent** if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that $U\pi(x) = \rho(x)U$, for all $x \in A$.

If $\pi : A \to \mathcal{B}(\mathcal{H})$ is a representation, and $\xi \in \mathcal{H}$, $\|\xi\| = 1$ then we obtain a state on A by the formula $\varphi_\xi(x) = \langle \pi(x)\xi, \xi \rangle$. Indeed, if $x \in A$ then $\langle \pi(x^*x)\xi, \xi \rangle = \|\pi(x)\xi\|^2 \geq 0$. We now show that every state arises in this way.

Theorem 2.3.6 (The GNS construction)**.** *Let A be a unital C^*-algebra, and consider $\varphi \in S(A)$, then there exists a Hilbert space $L^2(A, \varphi)$, and a unique (up to equivalence) representation $\pi : A \to \mathcal{B}(L^2(A, \varphi))$, with a unit cyclic vector $1_\varphi \in L^2(A, \varphi)$ such that $\varphi(x) = \langle \pi(x)1_\varphi, 1_\varphi \rangle$, for all $x \in A$.*

Proof. By Corollary 2.3.3 we may assume that A is unital. Consider $A_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$. By Lemma 2.3.1 we have that $A_\varphi = \{x \in A \mid \varphi(yx) = 0, y \in A\}$, and from this we see that N_φ is a closed linear subspace. We also see that N_φ is a left ideal since for $x \in N_\varphi$ Lemma 2.3.1 gives $\varphi((ax)^*(ax)) \leq \varphi(x^*x)^{1/2}\varphi(x^*(a^*a)^2x)^{1/2} = 0$.

We consider $\mathcal{H}_0 = A/N_\varphi$ which we endow with the inner product $\langle [x], [y] \rangle = \varphi(y^*x)$, where $[x]$ denotes the equivalence class of x in A/N_φ, (this is well defined since N_φ is a left ideal). This inner product is then positive definite, and hence we denote by $L^2(A,\varphi)$ the Hilbert space completion.

For $a \in A$ we consider the map $\pi_0(a) : \mathcal{H}_0 \to \mathcal{H}_0$ given by $\pi_0(a)[x] = [ax]$. Since N_φ is a left ideal this is well defined, and since $\|\pi_0(a)[x]\|^2 = \varphi((ax)^*(ax)) \leq \|a\|^2\varphi(x^*x)$ we have that this extends to a bounded operator $\pi(a) \in \mathcal{B}(L^2(A,\varphi))$ such that $\|\pi(a)\| \leq \|a\|$. The map $a \mapsto \pi(a)$ is clearly a homomorphism, and for $x, y \in A$ we have $\langle [x], \pi(a^*)[y]\rangle = \varphi(y^*a^*x) = \langle \pi(a)[x], [y]\rangle$, thus $\pi(a^*) = \pi(a)^*$. Also, if we consider $1_\varphi = [1] \in \mathcal{H}_0 \subset L^2(A,\varphi)$ then we have $\langle \pi(a)1_\varphi, 1_\varphi\rangle = \varphi(a)$.

If $\rho : A \to \mathcal{B}(\mathcal{K})$ and $\eta \in \mathcal{K}$ is a cyclic vector such that $\varphi(a) = \langle \rho(a)\eta, \eta\rangle$, then we can consider the map $U_0 : \mathcal{H}_0 \to \mathcal{K}$ given by $U_0([x]) = \rho(x)\eta$. We then have

$$\langle U_0([x]), U_0([y])\rangle = \langle \rho(x)\eta, \rho(y)\eta = \langle \rho(y^*x)\eta, \eta = \varphi(y^*x) = \langle [x], [y]\rangle$$

which shows that U_0 is well defined and isometric. Also, for $a, x \in A$ we have

$$U_0(\pi(a)[x]) = U_0([ax]) = \rho(ax)\eta = \rho(a)U_0([x]).$$

Hence, U_0 extends to an isometry $U : L^2(A,\varphi) \to \mathcal{K}$ such that $U\pi(a) = \rho(a)U$ for all $a \in A$. Since η is cyclic, and $\rho(A)\eta \subset U(L^2(A,\varphi))$ we have that U is unitary. ∎

Corollary 2.3.7. *Let A be a C^*-algebra, then there exists a faithful representation.*

Proof. If we let π be the direct sum over all GNS representations corresponding to states, then this follows easily from Proposition 2.3.5. Note also that if A is separable, then so is $S(A)$ and by considering a countable dense subset of $S(A)$ we can construct a faithful representation onto a separable Hilbert space. ∎

If φ and ψ are two Hermitian linear functionals, we write $\varphi \leq \psi$ if $\varphi(a) \leq \psi(a)$ for all $a \in A_+$, alternatively, this is if and only if $\psi - \varphi$ is a positive linear functional. The following is a Radon-Nikodym type theorem for positive linear functionals.

Proposition 2.3.8. *Suppose φ and ψ are positive linear functionals on a C^*-algebra A such that ψ is a state. Then $\varphi \leq \psi$, if and only if there exists a unique $y \in \pi_\psi(A)'$ such that $0 \leq y \leq 1$ and $\varphi(a) = \langle \pi_\psi(a)y1_\psi, 1_\psi\rangle$ for all $a \in A$.*

Proof. First suppose that $y \in \pi_\psi(A)'$, with $0 \leq y \leq 1$. Then for all $a \in A$, $a \geq 0$ we have $\pi_\psi(a)y = \pi_\psi(a)^{1/2}y\pi_\psi(a)^{1/2} \leq \pi_\psi(a)$, hence $\langle \pi_\psi(a)y1_\psi, 1_\psi\rangle \leq \langle \pi_\psi(a)1_\psi, 1_\psi\rangle = \psi(a)$.

Conversely, if $\varphi \leq \psi$, the Cauchy-Schwarz inequality implies

$$|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b) \leq \psi(a^*a)\psi(b^*b) = \|\pi_\psi(a)1_\psi\|^2\|\pi_\psi(b)1_\psi\|^2.$$

Thus $\langle \pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi\rangle_\varphi = \varphi(b^*a)$ is a well defined non-negative definite sesquilinear form on $\pi_\psi(A)1_\psi$ which is bounded by 1, and hence extends to the closure $L^2(A,\psi)$.

Therefore there is an operator $y \in \mathcal{B}(L^2(A, \psi))$, $0 \leq y \leq 1$, such that $\varphi(b^*a) = \langle y\pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle$, for all $a, b \in A$.

If $a, b, c \in A$ then

$$\begin{aligned} \langle y\pi_\psi(a)\pi_\psi(b)1_\psi, \pi_\psi(c)1_\psi \rangle &= \langle y\pi_\psi(ab)1_\psi, \pi_\psi(c)1_\psi \rangle = \varphi(c^*ab) \\ &= \langle y\pi_\psi(b)1_\psi, \pi_\psi(a^*)\pi_\psi(c)1_\psi \rangle \\ &= \langle \pi_\psi(a)y\pi_\psi(b)1_\psi, \pi_\psi(c)1_\psi \rangle. \end{aligned}$$

Thus, $y\pi_\psi(a) = \pi_\psi(a)y$, for all $a \in A$.

To see that y is unique, suppose that $0 \leq z \leq 1$, $z \in \pi_\psi(A)'$ such that $\langle \pi_\psi(a)z1_\psi, 1_\psi \rangle = \langle \pi_\psi(a)y1_\psi, 1_\psi \rangle$ for all $a \in A$. Then $\langle (z-y)1_\psi, \pi_\psi(a^*)1_\psi \rangle = 0$ for all $a \in A$ and hence $z - y = 0$ since 1_ψ is a cyclic vector for $\pi_\psi(A)$. ∎

2.3.2 Pure states

A state φ on a C^*-algebra A is said to be **pure** if it is an extreme point in $S(A)$.

Proposition 2.3.9. *A state φ on a C^*-algebra A is a pure state if and only if the corresponding GNS representation $\pi_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$ with corresponding cyclic vector 1_φ is irreducible.*

Proof. Suppose first that φ is pure. If $\mathcal{K} \subset L^2(A, \varphi)$ is a closed invariant subspace, then so is $\mathcal{K}^\perp$ and we may consider $\xi_1 = [\mathcal{K}](1_\varphi) \in \mathcal{K}$ and $\xi_2 = 1_\varphi - \xi_1 \in \mathcal{K}^\perp$. For $x \in A$ we have

$$\langle x\xi_1, \xi_1 \rangle + \langle x\xi_2, \xi_2 \rangle = \langle x1_\varphi, 1_\varphi \rangle = \varphi(x).$$

Thus, either $\xi_1 = 0$, or $\xi_2 = 0$, since φ is pure. Since 1_φ is cyclic, we have that ξ_1 is cyclic for $\mathcal{K}$ and ξ_2 is cyclic for $\mathcal{K}^\perp$ showing that either $\mathcal{K} = \{0\}$ or else $\mathcal{K}^\perp = \{0\}$.

Conversely, suppose that π_φ is irreducible and $\varphi = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$ where $\varphi_j \in S(A)$ for $j = 1, 2$, then we may consider the map $U : L^2(A, \varphi) \to L^2(A, \varphi_1) \oplus L^2(A, \varphi_2)$ such that $U(x1_\varphi) = (x\frac{1}{\sqrt{2}}1_{\varphi_1}) \oplus (x\frac{1}{\sqrt{2}}1_{\varphi_2})$, for all $x \in A$. It is not hard to see that U is a well defined isometry and $U\pi_\varphi(x) = (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))U$ for all $x \in A$. If we denote by $p_1 \in \mathcal{B}(L^2(A, \varphi_1) \oplus L^2(A, \varphi_2))$ the orthogonal projection onto $L^2(A, \varphi_1)$ then the positive operator $U^*p_1U \in \mathcal{B}(L^2(A, \varphi))$ commutes with $\pi_\varphi(A)$.

If the spectrum of U^*p_1U contained more than one point then we could take $t \in [0, 1]$ such that the spectrum of $U^*p_1U - t$ contained both strictly positive and strictly negative numbers. Using continuous functional calculus we can then write $U^*p_1U - t = a - b$ where a, b are non-zero, positive, satisfy $ab = ba = 0$, and both commute with $\pi_\varphi(A)$. It then follows that $\ker(a)$ gives a non-trivial closed subspace which is $\pi_\varphi(A)$-invariant, contradicting irreducibility. Thus, it must be the case that the spectrum of U^*p_1U contains only one point, i.e., $U^*p_1U = \alpha \in \mathbb{C}$.

We then have $U^*p_1U = \langle U^*p_1U1_\varphi, 1_\varphi \rangle = 1/2$. Thus, $u_1 = \sqrt{2}p_1U$ implements an isometry from $L^2(A, \varphi)$ to $L^2(A, \varphi_1)$ such that $u_11_\varphi = 1_{\varphi_1}$, and

$u_1\pi_\varphi(x) = \pi_{\varphi_1(x)}u_1$ for all $x \in A$. It then follows, in particular, that $\varphi_1(x) = \varphi(x)$, for all $x \in A$, hence $\varphi = \varphi_1 = \varphi_2$ showing that φ is pure. ■

Note that the previous proposition, together with Proposition 2.3.8 shows also that a state φ is pure if and only if for any positive linear functional ψ such that $\psi \leq \varphi$ there exists a constant $\alpha \geq 0$ such that $\psi = \alpha\varphi$.

Since irreducible representations of an abelian C^*-algebra must be one dimensional, the following corollary follows from the above Proposition.

Corollary 2.3.10. *Let A be an abelian C^*-algebra, then the pure states on A agree with the spectrum $\sigma(A)$.*

Theorem 2.3.11. *Let A be a C^*-algebra, then the convex hull of the pure states on A are weak*-dense in $S(A)$.*

Proof. If A is unital, then the state space $S(A)$ is a weak* compact convex subset of A^*, and hence the convex hull of extreme states are dense in $S(A)$ by the Krein-Milman theorem.

If A is not unital, then consider the unitization $\tilde{A}$. Any irreducible representation of A extends to an irreducible representation of $\tilde{A}$, and conversely, for any irreducible representation of $\tilde{A}$ we must have that its restriction to A is irreducible, or else contains A in its kernel and hence is the representation given by $\pi_0(x, \alpha) = \alpha$.

Thus, any pure state on A extends uniquely to a pure state on $\tilde{A}$, and the only pure state on $\tilde{A}$ which does not arise in this way is $\varphi_0(x, \alpha) = \alpha$. Since every state on A extends to a state on $\tilde{A}$ we may then use the Krein-Milman theorem on the state space of $\tilde{A}$ to conclude that any state on A is a weak* limit of convex combinations of pure states on A and 0. However, since states satisfy $\|\varphi\| = 1$, we see that there must be no contribution from 0. ■

Corollary 2.3.12. *Let A be a C^*-algebra and $x \in A$, $x \neq 0$, then there exists an irreducible representation $\pi : A \to \mathcal{B}(\mathcal{H})$ such that $\pi(x) \neq 0$.*

Proof. By Proposition 2.3.5 there exists a state φ on A such that $\varphi(x) \neq 0$, and hence by the previous theorem there exists a pure state φ_0 on A such that $\varphi_0(x) \neq 0$. Proposition 2.3.9 then shows that the corresponding GNS-representation gives an irreducible representation π such that $\pi(x) \neq 0$. ■

2.3.3 Jordan Decomposition

Theorem 2.3.13 (Jordan Decomposition)**.** *Let A be a C^*-algebra and $\varphi \in A^*$, a Hermitian linear functional, then there exist unique positive linear functionals $\varphi_+, \varphi_- \in A^*$ such that $\varphi = \varphi_+ - \varphi_-$, and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$.*

Proof. Suppose $\varphi \in A^*$ is Hermitian. Let Σ denote the set of positive linear functionals on A, then Σ is a compact Hausdorff space when given the weak$*$ topology. Consider the map $\gamma : A \to C(\Sigma)$ given by $\gamma(a)(\psi) = \psi(a)$, then by Proposition 2.3.4 γ is isometric, and we also have $\gamma(A_+) \subset C(\Sigma)_+$.

By the Hahn-Banach theorem there exists a linear functional $\tilde{\varphi} \in C(\Sigma)^*$ such that $\|\tilde{\varphi}\| = \|\varphi\|$, and $\tilde{\varphi}(\gamma(a)) = \varphi(a)$, for all $a \in A$. By replading $\tilde{\varphi}$ with $\frac{1}{2}(\tilde{\varphi}+\tilde{\varphi}^*)$ we may assume that $\tilde{\varphi}$ is also Hermitian. By the Reisz representation theorem there then exists a signed Radon measure ν on Σ such that $\tilde{\varphi}(f) = \int f \, d\nu$ for all $f \in C(\Sigma)$. By the Jordan decomposition of measures there exist positive measures ν_+, and ν_- such that $\nu = \nu_+ - \nu_-$, and $\|\nu\| = \|\nu_+\| + \|\nu_-\|$.

Define the linear functionals φ_+, and φ_- by setting $\varphi_+(a) = \int \gamma(a) \, d\nu_+$, and $\varphi_-(a) = \int \gamma(a) \, d\nu_-$, for all $a \in A$. Then since $\gamma(A_+) \subset C(\Sigma)_+$ it follows that φ_+, and φ_- are positive. Moreover, we have $\varphi = \varphi_+ - \varphi_-$, and we have $\|\varphi\| \leq \|\varphi_+\| + \|\varphi_-\| \leq \|\nu_+\| + \|\nu_-\| = \|\nu\| = \|\varphi\|$.

Suppose now that we are given another Jordan decomposition $\varphi = \psi_+ - \psi_-$ where ψ_+, ψ_- are positive linear functionals which satisfy $\|\varphi\| = \|\psi_+\| + \|\psi_-\|$. Fix $\varepsilon > 0$. Since φ is Hermitian there exists $x \in (A)_1$ with $x = x^*$ such that $\varphi(x) + \varepsilon \geq \|\varphi\|$. Since $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\| = \varphi_+(1) + \varphi_-(1)$ we can then rearrange this inequality to obtain

$$\varphi_+(1 - x) + \varphi_-(1 + x) \leq \varepsilon.$$

For the same reason we also have $\psi_+(1 - x) + \psi_-(1 + x) \leq \varepsilon$. If we set $y_1 = \frac{1}{2}(1 - x)$ and $y_2 = \frac{1}{2}(1 + x)$ then we have $0 \leq y_1, y_2 \leq 1$, $y_1 + y_2 = 1$ and

$$\varphi_+(y_1), \varphi_-(y_2), \psi_+(y_1), \psi_-(y_2) < \varepsilon.$$

Since $\varphi_+ - \psi_+ = \varphi_- - \psi_-$ we may then use Cauchy-Schwarz to show that for all $a \in A$ we have

$$\begin{aligned}
|\varphi_+(a) - \psi_+(a)| &\leq |\varphi_+(ay_1) - \psi_+(ay_1)| + |\varphi_+(ay_2) - \psi_+(ay_2)| \\
&\leq |\varphi_+(ay_1)| + |\psi_+(ay_1)| + |\varphi_-(ay_2)| + |\psi_-(ay_2)| \\
&\leq \varphi_+(ay_1a^*)^{1/2}\varphi_+(y_1)^{1/2} + \psi_+(ay_1a^*)^{1/2}\psi_+(y_1)^{1/2} \\
&\qquad + \varphi_-(ay_2a^*)^{1/2}\varphi_-(y_1)^{1/2} + \psi_-(ay_2a^*)^{1/2}\psi_-(y_2)^{1/2} \\
&\leq \|a\|\|\varphi\|^{1/2}(\varphi_+(y_1)^{1/2} + \psi_+(y_1)^{1/2} + \varphi_-(y_2)^{1/2} + \psi_-(y_2)^{1/2}) \\
&\leq 4\|a\|\varphi\|^{1/2}\varepsilon^{1/2}.
\end{aligned}$$

Since $\varepsilon > 0$ and $a \in A$ were arbitrary we then have $\varphi_+ = \psi_+$, and also $\varphi_- = \psi_-$. ∎

Corollary 2.3.14. *Let A be a C^*-algebra, then A^* is the span of positive linear functionals.*

Corollary 2.3.15. *Let A be a C^*-algebra and $\varphi \in A^*$, then there exists a representation $\pi : A \to \mathcal{B}(\mathcal{H})$, and vectors $\xi, \eta \in \mathcal{H}$, such that $\varphi(a) = \langle \pi(a)\xi, \eta \rangle$, for all $a \in A$.*

Proof. Let $\varphi \in A^*$ be given. By the previous corollary we have that $\varphi = \sum_{i=1}^n \alpha_i \psi_i$ for some $\alpha_i \in \mathbb{C}$, and ψ_i states. If we consider the GNS-representations $\pi_i : A \to \mathcal{B}(L^2(A, \psi_i))$, then setting $\pi = \oplus_{i=1}^n \pi_i$, $\xi = \oplus_{i=1}^n \alpha_i 1_{\psi_i}$, and $\eta = \oplus_{i=1}^n 1_{\psi_i}$, we have $\varphi(a) = \langle \pi(a)\xi, \eta \rangle$, for all $a \in A$. ∎

Note that $\|\varphi\| \leq \|\xi\|\|\eta\|$, however equality need not hold. We'll show in Theorem **??** below that we may also choose a representation π and vectors $\xi, \eta \in \mathcal{H}$ which additionally satisfy $\|\varphi\| = \|\xi\|\|\eta\|$.

Chapter 3

Bounded linear operators

Recall that if $\mathcal{H}$ is a Hilbert space then $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators is a C^*-algebra with norm

$$\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|,$$

and involution given by the adjoint, i.e., x^* is the unique bounded linear operator such that

$$\langle \xi, x^*\eta \rangle = \langle x\xi, \eta \rangle,$$

for all $\xi, \eta \in \mathcal{H}$.

Lemma 3.0.16. *Let $\mathcal{H}$ be a Hilbert space and consider $x \in \mathcal{B}(\mathcal{H})$, then* $\ker(x) = R(x^*)^\perp$.

Proof. If $\xi \in \ker(x)$, and $\eta \in \mathcal{H}$, then $\langle \xi, x^*\eta \rangle = \langle x\xi, \eta \rangle = 0$, hence $\ker(x) \subset R(x^*)^\perp$. If $\xi \in R(x^*)^\perp$ then for any $\eta \in \mathcal{H}$ we have $\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle = 0$, hence $\xi \in \ker(x)$. ∎

The **point spectrum** $\sigma_p(x)$ of an operator $x \in \mathcal{B}(\mathcal{H})$ consists of all points $\lambda \in \mathbb{C}$ such that $x - \lambda$ has a non-trivial kernel. The operator x has an **approximate kernel** if there exists a sequence of unit vectors $\xi_n \in \mathcal{H}$ such that $\|x\xi_n\| \to 0$. The **approximate point spectrum** $\sigma_{ap}(x)$ consists of all points $\lambda \in \mathbb{C}$ such that $x - \lambda$ has an approximate kernel. Note that we have $\sigma_p(x) \subset \sigma_{ap}(x) \subset \sigma(x)$.

Proposition 3.0.17. *Let $x \in \mathcal{B}(\mathcal{H})$ be a normal operator, then $\sigma_p(x^*) = \overline{\sigma_p(x)}$. Moreover, eigenspaces for x corresponding to distinct eigenvalues are orthogonal.*

Proof. As x is normal, so is $x - \lambda$, and hence for each $\xi \in \mathcal{H}$ we have $\|(x-\lambda)\xi\| = \|(x^* - \overline{\lambda})\xi\|$. The first implication then follows.

If $\xi, \eta \in \mathcal{H}$ are eigenvectors for x with respective eigenvalues λ, μ, such that $\lambda \neq \mu$, Then we have $x^*\eta = \overline{\mu}\eta$, and so $\lambda\langle \xi, \eta \rangle = \langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle = \mu\langle \xi, \eta \rangle$. As $\lambda \neq \mu$, it then follows $\langle \xi, \eta \rangle = 0$. ∎

Proposition 3.0.18. *Let $x \in \mathcal{B}(\mathcal{H})$, then $\partial\sigma(x) \subset \sigma_{ap}(x)$.*

Proof. Suppose $\lambda \in \partial\sigma(x)$. Then there exists a sequence $\lambda_n \in \rho(x)$, such that $\lambda_n \to \lambda$. By Lemma 1.1.12 we then have that $\|(x - \lambda_n)^{-1}\| \to \infty$, hence there exists a sequence of unit vectors $\xi_n \in \mathcal{H}$, such that $\|\xi_n\| \to 0$, and $\|(x - \lambda_n)^{-1}\xi_n\| = 1$. We then have $\|(x-\lambda)(x-\lambda_n)^{-1}\xi_n\| \leq |\lambda - \lambda_n|\|(x-\lambda_n)^{-1}\xi_n\| + \|\xi_n\| \to 0$. Hence, $\lambda \in \sigma_{ap}(x)$. ∎

Lemma 3.0.19. *Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$, then x is invertible in $\mathcal{B}(\mathcal{H})$ if and only if neither x nor x^* has an approximate kernel. Consequently, $\sigma(x) = \sigma_{ap}(x) \cup \overline{\sigma_{ap}(x^*)}$, for all $x \in \mathcal{B}(\mathcal{H})$.*

Proof. If x is invertible, then for all $\xi \in \mathcal{H}$ we have $\|x^{-1}\|\|x\xi\| \geq \|x^{-1}x\xi\| = \|\xi\|$, and hence x cannot have an approximate kernel. Neither can x^* since it is then also invertible.

Conversely, if neither x nor x^* has an approximate kernel then x is injective, and the previous lemma applied to x^* shows that x has dense range. If $\{x\xi_n\} \subset R(x)$ is Cauchy then we have $\lim_{n,m\to\infty} \|x(\xi_n - \xi_m)\| \to 0$. Since x does not have an approximate kernel it then follows that $\{\xi_n\}$ is also Cauchy (Otherwise an approximate kernel of the form $(\xi_n - \xi_m)/\|\xi_n - \xi_m\|$ may be found) and hence converges to a vector ξ. We then have $\lim_{n\to\infty} x\xi_n = x\xi \in R(x)$, thus $R(x)$ is closed and hence x is surjective. The open mapping theorem then implies that x has a bounded inverse. ∎

The **numerical range** $W(x)$ of an operator $x \in \mathcal{B}(\mathcal{H})$ is the closure of the set $\{\langle x\xi, \xi\rangle \mid \xi \in \mathcal{H}, \|\xi\| = 1\}$.

Lemma 3.0.20. *Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$, then $\sigma(x) \subset W(x)$.*

Proof. Suppose $\lambda \in \sigma(x)$. Then from the previous lemma either $x-\lambda$ or $(x-\lambda)^*$ has an approximate kernel. In either case there then exists a sequence of unit vectors $\xi_n \in \mathcal{H}$ such that $\langle (x-\lambda)\xi_n, \xi_n\rangle \to 0$. Hence, $\lambda \in W(x)$. ∎

Proposition 3.0.21. *Let $\mathcal{H}$ be a Hilbert space, then an operator $x \in \mathcal{B}(\mathcal{H})$ is*

(*i*) *normal if and only if $\|x\xi\| = \|x^*\xi\|$, for all $\xi \in \mathcal{H}$.*

(*ii*) *self-adjoint if and only if $\langle x\xi, \xi\rangle \in \mathbb{R}$, for all $\xi \in \mathcal{H}$.*

(*iii*) *positive if and only if $\langle x\xi, \xi\rangle \geq 0$, for all $\xi \in \mathcal{H}$.*

(*iv*) *an isometry if and only if $\|x\xi\| = \|\xi\|$, for all $\xi \in \mathcal{H}$.*

(*v*) *a projection if and only if x is the orthogonal projection onto some closed subspace of $\mathcal{H}$.*

(*vi*) *a partial isometry if and only if there is a closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $x_{|\mathcal{K}}$ is an isometry while $x_{|\mathcal{K}^\perp} = 0$.*

Proof.

(i) If x is normal than for all $\xi \in \mathcal{H}$ we have $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \langle xx^*\xi, \xi\rangle = \|x^*\xi\|^2$. Conversely, is $\langle (x^*x - xx^*)\xi, \xi\rangle = 0$, for all $\xi \in \mathcal{H}$, then for all $\xi, \eta \in \mathcal{H}$, by polarization we have

$$\langle (x^*x - xx^*)\xi, \eta\rangle = \sum_{k=0}^{3} i^k \langle (x^*x - xx^*)(\xi + i^k\eta), (\xi + i^k\eta)\rangle = 0.$$

Hence $x^*x = xx^*$.

(ii) If $x = x^*$ then $\overline{\langle x\xi, \xi\rangle} = \langle \xi, x\xi\rangle = \langle x\xi, \xi\rangle$. The converse follows again by a polarization argument.

(iii) If $x = y^*y$, then $\langle x\xi, \xi\rangle = \|y\xi\|^2 \geq 0$. Conversely, if $\langle x\xi, \xi\rangle \geq 0$, for all $\xi \in \mathcal{H}$ then we know from part (ii) that x is normal. From Lemma 3.0.20 we have that $\sigma(x) \subset [0, \infty)$ and hence x is positive by functional calculus.

(iv) If x is an isometry then $x^*x = 1$ and hence $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \|\xi\|^2$ for all $\xi \in \mathcal{H}$. The converse again follows from the polarization identity.

(v) If x is a projection then let $\mathcal{K} = \overline{R(x)} = \ker(x)^\perp$, and note that for all $\xi \in \mathcal{K}, \eta \in \ker(x), x\zeta \in R(x)$ we have $\langle x\xi, \eta + x\zeta\rangle = \langle \xi, x\zeta\rangle$, hence $x\xi \in \mathcal{K}$, and $x\xi = \xi$. This shows that x is the orthogonal projection onto the subspace $\mathcal{K}$.

(vi) This follows directly from (iv) and (v). ∎

Proposition 3.0.22 (Polar decomposition)**.** *Let $\mathcal{H}$ be a Hilbert space, and $x \in \mathcal{B}(\mathcal{H})$, then there exists a partial isometry v such that $x = v|x|$, and $\ker(v) = \ker(|x|) = \ker(x)$. Moreover, this decomposition is unique, in that if $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$ then $y = |x|$, and $w = v$.*

Proof. We define a linear operator $v_0 : R(|x|) \to R(x)$ by $v_0(|x|\xi) = x\xi$, for $\xi \in \mathcal{H}$. Since $\||x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$ it follows that v_0 is well defined and extends to a partial isometry v from $\overline{R(|x|)}$ to $\overline{R(x)}$, and we have $v|x| = x$. We also have $\ker(v) = R(|x|)^\perp = \ker(|x|) = \ker(x)$.

To see the uniqueness of this decomposition suppose $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$. Then $|x|^2 = x^*x = yw^*wy = y^2$, and hence $|x| = (|x|^2)^{1/2} = (y^2)^{1/2} = y$. We then have $\ker(w) = \overline{R(|x|)}^\perp$, and $\|w|x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$, hence $w = v$. ∎

3.1 Trace class operators

Given a Hilbert space $\mathcal{H}$, an operator $x \in \mathcal{B}(\mathcal{H})$ has finite rank if $\overline{R(x)} = \ker(x^*)^\perp$ is finite dimensional, the **rank** of x is $\dim(\overline{R(x)})$. We denote the space of finite rank operators by $\mathcal{FR}(\mathcal{H})$. If x is finite rank than $R(x^*) = R(x^*_{|\ker(x^*)^\perp})$ is also finite dimensional being the image of a finite dimensional space, hence we see that x^* also has finite rank. If $\xi, \eta \in \mathcal{H}$ are vectors we denote by $\xi \otimes \overline{\eta}$ the operator given by

$$(\xi \otimes \overline{\eta})(\zeta) = \langle \zeta, \eta \rangle \xi.$$

Note that $(\xi \otimes \overline{\eta})^* = \eta \otimes \overline{\xi}$, and if $\|\xi\| = \|\eta\| = 1$ then $\xi \otimes \overline{\eta}$ is a rank one partial isometry from $C\eta$ to $\mathbb{C}\xi$. Also note that if $x, y \in \mathcal{B}(\mathcal{H})$, then we have $x(\xi \otimes \overline{\eta})y = (x\xi) \otimes (\overline{y^*\eta})$.

From above we see that any finite rank operator is of the form pxq where $p, q \in \mathcal{B}(\mathcal{H})$ are projections onto finite dimensional subspaces. In particular this shows that $\mathcal{FR}(\mathcal{H}) = \text{sp}\{\xi \otimes \overline{\eta} \mid \xi, \eta \in \mathcal{H}\}$

Lemma 3.1.1. *Suppose $x \in \mathcal{B}(\mathcal{H})$ has polar decomposition $x = v|x|$. Then for all $\xi \in \mathcal{H}$ we have*

$$2|\langle x\xi, \xi \rangle| \leq \langle |x|\xi, \xi \rangle + \langle |x|v^*\xi, v^*\xi \rangle.$$

Proof. If $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, then we have

$$\begin{aligned} 0 &\leq \|(|x|^{1/2} - \lambda |x|^{1/2} v^*)\xi\|^2 \\ &= \||x|^{1/2}\xi\|^2 - 2\text{Re}(\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi \rangle) + \||x|^{1/2}v^*\xi\|^2. \end{aligned}$$

Taking λ such that $\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi \rangle \geq 0$, the inequality follows directly. ■

If $\{\xi_i\}$ is an orthonormal basis for $\mathcal{H}$, and $x \in \mathcal{B}(\mathcal{H})$ is positive, then we define the trace of x to be

$$\text{Tr}(x) = \sum_i \langle x\xi_i, \xi_i \rangle.$$

Lemma 3.1.2. *If $x \in \mathcal{B}(\mathcal{H})$ then $\text{Tr}(x^*x) = \text{Tr}(xx^*)$.*

Proof. By Parseval's identity and Fubini's theorem we have

$$\begin{aligned} \sum_i \langle x^*x\xi_i, \xi_i \rangle &= \sum_i \sum_j \langle x\xi_i, \xi_j \rangle \overline{\langle x\xi_i, \xi_j \rangle} \\ &= \sum_j \sum_i \langle \xi_i, x^*\xi_j \rangle \langle x^*\xi_j, \xi_i \rangle = \sum_j \langle xx^*\xi_j, \xi_j \rangle. \end{aligned}$$ ■

Corollary 3.1.3. *If $x \in \mathcal{B}(\mathcal{H})$ is positive and u is a unitary, then $\text{Tr}(u^*xu) = \text{Tr}(x)$. In particular, the trace is independent of the chosen orthonormal basis.*

Proof. If we write $x = y^*y$, then from the previous lemma we have

$$\text{Tr}(y^*y) = \text{Tr}(yy^*) = \text{Tr}((yu)(u^*y^*)) = \text{Tr}(u^*(y^*y)u).$$ ■

An operator $x \in \mathcal{B}(\mathcal{H})$ is said to be of **trace class** if $\|x\|_1 := \text{Tr}(|x|) < \infty$. We denote the set of trace class operators by $L^1(\mathcal{B}(\mathcal{H}))$ or $L^1(\mathcal{B}(\mathcal{H}), \text{Tr})$.

Given an orthonormal basis $\{\xi_i\}$, and $x \in L^1(\mathcal{B}(\mathcal{H}))$ we define the **trace** of x by

$$\text{Tr}(x) = \Sigma_i \langle x\xi_i, \xi_i \rangle.$$

By Lemma 3.1.1 this is absolutely summable, and

$$2|\text{Tr}(x)| \leq \text{Tr}(|x|) + \text{Tr}(v|x|v^*) \leq 2\|x\|_1.$$

Lemma 3.1.4. *$L^1(\mathcal{B}(\mathcal{H}))$ is a two sided self-adjoint ideal in $\mathcal{B}(\mathcal{H})$ which coincides with the span of the positive operators with finite trace. The trace is independent of the chosen basis, and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.*

Proof. If $x, y \in L^1(\mathcal{B}(\mathcal{H}))$ and we let $x+y = w|x+y|$ be the polar decomposition, then we have $w^*x, w^*y \in L^1(\mathcal{B}(\mathcal{H}))$, therefore $\sum_i \langle |x+y|\xi_i, \xi_i\rangle = \sum_i \langle w^*x\xi_i, \xi_i\rangle + \langle w^*y\xi_i, \xi_i\rangle$ is absolutely summable. Thus $x + y \in L^1(\mathcal{B}(\mathcal{H}))$ and

$$\|x+y\|_1 \leq \|w^*x\|_1 + \|w^*y\|_1 \leq \|x\|_1 + \|y\|_1.$$

Thus, it follows that $L^1(\mathcal{B}(\mathcal{H}))$ is a linear space which contains the span of the positive operators with finite trace, and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.

If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ then

$$4a|x| = \sum_{k=0}^{3} i^k (a + i^k)|x|(a + i^k)^*,$$

and for each k we have

$$\mathrm{Tr}((a+i^k)|x|(a+i^k)^*) = \mathrm{Tr}(|x|^{1/2}|a+i^k|^2|x|^{1/2}) \leq \|a+i^k\|^2 \,\mathrm{Tr}(|x|).$$

Thus if we take a to be the partial isometry in the polar decomposition of x we see that x is a linear combination of positive operators with finite trace, (in particular, the trace is independent of the basis). This also shows that $L^1(\mathcal{B}(\mathcal{H}))$ is a self-adjoint left ideal, and hence is also a right ideal. ∎

Theorem 3.1.5. *If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a, b \in \mathcal{B}(\mathcal{H})$ then*

$$\|x\| \leq \|x\|_1$$

$$\|axb\|_1 \leq \|a\| \; \|b\| \; \|x\|_1,$$

and

$$\mathrm{Tr}(ax) = \mathrm{Tr}(xa).$$

Proof. Since the trace is independent of the basis, and $\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|$ it follows easily that $\|x\| \leq \|x\|_1$.

Since for $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ we have $|ax| \leq \|a\||x|$ it follows that $\|ax\|_1 \leq \|a\|\|x\|_1$. Since $\|x\|_1 = \|x^*\|_1$ we also have $\|xb\|_1 \leq \|b\|\|x\|_1$.

Since the definition of the trace is independent of the chosen basis, if $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $u \in \mathcal{U}(\mathcal{H})$ we have

$$\mathrm{Tr}(xu) = \sum_i \langle xu\xi_i, \xi_i\rangle = \sum_i \langle uxu\xi_i, u\xi_i\rangle = \mathrm{Tr}(ux).$$

Since every operator $a \in \mathcal{B}(\mathcal{H})$ is a linear combination of four unitaries this also gives

$$\mathrm{Tr}(xa) = \mathrm{Tr}(ax).$$ ∎

We also remark that for all $\xi, \eta \in \mathcal{H}$, the operators $\xi \otimes \overline{\eta}$ satisfy $\mathrm{Tr}(\xi \otimes \overline{\eta}) = \langle \xi, \eta \rangle$. Also, it's easy to check that $\mathcal{FR}(\mathcal{H})$ is a dense subspace of $L^1(\mathcal{B}(\mathcal{H}))$, endowed with the norm $\| \cdot \|_1$.

Proposition 3.1.6. *The space of trace class operators $L^1(\mathcal{B}(\mathcal{H}))$, with the norm $\| \cdot \|_1$ is a Banach space.*

Proof. From Lemma 3.1.4 we know that $\| \cdot \|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$ and hence we need only show that $L^1(\mathcal{B}(\mathcal{H}))$ is complete. Suppose x_n is Cauchy in $L^1(\mathcal{B}(\mathcal{H}))$. Since $\|x_n - x_m\| \leq \|x_n - x_m\|_1$ it follows that x_n is also Cauchy in $\mathcal{B}(\mathcal{H})$, therefore we have $\|x - x_n\| \to 0$, for some $x \in \mathcal{B}(\mathcal{H})$, and by continuity of functional calculus we also have $\||x| - |x_n|\| \to 0$. Thus for any finite orthonormal set $\eta_1, \ldots, \eta_k$ we have

$$\begin{aligned}\sum_{i=1}^{k} \langle |x| \eta_i, \eta_i \rangle &= \lim_{n \to \infty} \sum_{i=1}^{k} \langle |x_n| \eta_i, \eta_i \rangle \\ &\leq \lim_{n \to \infty} \|x_n\|_1 < \infty.\end{aligned}$$

Hence $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $\|x\|_1 \leq \lim_{n \to \infty} \|x_n\|_1$.

If we let $\varepsilon > 0$ be given and consider $N \in \mathbb{N}$ such that for all $n > N$ we have $\|x_n - x_N\|_1 < \varepsilon/3$, and then take $\mathcal{H}_0 \subset \mathcal{H}$ a finite dimensional subspace such that $\|x_N P_{\mathcal{H}_0^\perp}\|_1, \|x P_{\mathcal{H}_0^\perp}\|_1 < \varepsilon/3$. Then for all $n > N$ we have

$$\begin{aligned}&\|x - x_n\|_1 \\ &\quad \leq \|(x - x_n) P_{\mathcal{H}_0}\|_1 + \|x P_{\mathcal{H}_0^\perp} - x_N P_{\mathcal{H}_0^\perp}\|_1 + \|(x_N - x_n) P_{\mathcal{H}_0^\perp}\|_1 \\ &\quad \leq \|(x - x_n) P_{\mathcal{H}_0}\|_1 + \varepsilon.\end{aligned}$$

Since $\|x - x_n\| \to 0$ it follows that $\|(x - x_n) P_{\mathcal{H}_0}\|_1 \to 0$, and since $\varepsilon > 0$ was arbitrary we then have $\|x - x_n\|_1 \to 0$. ■

Theorem 3.1.7. *The map $\psi : \mathcal{B}(\mathcal{H}) \to L^1(\mathcal{B}(\mathcal{H}))^*$ given by $\psi_a(x) = \mathrm{Tr}(ax)$, for $a \in \mathcal{B}(\mathcal{H})$, $x \in L^1(\mathcal{B}(\mathcal{H}))$, is a Banach space isomorphism.*

Proof. From Theorem 3.1.5 we have that ψ is a linear contraction.

Suppose $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$, then $(\xi, \eta) \mapsto \varphi(\xi \otimes \overline{\eta})$ defines a bounded sesquilinear form on $\mathcal{H}$ and hence there exists a bounded operator $a \in \mathcal{B}(\mathcal{H})$ such that $\langle a\xi, \eta \rangle = \varphi(\xi \otimes \overline{\eta})$, for all $\xi, \eta \in \mathcal{H}$. Since the finite rank operators is dense in $L^1(\mathcal{B}(\mathcal{H}))$, and since operators of the form $\xi \otimes \overline{\eta}$ span the finite rank operators we have $\varphi = \psi_a$, thus we see that ψ is bijective.

We also have

$$\begin{aligned}\|a\| &= \sup_{\substack{\xi, \eta \in \mathcal{H}, \\ \|\xi\|, \|\eta\| \leq 1}} |\langle a\xi, \eta \rangle| \\ &= \sup_{\substack{\xi, \eta \in \mathcal{H}, \\ \|\xi\|, \|\eta\| \leq 1}} |\mathrm{Tr}(a(\xi \otimes \overline{\eta}))| \leq \|\psi_a\|.\end{aligned}$$

Hence ψ is isometric. ■

3.2 Hilbert-Schmidt operators

Given a Hilbert space $\mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$, we say that x is a Hilbert-Schmidt operator on $\mathcal{H}$ if $|x|^2 \in L^1(\mathcal{B}(\mathcal{H}))$. We define the set of Hilbert-Schmidt operators by $L^2(\mathcal{B}(\mathcal{H}))$, or $L^2(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$.

Lemma 3.2.1. *$L^2(\mathcal{B}(\mathcal{H}))$ is a self-adjoint ideal in $\mathcal{B}(\mathcal{H})$, and if $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ then $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$, and*

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx).$$

Proof. Since $|x+y|^2 \leq |x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$ we see that $L^2(\mathcal{B}(\mathcal{H}))$ is a linear space, also since $|ax|^2 \leq \|a\|^2 |x|^2$ we have that $L^2(\mathcal{B}(\mathcal{H}))$ is a left ideal. Moreover, since we have $\mathrm{Tr}(xx^*) = \mathrm{Tr}(x^*x)$ we see that $L^2(\mathcal{B}(\mathcal{H}))$ is self-adjoint. In particular, $L^2(\mathcal{B}(\mathcal{H}))$ is also a right ideal.

By the polarization identity

$$4y^*x = \sum_{k=0}^{3} i^k |x + i^k y|^2,$$

we have that $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$ for $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, and

$$\begin{aligned} 4\,\mathrm{Tr}(y^*x) &= \sum_{k=0}^{3} i^k\, \mathrm{Tr}((x + i^k y)^*(x + i^k y)) \\ &= \sum_{k=0}^{3} i^k\, \mathrm{Tr}((x + i^k y)(x + i^k y)^*) = 4\,\mathrm{Tr}(xy^*). \end{aligned}$$

■

From the previous lemma we see that the sesquilinear form on $L^2(\mathcal{B}(\mathcal{H}))$ give by

$$\langle x, y \rangle_2 = \mathrm{Tr}(y^*x)$$

is well defined and positive definite. We again have $\|axb\|_2 \leq \|a\|\,\|b\|\,\|x\|_2$, and any $x \in L^2(\mathcal{B}(\mathcal{H}))$ can be approximated in $\|\cdot\|_2$ by operators px where p is a finite rank projection. Thus, the same argument as for the trace class operators shows that the Hilbert-Schmidt operators is complete in the Hilbert-Schmidt norm.

Also, note that if $x \in L^2(\mathcal{B}(\mathcal{H}))$ then since $\|y\| \leq \|y\|_2$ for all $y \in L^2(\mathcal{B}(\mathcal{H}))$ it follows that

$$\begin{aligned} \|x\|_2 &= \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})), \\ \|y\|_2 \leq 1}} |\,\mathrm{Tr}(y^*x)| \\ &\leq \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})), \\ \|y\|_2 \leq 1}} \|y\|\|x\|_1 \leq \|x\|_1. \end{aligned}$$

Proposition 3.2.2. *Let $\mathcal{H}$ be a Hilbert space and suppose $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then*

$$\|xy\|_1 \leq \|x\|_2 \|y\|_2.$$

Proof. If we consider the polar decomposition $xy = v|xy|$, then by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\|xy\|_1 &= |\operatorname{Tr}(v^*xy)| = |\langle y, x^*v\rangle_2| \\ &\le \|x^*v\|_2\|y\|_2 \le \|x\|_2\|y\|_2.\end{aligned}$$ ■

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then we may extend a bounded operator $x : \mathcal{H} \to \mathcal{K}$ to a bounded operator $\tilde{x} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ by $\tilde{x}(\xi \oplus \eta) = 0 \oplus x\xi$. We define $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ as the bounded operators $x : \mathcal{H} \to \mathcal{K}$ such that $\tilde{x} \in L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$. In this way $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ forms a closed subspace of $L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$.

Note that $\mathrm{HS}(\mathcal{H}, \mathbb{C})$ is the dual Banach space of $\mathcal{H}$, and is naturally anti-isomorphic to $\mathcal{H}$, we denote this isomorphism by $\xi \mapsto \overline{\xi}$. We call this the **conjugate Hilbert space** of $\mathcal{H}$, and denote it by $\overline{\mathcal{H}}$. Note that we have the natural identification $\overline{\overline{\mathcal{H}}} = \mathcal{H}$. Also, we have a natural anti-linear map $x \mapsto \overline{x}$ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\overline{\mathcal{H}})$ given by $\overline{x}\overline{\xi} = \overline{x\xi}$.

If we wish to emphasize that we are considering only the Hilbert space aspects of the Hilbert-Schmidt operators, we often use the notation $\mathcal{H}\overline{\otimes}\mathcal{K}$ for the Hilbert-Schmidt operators $\mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$. In this setting we call $\mathcal{H}\overline{\otimes}\mathcal{K}$ the **Hilbert space tensor product** of $\mathcal{H}$ with $\mathcal{K}$. Note that if $\{\xi_i\}_i$ and $\{\eta_j\}_j$ form orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$ respectively, then $\{\xi_i \otimes \eta_j\}_{i,j}$ forms an orthonormal basis for $\mathcal{H}\overline{\otimes}\mathcal{K}$. We see that the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ of $\mathcal{H}$ and $\mathcal{K}$ can be realized as the subspace of finite rank operators, i.e., we have $\mathcal{H} \otimes \mathcal{K} = \mathrm{sp}\{\xi \otimes \eta \mid \xi \in \mathcal{H}, \eta \in \mathcal{K}\}$.

If $x \in \mathcal{B}(\mathcal{H})$ and $y \in \mathcal{B}(\mathcal{K})$ then we obtain an operator $x \otimes y \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ which is given by $(x \otimes y)\Xi = x\Xi\overline{y}^*$. We then have that $\|x \otimes y\| \le \|x\|\|y\|$, and $(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$ for all $\xi \in \mathcal{H}$, and $\eta \in \mathcal{K}$. We also have $(x \otimes y)^* = x^* \otimes y^*$, and the map $(x, y) \mapsto x \otimes y$ is separately linear in each variable. If $A \subset \mathcal{B}(\mathcal{H})$ and $B \subset \mathcal{B}(\mathcal{K})$ are algebras then the tensor product $A \otimes B$ is the algebra generated by operators of the form $a \otimes b$ for $a \in A$ and $b \in B$.

If (X, μ) is a measure space then we have a particularly nice description of the Hilbert-Schmidt operators on $L^2(X, \mu)$.

Theorem 3.2.3. *For each $k \in L^2(X \times X, \mu \times \mu)$ the integral operator T_k defined by*

$$T_k\xi(x) = \int k(x, y)\xi(y)d\mu(y), \qquad \xi \in L^2(X, \mu),$$

is a Hilbert-Schmidt operator on $L^2(X, \mu)$. Moreover, the map $k \mapsto T_k$ is a unitary operator from $L^2(X \times X, \mu \times \mu)$ to $L^2(\mathcal{B}(L^2(X, \mu)))$. Moreover, if we define $k^*(x, y) = \overline{k(x, y)}$ then we have $T_k^* = T_{k^*}$.

Proof. For all $\eta \in L^2(X, \mu)$, the Cauchy-Schwarz inequality gives

$$\|k(x, y)\xi(y)\eta(x)\|_1 \le \|k\|_2\|\|\xi\|_{L^2(X,\mu)}\|\eta\|_2.$$

This shows that T_k is a well defined operator on $L^2(X, \mu)$ and $\|T_k\| \le \|k\|_2$. If $\{\xi_i\}_i$ gives an orthonormal basis for $L^2(X, \mu)$ and $k(x, y) = \sum \alpha_{i,j}\xi_i(x)\xi_j(y)$ is a finite sum then for $\eta \in L^2(X, \mu)$ we have

$$T_k\eta = \sum \alpha_{i,j}\langle \xi, \xi_j\rangle \xi_i = (\sum \alpha_{i,j}\xi_i \otimes \overline{\xi_j})\eta.$$

Thus, $\|T_k\|_2 = \|\sum \alpha_{i,j}\xi_i \otimes \overline{\xi_j}\|_2 = \|k\|_2$, which shows that $k \mapsto T_k$ is a unitary operator.

The same formula above also shows that $T_k^* = T_{k^*}$. ■

3.3 Compact operators

We denote by $(\mathcal{H})_1$ the unit ball in $\mathcal{H}$.

Theorem 3.3.1. *For $x \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent:*

(i) $x \in \overline{\mathcal{FR}(\mathcal{H})}^{\|\cdot\|}$.

(ii) *x restricted to $(\mathcal{H})_1$ is continuous from the weak to the norm topology.*

(iii) *$x(\mathcal{H})_1$ is compact in the norm topology.*

(iv) *$x(\mathcal{H})_1$ has compact closure in the norm topology.*

Proof. (i) $\implies$ (ii) Let $\{\xi_\alpha\}_\alpha$ be net in $(\mathcal{H})_1$ which weakly converges to ξ. By hypothesis for every $\varepsilon > 0$ there exists $y \in \mathcal{FR}(\mathcal{H})$ such that $\|x - y\| < \varepsilon$. We then have

$$\|x\xi - x\xi_\alpha\| \leq \|y\xi - y\xi_\alpha\| + 2\varepsilon.$$

Thus, it is enough to consider the case when $x \in \mathcal{FR}(\mathcal{H})$. This case follows easily since then the range of x is then finite dimensional where the weak and norm topologies agree.

(ii) $\implies$ (iii) $(\mathcal{H})_1$ is compact in the weak topology and hence $x(\mathcal{H})_1$ is compact being the continuous image of a compact set.

(iii) $\implies$ (iv) This implication is obvious.

(iv) $\implies$ (i) Let P_α be a net of finite rank projections such that $\|P_\alpha\xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}$. Then $P_\alpha x$ are finite rank and if $\|P_\alpha x - x\| \not\to 0$ then there exists $\varepsilon > 0$, and $\xi_\alpha \in (\mathcal{H})_1$ such that $\|x\xi_\alpha - P_\alpha x\xi_\alpha\| \geq \varepsilon$. By hypothesis we may pass to a subnet and assume that $x\xi_\alpha$ has a limit ξ in the norm topology. We then have

$$\begin{aligned}\varepsilon &\leq \|x\xi_\alpha - P_\alpha x\xi_\alpha\| \leq \|\xi - P_\alpha\xi\| + \|(1 - P_\alpha)(x\xi_\alpha - \xi)\| \\ &\leq \|\xi - P_\alpha\xi\| + \|x\xi_\alpha - \xi\| \to 0,\end{aligned}$$

which gives a contradiction. ■

If any of the above equivalent conditions are satisfied we say that x is a **compact operator**. We denote the space of compact operators by $\mathcal{K}(\mathcal{H})$. Clearly $\mathcal{K}(\mathcal{H})$ is a norm closed two sided ideal in $\mathcal{B}(\mathcal{H})$.

Lemma 3.3.2. *Let $x \in \mathcal{K}(\mathcal{H})$ be a compact operator, then $\sigma_{ap}(x) \setminus \{0\} \subset \sigma_p(x)$.*

Proof. Suppose $\lambda \in \sigma_{ap}(x) \setminus \{0\}$ and let $\xi_n \in \mathcal{H}$ be a sequence of unit vectors such that $\|(x-\lambda)\xi_n\| \to 0$. Since x is compact, by taking a subsequence we may assume that $x\xi_n$ converges to a vector ξ. We then have $x\xi = \lim_{n\to\infty} x^2\xi_n = \lim_{n\to\infty} \lambda x\xi_n = \lambda\xi$. Moreover, ξ is non-zero since $\|\xi\| = \lim_{n\to\infty} \|x\xi_n\| = \lim_{n\to\infty} \|\lambda\xi_n\| = |\lambda| \neq 0$. Hence, $\lambda \in \sigma_p(x)$. ∎

Lemma 3.3.3. *Let $x \in \mathcal{K}(\mathcal{H})$ be a compact operator, then each point in $\sigma(x) \setminus \{0\}$ is isolated.*

Proof. Suppose that $\{\lambda_n\}_n \subset \sigma(x) \setminus \{0\}$ is a sequence of pairwise distinct values such that $\lambda_n \to \lambda$. From Lemma 3.0.19 we have $\sigma(x) = \sigma_{ap}(x) \cup \overline{\sigma_{ap}(x^*)}$, and hence by taking a further subsequence, and replacing x with x^* if necessary, we will assume that $\lambda_n \in \sigma_{ap}(x)$, for each n, and then from the previous lemma we have $\lambda_n \in \sigma_p(x)$, for each n.

Thus, there exists a sequence of unit eigenvectors $\{\xi_n\} \subset \mathcal{H}$, whose corresponding eigenvalues are $\{\lambda_n\}$. Note that since $\{\lambda_n\}$ are distinct values we have that $\{\xi_n\}$ is a linearly independent set. Let $Y_n = \mathrm{sp}\{\xi_1, \ldots, \xi_n\}$, and choose unit vectors $\eta_n \in Y_n$, so that $\|P_{Y_{n-1}}(\eta_n)\| = 0$, for all n. Then for $n < m$ we have

$$\|x\eta_n - x\eta_m\| = \|x\eta_n - (x - \lambda_m)\eta_m + \lambda_m\eta_m\|,$$

and since $x\eta_n - (x-\lambda_m)\eta_m \in Y_{m-1}$ we conclude that $\|x\eta_n - x\eta_m\| \geq |\lambda_m|\|\eta_m\|$. Since x is compact, and $\{\eta_n\}$ are pairwise orthogonal unit vectors it then follows that $|\lambda| = \lim_{m\to\infty} |\lambda_m| = 0$, and hence 0 is the only possible accumulation point of $\sigma(x)$. ∎

Theorem 3.3.4 (The Fredholm Alternative)**.** *Let $x \in \mathcal{K}(\mathcal{H})$ be a compact operator, then for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$, either $\lambda \in \sigma_p(x)$, or else $\lambda \in \rho(x)$, i.e., $\sigma(x) \setminus \{0\} \subset \sigma_p(x)$.*

Proof. By the previous lemma, each point in $\sigma(x) \setminus \{0\}$ is isolated. It then follows from Proposition 3.0.18, that $\sigma(x) \setminus \{0\} = (\partial\sigma(x)) \setminus \{0\} \subset \sigma_{ap}(x)$, and then from Lemma 3.3.2 it follows that $\sigma(x) \setminus \{0\} \subset \sigma_p(x)$. ∎

Theorem 3.3.5 (The spectral theorem for compact operators)**.** *Let $x \in \mathcal{K}(\mathcal{H})$ be a normal compact operator. For each eigenvalue λ for x, denote by E_λ the corresponding eigenspace. Then we have $x = \sum_{\lambda\in\sigma(x)\setminus\{0\}} \lambda P_{E_\lambda}$, where the convergence is in the uniform norm.*

Proof. If ξ_n is an orthogonal sequence of unit eigenvectors for x, then $\xi_n \to 0$ weakly, and since x is compact we then have that $x\xi_n \to 0$ in norm. Thus, the eigenvalues corresponding to ξ_n must converge to 0. This shows, in particular, that each eigenspace corresponding to a non-zero eigenvalue must be finite dimensional. Also, since eigenspaces corresponding to distinct eigenvectors are orthogonal by Lemma 3.0.17, it follows that x can have at most finitely many eigenvalues with modulus greater than any fixed positive number. Thus, by Theorem 3.3.4 we have that $\sigma(x)$ is countable and has no non-zero accumulation points.

If we set $y = \sum_{\lambda \in \sigma(x) \setminus \{0\}} \lambda P_{E_\lambda}$ then y is compact since it is a norm limit of finite rank operators. Thus, $x - y$ is compact and normal, and $\mathrm{sp}\{E_\lambda\} \subset \ker(x - y)$. If $\xi \in \mathcal{H}$ is an eigenvector for $x - y$ with non-zero eigenvalue, then ξ must be orthogonal to $\ker(x - y)$, and hence ξ is orthogonal to E_λ for each $\lambda \in \sigma(x) \setminus \{0\}$. Hence, we would have $y\xi = 0$, and so ξ would be an eigenvector for x. But then $\xi \in E_\lambda$ for some $\lambda \in \sigma(x) \setminus \{0\}$ giving a contradiction.

Thus, we conclude that $x - y$ is a compact operator without non-zero eigenvalues. From Theorem 3.3.4 we than have that $\sigma(x - y) = \{0\}$. Since $x - y$ is normal we then have $x = y = \sum_{\lambda \in \sigma(x) \setminus \{0\}} \lambda P_{E_\lambda}$. ■

Theorem 3.3.6 (Alternate form of the spectral theorem for compact operators)**.** *Let $x \in \mathcal{K}(\mathcal{H})$ be a normal compact operator. Then there exists a set X, a function $f \in c_0(X)$, and a unitary operator $U : \ell^2 X \to \mathcal{H}$, such that $x = U M_f U^*$, where $M_f \in \mathcal{B}(\ell^2 X)$ is the operator defined by $M_f \xi = f\xi$.*

Proof. From the previous theorem we may write $x = \sum_{\lambda \in \sigma(x) \setminus \{0\}} \lambda P_{E_\lambda}$. We then have $\ker(x) = \ker(x^*)$, and $\mathcal{H} = \ker(x) \oplus \bigoplus_{\lambda \in \sigma(x) \setminus \{0\}} E_\lambda$. Thus, there exists an orthonormal basis $\{\xi_x\}_{x \in X} \subset \mathcal{H}$, which consists of eigenvectors for x. If we consider the unitary operator $U : \ell^2 X \to \mathcal{H}$, which sends δ_x to ξ_x, and we consider the function $f \in c_0(X)$ by letting $f(x)$ be the eigenvalue corresponding to the eigenvector ξ_x, then it is easy to see that $x = U M_f U^*$. ■

Exercise 3.3.7. Show that the map $\psi : L^1(\mathcal{B}(\mathcal{H})) \to \mathcal{K}(\mathcal{H})^*$ given by $\psi_x(a) = \mathrm{Tr}(ax)$ implements an isometric Banach space isomorphism between $L^1(\mathcal{B}(\mathcal{H}))$ and $\mathcal{K}(\mathcal{H})^*$.

3.4 Locally convex topologies on the space of operators

Let $\mathcal{H}$ be a Hilbert space. On $\mathcal{B}(\mathcal{H})$ we define the following locally convex topologies:

- The **weak operator topology** (WOT) is defined by the family of seminorms $T \mapsto |\langle T\xi, \eta \rangle|$, for $\xi, \eta \in \mathcal{H}$.
- The **strong operator topology** (SOT) is defined by the family of seminorms $T \mapsto \|T\xi\|$, for $\xi \in \mathcal{H}$.

Note that the from coarsest to finest topologies we have

$$\text{WOT} \prec \text{SOT} \prec \text{Uniform}.$$

Also note that since an operator T is normal if and only if $\|T\xi\| = \|T^*\xi\|$ for all $\xi \in \mathcal{H}$, it follows that the adjoint is SOT continuous on the set of normal operators.

Lemma 3.4.1. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(*i*) *There exists* $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in \mathcal{H}$ *such that* $\varphi(T) = \sum_{i=1}^n \langle T\xi_i, \eta_i \rangle$, *for all* $T \in \mathcal{B}(\mathcal{H})$.

(*ii*) φ *is WOT continuous.*

(*iii*) φ *is SOT continuous.*

Proof. The implications (i) $\implies$ (ii) and (ii) $\implies$ (iii) are clear and so we will only show (iii) $\implies$ (i). Suppose φ is SOT continuous. Thus, the inverse image of the open ball in $\mathbb{C}$ is open in the SOT and hence by considering the semi-norms which define the topology we have that there exists a constant $K > 0$, and $\xi_1, \dots, \xi_n \in \mathcal{H}$ such that

$$|\varphi(T)|^2 \leq K \sum_{i=1}^n \|T\xi_i\|^2.$$

If we then consider $\{\oplus_{i=1}^n T\xi_i \mid T \in \mathcal{B}(\mathcal{H})\} \subset \mathcal{H}^{\oplus n}$, and let $\mathcal{H}_0$ be its closure, we have that

$$\oplus_{i=1}^n T\xi_i \mapsto \varphi(T)$$

extends to a well defined, continuous linear functional on $\mathcal{H}_0$ and hence by the Riesz representation theorem there exists $\eta_1, \dots, \eta_n \in \mathcal{H}$ such that

$$\varphi(T) = \sum_{i=1}^n \langle T\xi_i, \eta_i \rangle,$$

for all $T \in \mathcal{B}(\mathcal{H})$. ∎

Corollary 3.4.2. *Let* $K \subset \mathcal{B}(\mathcal{H})$ *be a convex set, then the WOT, SOT, and closures of* K *coincide.*

Proof. By Lemma 3.4.1 the three topologies above give rise to the same dual space, hence this follows from the the Hahn-Banach separation theorem. ∎

If $\mathcal{H}$ is a Hilbert space then the map $\mathrm{id} \otimes 1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}\overline{\otimes}\ell^2\mathbb{N})$ defined by $(\mathrm{id} \otimes 1)(x) = x \otimes 1$ need not be continuous in either of the locally convex topologies defined above even though it is an isometric C^*-homomorphism with respect to the uniform topology. Thus, on $\mathcal{B}(\mathcal{H})$ we define the following additional locally convex topologies:

- The **σ-weak operator topology** (σ-WOT) is defined by pulling back the WOT of $\mathcal{B}(\mathcal{H}\overline{\otimes}\ell^2\mathbb{N})$ under the map $\mathrm{id}\otimes 1$.
- The **σ-strong operator topology** (σ-SOT) is defined by pulling back the SOT of $\mathcal{B}(\mathcal{H}\overline{\otimes}\ell^2\mathbb{N})$ under the map $\mathrm{id}\otimes 1$.

Note that the σ-weak operator topology can alternately be defined by the family of semi-norms $T \mapsto |\operatorname{Tr}(Ta)|$, for $a \in L^1(\mathcal{B}(\mathcal{H}))$. Hence, under the identification $\mathcal{B}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))^*$, we have that the weak*-topology on $\mathcal{B}(\mathcal{H})$ agrees with the σ-WOT.

Lemma 3.4.3. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(i) *There exists a trace class operator $a \in L^1(\mathcal{B}(\mathcal{H}))$ such that $\varphi(x) = \mathrm{Tr}(xa)$ for all $x \in \mathcal{B}(\mathcal{H})$*

(ii) *φ is σ-WOT continuous.*

(iii) *φ is σ-SOT continuous.*

Proof. Again, we need only show the implication (iii) $\implies$ (i), so suppose φ is σ-SOT continuous. Then by the Hahn-Banach theorem, considering $\mathcal{B}(\mathcal{H})$ as a subspace of $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$ through the map $\mathrm{id} \otimes 1$, we may extend φ to a SOT continuous linear functional on $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$. Hence by Lemma 3.4.1 there exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}\overline{\otimes}\ell^2\mathbb{N}$ such that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\varphi(x) = \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle.$$

For each $1 \leq i \leq n$ we may define $a_i, b_i \in \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$ as the operators corresponding to ξ_i, η_i in the Hilbert space isomorphism $\mathcal{H} \otimes \ell^2\mathbb{N} \cong \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$. By considering $a = \sum_{i=1}^n b_i^* a_i \in L^1(\mathcal{B}(\mathcal{H}))$, it then follows that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned} \mathrm{Tr}(xa) &= \sum_{i=1}^{n} \langle a_i x, b_i \rangle_2 \\ &= \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle = \varphi(x). \end{aligned}$$

■

Corollary 3.4.4. *The unit ball in $\mathcal{B}(\mathcal{H})$ is compact in the σ-WOT.*

Proof. This follows from Theorem 3.1.7 and the Banach-Alaoglu theorem. ■

Corollary 3.4.5. *The WOT and the σ-WOT agree on bounded sets.*

Proof. The identity map is clearly continuous from the σ-WOT to the WOT. Since both spaces are Hausdorff it follows that this is a homeomorphism from the σ-WOT compact unit ball in $\mathcal{B}(\mathcal{H})$. By scaling we therefore have that this is a homeomorphism on any bounded set. ■

Exercise 3.4.6. Show that the adjoint $T \mapsto T^*$ is continuous in the WOT, and when restricted to the space of normal operators is continuous in the SOT, but is not continuous in the SOT on the space of all bounded operators.

Exercise 3.4.7. Show that operator composition is jointly continuous in the SOT on bounded subsets.

Exercise 3.4.8. Show that the SOT agrees with the σ-SOT on bounded subsets of $\mathcal{B}(\mathcal{H})$.

Exercise 3.4.9. Show that pairing $\langle x, a \rangle = \mathrm{Tr}(a^* x)$ gives an identification between $\mathcal{K}(\mathcal{H})^*$ and $(L^1(\mathcal{B}(\mathcal{H})), \| \cdot \|_1)$.

3.5 Von Neumann algebras and the double commutant theorem

A **von Neumann algebra** (over a Hilbert space $\mathcal{H}$) is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains 1 and is closed in the weak operator topology.

Note that since subalgebras are of course convex, it follows from Corollary 3.4.2 that von Neumann algebras are also closed in the strong operator topology.

If $A \subset \mathcal{B}(\mathcal{H})$ then we denote by $W^*(A)$ the von Neumann subalgebra which is generated by A, i.e., $W^*(A)$ is the smallest von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ which contains A.

Lemma 3.5.1. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $(A)_1$ is compact in the WOT.*

Proof. This follows directly from Corollary 3.4.4. ∎

Corollary 3.5.2. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then $(A)_1$ and $A_{\text{s.a.}}$ are closed in the weak and strong operator topologies.*

Proof. Since taking adjoints is continuous in the weak operator topology it follows that $A_{\text{s.a.}}$ is closed in the weak operator topology, and by the previous result this is also the case for $(A)_1$. ∎

If $B \subset \mathcal{B}(\mathcal{H})$, the **commutant** of B is

$$B' = \{T \in \mathcal{B}(\mathcal{H}) \mid TS = ST, \text{ for all } S \in B\}.$$

We also use the notation $B'' = (B')'$ for the **double commutant**.

Theorem 3.5.3. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint set, then A' is a von Neumann algebra.*

Proof. It is easy to see that A' is a self-adjoint algebra containing 1. To see that it is closed in the weak operator topology just notice that if $x_\alpha \in A'$ is a net such that $x_\alpha \to x \in \mathcal{B}(\mathcal{H})$ then for any $a \in A$, and $\xi, \eta \in \mathcal{H}$, we have

$$\langle [x, a]\xi, \eta \rangle = \langle xa\xi, \eta \rangle - \langle x\xi, a^*\eta \rangle$$

$$= \lim_{\alpha \to \infty} \langle x_\alpha a\xi, \eta \rangle - \langle x_\alpha \xi, a^*\eta \rangle = \lim_{\alpha \to \infty} \langle [x_\alpha, a]\xi, \eta \rangle = 0.$$ ∎

Corollary 3.5.4. *A self-adjoint maximal abelian subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra.*

Proof. Since A is maximal abelian we have $A = A'$. ∎

Lemma 3.5.5. *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing 1. Then for all $\xi \in \mathcal{H}$, and $x \in A''$ there exists $x_\alpha \in A$ such that $\lim_{\alpha \to \infty} \|(x - x_\alpha)\xi\| = 0$.*

Proof. Consider the closed subspace $\mathcal{K} = \overline{A\xi} \subset \mathcal{H}$, and denote by p the projection onto this subspace. Since for all $a \in A$ we have $a\mathcal{K} \subset \mathcal{K}$, it follows that $ap = pap$. But since A is self-adjoint it then also follows that for all $a \in A$ we have $pa = (a^*p)^* = (pa^*p)^* = pap = ap$, and hence $p \in A'$.

We therefore have that $xp = xp^2 = pxp$ and hence $x\mathcal{K} \subset \mathcal{K}$. Since $1 \in A$ it follows that $\xi \in \mathcal{K}$ and hence also $x\xi \in \overline{A\xi}$. ■

Theorem 3.5.6 (Von Neumann's double commutant theorem)**.** *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing* 1*. Then A'' is equal to the weak operator topology closure of A.*

Proof. By Theorem 3.5.3 we have that A'' is closed in the weak operator topology, and we clearly have $A \subset A''$, so we just need to show that $A \subset A''$ is dense in the weak operator topology. For this we use the previous lemma together with a matrix trick.

Let $\xi_1, \ldots, \xi_n \in \mathcal{H}$, $x \in A''$ and consider the subalgebra $\tilde{A}$ of $\mathcal{B}(\mathcal{H}^n) \cong \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$ consisting of diagonal matrices with constant diagonal coefficients contained in A. Then the diagonal matrix whose diagonal entries are all x is easily seen to be contained in $\tilde{A}''$, hence the previous lemma applies and so there exists a net $a_\alpha \in A$ such that $\lim_{\alpha \to \infty} \|(x - a_\alpha)\xi_k\| = 0$, for all $1 \leq k \leq n$. This shows that $A \subset A''$ is dense in the strong operator topology. ■

We also have the following formulation which is easily seen to be equivalent.

Corollary 3.5.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra. Then A is a von Neumann algebra if and only if $A = A''$.*

Corollary 3.5.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $x \in A$, and consider the polar decomposition $x = v|x|$. Then $v \in A$.*

Proof. Note that $\ker(v) = \ker(|x|)$, and if $a \in A'$ then we have $a\ker(|x|) \subset \ker(|x|)$. Also, we have

$$\|(av - va)|x|\xi\| = \|ax\xi - xa\xi\| = 0,$$

for all $\xi \in \mathcal{H}$. Hence av and va agree on $\ker(|x|) + \overline{R(|x|)} = \mathcal{H}$, and so $v \in A'' = A$. ■

Proposition 3.5.9. *Let (X, μ) be a σ-finite*[1] *measure space. Consider the Hilbert space $L^2(X, \mu)$, and the map $M : L^\infty(X, \mu) \to \mathcal{B}(L^2(X, \mu))$ defined by $(M_g\xi)(x) = g(x)\xi(x)$, for all $\xi \in L^2(X, \mu)$. Then M is an isometric $*$-isomorphism from $L^\infty(X, \mu)$ onto a maximal abelian von Neumann subalgebra of $\mathcal{B}(L^2(X, \mu))$.*

Proof. The fact that M is a $*$-isomorphism onto its image is clear. If $g \in L^\infty(X, \mu)$ then by definition of $\|g\|_\infty$ we can find a sequence E_n of measurable

[1] For technical reasons we restrict ourselves to σ-finite spaces, although here and throughout most of these notes the proper setting is really that of localizable spaces [**?**]

subsets of X such that $0 < \mu(E_n) < \infty$, and $|g|_{|E_n} \geq \|g\|_\infty - 1/n$, for all $n \in \mathbb{N}$. We then have

$$\|M_g\| \geq \|M_g 1_{E_n}\|_2 / \|1_{E_n}\|_2 \geq \|g\|_\infty - 1/n.$$

The inequality $\|g\|_\infty \leq \|M_g\|$ is also clear and hence M is isometric.

To see that $M(L^\infty(X,\mu))$ is maximal abelian let's suppose $T \in \mathcal{B}(L^2(X,\mu))$ commutes with M_f for all $f \in L^\infty(X,\mu)$. We take $E_n \subset X$ measurable sets such that $0 < \mu(E_n) < \infty$, $E_n \subset E_{n+1}$, and $X = \cup_{n\in\mathbb{N}} E_n$. Define $f_n \in L^2(X,\mu)$ by $f_n = T(1_{E_n})$.

For each $g, h \in L^\infty(X,\mu) \cap L^2(X,\mu)$, we have

$$|\int f_n g \overline{h} d\mu| = |\langle M_g T(1_{E_n}), h\rangle| = |\langle T(g_{|E_n}), h\rangle| \leq \|T\| \|g\|_2 \|h\|_2.$$

Since $L^\infty(X,\mu) \cap L^2(X,\mu)$ is dense in $L^2(X,\mu)$, it then follows from Hölder's inequality that $f_n \in L^\infty(X,\mu)$ with $\|f_n\|_\infty \leq \|T\|$, and that $M_{1_{E_n}} T = M_{f_n}$. Note that for $m \geq n$, $1_{E_m} f_n = 1_{E_m} T(1_{E_n}) = T(1_{E_n}) = f_n$. Hence, $\{f_n\}$ converges almost every where to a measurable function f. Since $\|f_n\|_\infty \leq \|T\|$ for each n, we have $\|f\|_\infty \leq \|T\|$. Moreover, if $g, h \in L^2(X,\mu)$ then we have

$$\int f g \overline{h} d\mu = \lim_{n\to\infty} \int f_n g \overline{h} d\mu = \lim_{n\to\infty} \langle 1_{E_n} T(g), h\rangle = \langle T(g), h\rangle.$$

Thus, $T = M_f$. ∎

Because of the previous result we will often identify $L^\infty(X,\mu)$ with the subalgebra of $\mathcal{B}(L^2(X,\mu))$ as described above. This should not cause any confusion.

Exercise 3.5.10. Let X be an uncountable set, $\mathcal{B}_1$ the set of all subsets of X, $\mathcal{B}_2 \subset B_1$ the set consisting of all sets which are either countable or have countable complement, and μ the counting measure on X. Show that the identity map implements a unitary operator $\mathrm{id} : L^2(X, \mathcal{B}_1, \mu) \to L^2(X, \mathcal{B}_2, \mu)$, and we have $L^\infty(X, \mathcal{B}_2, \mu) \subsetneq L^\infty(X, \mathcal{B}_2, \mu)'' = \mathrm{id}\, L^\infty(X, \mathcal{B}_1, \mu)\, \mathrm{id}^*$.

3.6 Kaplansky's density theorem

Proposition 3.6.1. *If $f \in C(\mathbb{C})$ then $x \mapsto f(x)$ is continuous in the strong operator topology on any bounded set of normal operators in $\mathcal{B}(\mathcal{H})$.*

Proof. By the Stone-Weierstrass theorem we can approximate f uniformly well by polynomials on any compact set. Since multiplication is jointly SOT continuous on bounded sets, and since taking adjoints is SOT continuous on normal operators, the result follows easily. ∎

Proposition 3.6.2 (The Cayley transform)**.** *The map $x \mapsto (x - i)(x + i)^{-1}$ is strong operator topology continuous from the set of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ into the unitary operators in $\mathcal{B}(\mathcal{H})$.*

Proof. Suppose $\{x_k\}_k$ is a net of self-adjoint operators such that $x_k \to x$ in the SOT. By the spectral mapping theorem we have $\|(x_k+i)^{-1}\| \leq 1$ and hence for all $\xi \in \mathcal{H}$ we have

$$\begin{aligned} &\|(x-i)(x+i)^{-1}\xi - (x_k-i)(x_k+i)^{-1}\xi\| \\ &= \|(x_k+i)^{-1}((x_k+i)(x-i) - (x_k-i)(x+i))(x+i)^{-1}\xi\| \\ &= \|2i(x_k+i)^{-1}(x-x_k)(x+i)^{-1}\xi\| \leq 2\|(x-x_k)(x+i)^{-1}\xi\| \to 0. \end{aligned}$$

■

Corollary 3.6.3. *If $f \in C_0(\mathbb{R})$ then $x \mapsto f(x)$ is strong operator topology continuous on the set of self-adjoint operators.*

Proof. Since f vanishes at infinity, we have that $g(t) = f\left(i\frac{1+t}{1-t}\right)$ defines a continuous function on $\mathbb{T}$ if we set $g(1) = 0$. By Proposition 3.6.1 $x \mapsto g(x)$ is then SOT continuous on the space of unitaries. If $U(z) = \frac{z-i}{z+i}$ is the Cayley transform, then by Proposition 3.6.2 it follows that $f = g \circ U$ is SOT continuous being the composition of two SOT continuous functions. ■

Theorem 3.6.4 (Kaplansky's density theorem)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ and denote by B the strong operator topology closure of A.*

(*i*) *The strong operator topology closure of $A_{\text{s.a.}}$ is $B_{\text{s.a.}}$.*

(*ii*) *The strong operator topology closure of $(A)_1$ is $(B)_1$.*

Proof. We may assume that A is a C^*-algebra. If $\{x_k\}_k \subset A$ is a net of elements which converge in the SOT to a self-adjoint element x_k, then since taking adjoints is WOT continuous we have that $\frac{x_k+x_k^*}{2} \to x$ in the WOT. But $A_{\text{s.a.}}$ is convex and so the WOT and SOT closures coincide, showing (a). Moreover, if $\{y_k\}_k \subset A_{\text{s.a.}}$ such that $y_k \to x$ in the SOT then by considering a function $f \in C_0(\mathbb{R})$ such that $f(t) = t$ for $|t| \leq \|x\|$, and $|f(t)| \leq \|x\|$, for $t \in \mathbb{R}$, we have $\|f(y_k)\| \leq \|x\|$, for all k and $f(y_k) \to f(x)$ in the SOT by Corollary 3.6.3. Hence $(A)_1 \cap A_{\text{s.a.}}$ is SOT dense in $(B)_1 \cap B_{\text{s.a.}}$.

Note that $\mathbb{M}_2(A)$ is SOT dense in $\mathbb{M}_2(B) \subset \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Therefore if $x \in (B)_1$ then $\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (\mathbb{M}_2(B))_1$ is self-adjoint. Hence from above there exists a net of operators $\tilde{x}_n \in (\mathbb{M}_2(A))_1$ such that $\tilde{x}_n \to \tilde{x}$ in the SOT. Writing $\tilde{x}_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ we then have that $\|b_n\| \leq 1$ and $b_n \to x$ in the SOT. ■

Corollary 3.6.5. *A self-adjoint unital subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $(A)_1$ is closed in the SOT.*

Corollary 3.6.6. *A self-adjoint unital subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if A is closed in the σ-WOT.*

3.7 The spectral theorem and Borel functional calculus

For $T \in \mathcal{K}(\mathcal{H})$ a compact normal operator, there were two different perspectives we could take when describing the spectral theorem for T. The first (Theorem 3.3.5) was a basis free approach, we considered the eigenvalues $\sigma_p(T)$ for T, and to each eigenvalue λ associated to it the projection $E(\lambda)$ onto the corresponding eigenspace. Since T is normal we have that the $E(\lambda)$'s are pairwise orthogonal and we showed

$$T = \sum_{\lambda \in \sigma(T)} \lambda E(\lambda).$$

The second approach (Theorem 3.3.6) was to use that since T is normal, it is diagonalizable with respect to a given basis, i.e., we produced a set X unitary matrix $U : \ell^2 X \to \mathcal{H}$ such that UTU^* is a multiplication operator corresponding to some function $f \in \ell^\infty X$.

For bounded normal operators there are two similar approaches to the spectral theorem. The first approach is to find a substitute for the projections $E(\lambda)$ and this leads naturally to the notion of a spectral measure. For the second approach, this naturally leads to the interpretation of diagonal matrices corresponding to multiplication by essentially bounded functions on a measure space.

Lemma 3.7.1. *Let $x_\alpha \in \mathcal{B}(\mathcal{H})$ be an increasing net of positive operators such that $\sup_\alpha \|x_\alpha\| < \infty$, then there exists a bounded operator $x \in \mathcal{B}(\mathcal{H})$ such that $x_\alpha \to x$ in the SOT.*

Proof. We may define a quadratic form on $\mathcal{H}$ by $\xi \mapsto \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$. Since $\sup_\alpha \|x_\alpha\| < \infty$ we have that this quadratic form is bounded and hence there exists a bounded positive operator $x \in \mathcal{B}(\mathcal{H})$ such that $\|\sqrt{x}\xi\|^2 = \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$, for all $\xi \in \mathcal{H}$. Note that $x_\alpha \leq x$ for all α, and $\sup_\alpha \|(x - x_\alpha)^{1/2}\| < \infty$. Thus for each $\xi \in \mathcal{H}$ we have

$$\begin{aligned}\|(x - x_\alpha)\xi\|^2 &\leq \|(x - x_\alpha)^{1/2}\|^2 \|(x - x_\alpha)^{1/2}\xi\|^2 \\ &= \|(x - x_\alpha)^{1/2}\|^2 (\|\sqrt{x}\xi\|^2 - \|\sqrt{x_\alpha}\xi\|^2) \to 0.\end{aligned}$$

Hence, $x_\alpha \to x$ in the SOT. ■

Corollary 3.7.2. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\{p_\iota\}_{\iota \in I} \subset A$ is a collection of pairwise orthogonal projections then $p = \sum_{\iota \in I} p_\iota \in A$ is well defined as a SOT limit of finite sums.*

3.7.1 Spectral measures

Let K be a locally compact Hausdorff space and let $\mathcal{H}$ be a Hilbert space. A **spectral measure** E on K relative to $\mathcal{H}$ is a mapping from the Borel subsets of K to the set of projections[2] in $\mathcal{B}(\mathcal{H})$ such that

[2]In Chapter 8 we'll consider the more general situation where we map into the space of bounded operators. See Example 8.1.9 and Corollary 8.2.2

(*i*) $E(\emptyset) = 0, E(K) = 1$.

(*ii*) $E(B_1 \cup B_2) = E(B_1) + E(B_2)$ for all disjoint Borel sets B_1 and B_2.

(*iii*) For all $\xi, \eta \in \mathcal{H}$ the function

$$B \mapsto E_{\xi,\eta}(B) = \langle E(B)\xi, \eta\rangle$$

is a finite Radon measure on K.

Example 3.7.3. If K is a locally compact Hausdorff space and μ is a σ-finite Radon measure on K, then the map $E(B) = 1_B \in L^\infty(K, \mu) \subset \mathcal{B}(L^2(K, \mu))$ defines a spectral measure on K relative to $L^2(K, \mu)$.

We denote by $B_\infty(K)$ the space of all bounded Borel functions on K. This is clearly a C^*-algebra with the sup norm.

For each $f \in B_\infty(K)$ it follows that the map

$$(\xi, \eta) \mapsto \int f \, dE_{\xi,\eta}$$

gives a continuous sesqui-linear form on $\mathcal{H}$ and hence it follows that there exists a bounded operator T such that $\langle T\xi, \eta\rangle = \int f \, dE_{\xi,\eta}$. We denote this operator T by $\int f \, dE$ so that we have the formula $\langle (\int f \, dE)\xi, \eta\rangle = \int f \, dE_{\xi,\eta}$, for each $\xi, \eta \in \mathcal{H}$.

Theorem 3.7.4. *Let K be a locally compact Hausdorff space, let $\mathcal{H}$ be a Hilbert space, and suppose that E is a spectral measure on K relative to $\mathcal{H}$. Then the association*

$$f \mapsto \int f \, dE$$

defines a continuous unital $*$-homomorphism from $B_\infty(K)$ to $\mathcal{B}(\mathcal{H})$. Moreover, the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, and if $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty$, then $\int f_n dE \to \int f dE$ in the SOT.

Proof. It is easy to see that this map defines a linear contraction which preserves the adjoint operation. If $A, B \subset K$ are Borel subsets, and $\xi, \eta \in \mathcal{H}$, then denoting $x = \int 1_A dE$, $y = \int 1_B dE$, and $z = \int 1_{A\cap B} dE$ we have

$$\begin{aligned} \langle xy\xi, \eta\rangle &= \langle E(A)y\xi, \eta\rangle = \langle E(B)\xi, E(A)\eta\rangle \\ &= \langle E(B \cap A)\xi, \eta\rangle = \langle z\xi, \eta\rangle. \end{aligned}$$

Hence $xy = z$, and by linearity we have that $(\int f \, dE)(\int g \, dE) = \int fg \, dE$ for all simple functions $f, g \in B_\infty(K)$. Since every function in $B_\infty(K)$ can be approximated uniformly by simple functions this shows that this is indeed a $*$-homomorphism.

To see that the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, note that if a commutes with all operators of the form $\int f\,dE$ for $f \in C(K)$ then for all $\xi, \eta \in \mathcal{H}$ we have

$$0 = \langle (a(\int f\,dE) - (\int f\,dE)a)\xi, \eta\rangle = \int f\,dE_{\xi, a^*\eta} - \int f\,dE_{a\xi,\eta}.$$

Thus $E_{\xi,a^*\eta} = E_{a\xi,\eta}$ and hence we have that a also commutes with operators of the form $\int g\,dE$ for any $g \in B_\infty(K)$. Therefore by Theorem 3.5.6 $\int g\,dE$ is contained in the von Neumann algebra generated by the image of $C(K)$.

Now suppose $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty(K)$. For each $\xi, \eta \in \mathcal{H}$ we have

$$\int f_n\,dE_{\xi,\eta} \to \int f\,dE_{\xi,\eta},$$

hence $\int f_n\,dE$ converges in the WOT to $\int f\,dE$. However, since $\int f_n\,dE$ is an increasing sequence of bounded operators with $\| \int f_n\,dE\| \leq \|f\|_\infty$, Lemma 3.7.1 shows that $\int f_n\,dE$ converges in the SOT to some operator $x \in \mathcal{B}(\mathcal{H})$ and we must then have $x = \int f\,dE$. ■

The previous theorem shows, in particular, that if A is an abelian C^*-algebra, and E is a spectral measure on $\sigma(A)$ relative to $\mathcal{H}$, then we obtain a unital $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ by the formula

$$\pi(x) = \int \Gamma(x)dE.$$

We next show that in fact every unital $*$-representation arises in this way.

Theorem 3.7.5 (The spectral theorem)**.** *Let A be an abelian C^*-algebra, $\mathcal{H}$ a Hilbert space and $\pi : A \to \mathcal{B}(\mathcal{H})$ a $*$-representation, which is non-degenerate in the sense that $\xi = 0$ if and only if $\pi(x)\xi = 0$ for all $x \in A$. Then there is a unique spectral measure E on $\sigma(A)$ relative to $\mathcal{H}$ such that for all $x \in A$ we have*

$$\pi(x) = \int \Gamma(x)dE.$$

Proof. For each $\xi, \eta \in \mathcal{H}$ we have that $f \mapsto \langle \pi(\Gamma^{-1}(f))\xi, \eta\rangle$ defines a bounded linear functional on $\sigma(A)$ and hence by the Riesz representation therorem there exists a Radon measure $E_{\xi,\eta}$ such that for all $f \in C(\sigma(A))$ we have

$$\langle \pi(\Gamma^{-1}(f))\xi, \eta\rangle = \int f dE_{\xi,\eta}.$$

Since the Gelfand transform is a $*$-homomorphism we verify easily that $f dE_{\xi,\eta} = dE_{\pi(\Gamma^{-1}(f))\xi,\eta} = dE_{\xi,\pi(\Gamma^{-1}(\overline{f}))\eta}$.

Thus for each Borel set $B \subset \sigma(A)$ we can consider the sesquilinear form $(\xi, \eta) \mapsto \int 1_B dE_{\xi,\eta}$. We have $| \int f dE_{\xi,\eta}| \leq \|f\|_\infty \|\xi\| \|\eta\|$, for all $f \in C(\sigma(A))$

and hence this sesquilinear form is bounded and there exists a bounded operator $E(B)$ such that $\langle E(B)\xi, \eta\rangle = \int 1_B dE_{\xi,\eta}$, for all $\xi, \eta \in \mathcal{H}$. For all $f \in C(\sigma(A))$ we have

$$\langle \pi(\Gamma^{-1}(f))E(B)\xi, \eta\rangle = \int 1_B dE_{\xi, \pi(\Gamma^{-1}(\bar{f}))\eta} = \int 1_B f dE_{\xi,\eta}.$$

Thus it follows that $E(B)^* = E(B)$, and $E(B')E(B) = E(B' \cap B)$, for any Borel set $B' \subset \sigma(A)$. In particular, $E(B)$ is a projection and since π is non-degenerate it follows easily that $E(\sigma(A)) = 1$, thus E gives a spectral measure on $\sigma(A)$ relative to $\mathcal{H}$. The fact that for $x \in A$ we have $\pi(x) = \int \Gamma(x) dE$ follows easily from the way we constructed E. ∎

If $\mathcal{H}$ is a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$ is a normal operator, then by applying the previous theorem to the C^*-subalgebra A generated by x and 1, and using the identification $\sigma(A) = \sigma(x)$ we obtain a homomorphism from $B_\infty(\sigma(x))$ to $\mathcal{B}(\mathcal{H})$ and hence for $f \in B_\infty(\sigma(x))$ we may define

$$f(x) = \int f dE.$$

Note that it is straight forward to check that considering the function $f(z) = z$ we have

$$x = \int z dE(z).$$

We now summarize some of the properties of this functional calculus which follow easily from the previous results.

Theorem 3.7.6 (Borel functional calculus)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose $x \in A$ is a normal operator, then the Borel functional calculus defined by $f \mapsto f(x)$ satisfies the following properties:*

(i) $f \mapsto f(x)$ is a continuous unital $*$-homomorphism from $B_\infty(\sigma(x))$ into A.

(ii) If $f \in B_\infty(\sigma(x))$ then $\sigma(f(x)) \subset f(\sigma(x))$.

(iii) If $f \in C(\sigma(x))$ then $f(x)$ agrees with the definition given by continuous functional calculus.

Corollary 3.7.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then A is the uniform closure of the span of its projections.*

Proof. By decomposing an operator into its real and imaginary parts it is enough to check this for self-adjoint operators in the unit ball, and this follows from the previous theorem by approximating the function $f(t) = t$ uniformly by simple functions on $[-1, 1]$. ∎

Corollary 3.7.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then the unitary group $\mathcal{U}(A)$ is path connected in the uniform topology.*

Proof. If $u \in \mathcal{U}(A)$ is a unitary and we consider a branch of the log function $f(z) = \log z$, then from Borel functional calculus we have $u = e^{ix}$ where $x = -if(u)$ is self-adjoint. We then have that $u_t = e^{itx}$ is a uniform norm continuous path of unitaries such that $u_0 = 1$ and $u_1 = u$. ∎

Corollary 3.7.9. *If $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then $\mathcal{K}(\mathcal{H})$ is the unique non-zero proper norm closed two sided ideal in $\mathcal{B}(\mathcal{H})$.*

Proof. If $I \subset \mathcal{B}(\mathcal{H})$ is a norm closed two sided ideal and $x \in I \setminus \{0\}$, then for any $\xi \in R(x^*x)$, $\|\xi\| = 1$ we can consider $y = (\xi \otimes \overline{\xi})x^*x(\xi \otimes \overline{\xi}) \in I$ which is a rank one self-adjoint operator with $R(y) = \mathbb{C}\xi$. Thus y is a multiple of $(\xi \otimes \overline{\xi})$ and hence $(\xi \otimes \overline{\xi}) \in I$. For any $\zeta, \eta \in \mathcal{H}$, we then have $\zeta \otimes \overline{\eta} = (\zeta \otimes \overline{\xi})(\xi \otimes \overline{\xi})(\xi \otimes \overline{\eta}) \in I$ and hence I contains all finite rank operators. Since I is closed we then have that $\mathcal{K}(\mathcal{H}) \subset I$.

If $x \in I$ is not compact then for some $\varepsilon > 0$ we have that $\dim(1_{[\varepsilon,\infty)}(x^*x)\mathcal{H}) = \infty$. If we let $u \in \mathcal{B}(\mathcal{H})$ be an isometry from $\mathcal{H}$ onto $1_{[\varepsilon,\infty)}(x^*x)\mathcal{H}$, then we have that $\sigma(u^*x^*xu) \subset [\varepsilon, \infty)$. Hence, $u^*x^*xu \in I$ is invertible which shows that $I = \mathcal{B}(\mathcal{H})$. ∎

Exercise 3.7.10. Suppose that K is a compact Hausdorff space and E is a spectral measure for K relative to a Hilbert space $\mathcal{H}$, show that if $\{f_n\}_n \subset B_\infty(K)$ is a uniformly bounded sequence, and $f \in B_\infty(K)$ such that $f_n(k) \to f(k)$ for every $k \in K$, then $\int f_n \, \mathrm{d}E \to \int f \, \mathrm{d}E$ in the strong operator topology.

3.8 Abelian von Neumann algebras

Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $\xi \in \mathcal{H}$ is a non-zero vector. Then ξ is said to be **cyclic** for A if $A\xi$ is dense in $\mathcal{H}$. We say that ξ is **separating** for A if $x\xi \neq 0$, for all $x \in A$, $x \neq 0$.

Proposition 3.8.1. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then a non-zero vector $\xi \in \mathcal{H}$ is cyclic for A if and only if ξ is separating for A'.*

Proof. Suppose ξ is cyclic for A, and $x \in A'$ such that $x\xi = 0$. Then $xa\xi = ax\xi = 0$ for all $a \in A$, and since $A\xi$ is dense in $\mathcal{H}$ it follows that $x\eta = 0$ for all $\eta \in \mathcal{H}$. Conversely, if $A\xi$ is not dense, then the orthogonal projection p onto its complement is a nonzero operator in A' such that $p\xi = 0$. ∎

Corollary 3.8.2. *If $A \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann algebra and $\xi \in \mathcal{H}$ is cyclic, then ξ is also separating.*

Proof. Since ξ being separating passes to von Neumann subalgebras and $A \subset A'$ this follows. ∎

Infinite dimensional von Neumann algebras are never separable in the norm topology. For this reason we will say that a von Neumann algebra A is **separable** if A is separable in the SOT. Equivalently, A is separable if its predual A_* is separable.

Proposition 3.8.3. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable von Neumann algebra. Then there exists a separating vector for A.*

Proof. Since A is separable, it follows that there exists a countable collection of vectors $\{\xi_k\}_k \subset \mathcal{H}$ such that $x\xi_k = 0$ for all k only if $x = 0$. Also, since A is separable we have that $\mathcal{H}_0 = \overline{\text{sp}}(A\{\xi_k\}_k)$ is also separable. Thus, restricting A to $\mathcal{H}_0$ we may assume that $\mathcal{H}$ is separable.

By Zorn's lemma we can find a maximal family of non-zero unit vectors $\{\xi_\alpha\}_\alpha$ such that $A\xi_\alpha \perp A\xi_\beta$, for all $\alpha \neq \beta$. Since $\mathcal{H}$ is separable this family must be countable and so we may enumerate it $\{\xi_n\}_n$, and by maximality we have that $\{A\xi_n\}_n$ is dense in $\mathcal{H}$.

If we denote by p_n the orthogonal projection onto the closure of $A\xi_n$ then we have that $p_n \in A'$, hence, setting $\xi = \sum_n \frac{1}{2^n}\xi$ if $x \in A$ such that $x\xi = 0$, then for every $n \in \mathbb{N}$ we have $0 = 2^n p_n x\xi = 2^n x p_n \xi = x\xi_n$ and so $x = 0$ showing that ξ is a separating vector for A. ∎

Corollary 3.8.4. *Suppose $\mathcal{H}$ is separable, if $A \subset \mathcal{B}(\mathcal{H})$ is a maximal abelian self-adjoint subalgebra (masa), then there exists a cyclic vector for A.*

Proof. By Propostion 3.8.3 there exists a non-zero vector $\xi \in \mathcal{H}$ which is separating for A, and hence by Proposition 3.8.1 is cyclic for $A' = A$. ∎

The converse of the previous corollary also holds (without the separability hypothesis), which follows from Proposition 3.5.9, together with the following theorem.

Theorem 3.8.5. *Let $A \subset \mathcal{B}(\mathcal{H})$ be an abelian von Neumann algebra and suppose $\xi \in \mathcal{H}$ is a cyclic vector. Then for any SOT dense C^*-subalgebra $A_0 \subset A$ there exists a Radon probability measure μ on $K = \sigma(A_0)$ with $\text{supp}(\mu) = K$, and a unitary $U : L^2(K,\mu) \to \mathcal{H}$ such that $U^*AU = L^\infty(K,\mu) \subset \mathcal{B}(L^2(X,\mu))$, and such that $\int U^*aU \, d\mu = \langle a\xi, \xi\rangle$ for all $x \in A$.*

Proof. Fix a SOT dense C^*-algebra $A_0 \subset A$, then by the Riesz representation theorem we obtain a finite Radon measure μ on $K = \sigma(A_0)$ such that $\langle \Gamma(f)\xi, \xi\rangle = \int f d\mu$ for all $f \in C(K)$. Since the Gelfand transform takes positive operator to positive functions we see that μ is a probability measure.

We define a map $U_0 : C(K) \to \mathcal{H}$ by $f \mapsto \Gamma(f)\xi$, and note that $\|U_0(f)\|^2 = \langle \Gamma(\overline{f}f)\xi, \xi\rangle = \int \overline{f} f d\mu = \|f\|_2$. Hence U_0 extends to an isometry $U : L^2(K,\mu) \to \mathcal{H}$. Since ξ is cyclic we have that $A_0\xi \subset U(L^2(K,\mu))$ is dense and hence U is a unitary. If the support of μ were not K then there would exist a non-zero continuous function $f \in C(K)$ such that $0 = \int |f^2| d\mu = \|\Gamma(f)\xi\|^2$, but since by Corollary 3.8.2 we know that ξ is separating and hence this cannot happen.

If $f \in C(K) \subset \mathcal{B}(L^2(K,\mu))$, and $g \in C(K) \subset L^2(K,\mu)$ then we have

$$U^*\Gamma(f)Ug = U^*\Gamma(f)\Gamma(g)\xi = fg = M_f g.$$

Since $C(K)$ is $\|\cdot\|_2$-dense in $L^2(K,\mu)$ it then follows that $U^*\Gamma(f)U = M_f$, for all $f \in C(K)$ and thus $U^*A_0U \subset L^\infty(K,\mu)$. Since A_0 is SOT dense in A we then have that $U^*AU \subset L^\infty(K, \mu)$. But since $x \mapsto U^*xU$ is WOT continuous and $(A)_1$ is compact in the WOT it follows that $U^*(A)_1U = (L^\infty(K, \mu))_1$ and hence $U^*AU = L^\infty(K, \mu)$. This similarly shows that we have $\int U^*aU \, d\mu = \langle a\xi, \xi \rangle$ for all $x \in A$. ■

In general, if $A \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann algebra and $\xi \in \mathcal{H}$ is a non-zero vector, then we can consider the projection p onto the $\mathcal{K} = \overline{A\xi}$. We then have $p \in A'$, and $Ap \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann for which ξ is a cyclic vector, thus by the previous result Ap is $*$-isomorphic to $L^\infty(X, \mu)$ for some probability space (X, μ). An application of Zorn's Lemma can then be used to show that A is $*$-isomorphic to $L^\infty(Y, \nu)$ were (Y, ν) is a measure space which is a disjoint union of probability spaces. In the case when A is separable an even more concrete classification will be given below.

Theorem 3.8.6. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there exists a separable compact Hausdorff space K with a Radon probability measure μ on K such that A and $L^\infty(K, \mu)$ are $*$-isomorphic. Moreover, if φ is a normal faithful state on A, then the isomorphism $\theta : A \to L^\infty(K, \mu)$ may be chosen so that $\varphi(a) = \int \theta(a) \, d\mu$ for all $a \in A$.*

Proof. By Proposition 3.8.3 there exists a non-zero vector $\xi \in \mathcal{H}$ which is separating for A. Thus if we consider $\mathcal{K} = \overline{A\xi}$ we have that restricting each operator $x \in A$ to $\mathcal{K}$ is a C^*-algebra isomorphism and $\xi \in \mathcal{K}$ is then cyclic. Thus, the result follows from Theorem 3.8.5.

If φ is a normal faithful state on A, then considering the GNS-construction we may representation A on $L^2(A, \varphi)$ with a cyclic vector $\hat{1}$ which satisfies $\varphi(a) = \langle a\hat{1}, \hat{1} \rangle$. The result then follows as above. ■

If $x \in \mathcal{B}(\mathcal{H})$ is normal such that $A = W^*(x)$ is separable (e.g., if $\mathcal{H}$ is separable), then we may let A_0 be the C^*-algebra generated by x. We then obtain the following alternate version of the spectral theorem.

Corollary 3.8.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $x \in A$ is normal such that $W^*(x)$ is separable, then there exists a Radon probability measure μ on $\sigma(x)$ and a $*$-homomorphism $f \mapsto f(x)$ from $L^\infty(\sigma(x), \mu)$ into A which agrees with Borel functional calculus. Moreover, we have that $\sigma(f(x))$ is the essential range of f.*

Note that $W^*(x)$ need not be separable in general. For example, $\ell^\infty([0, 1]) \subset \mathcal{B}(\ell^2([0, 1]))$ is generated by the multiplication operator corresponding to the function $t \mapsto t$.

Lemma 3.8.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there exists a self-adjoint operator $x \in A$ such that $A = \{x\}''$.*

Proof. Since A is separable we have that A is countably generated as a von Neumann algebra. Indeed, just take a countable family in A which is dense in the SOT. By functional calculus we can approximate any self-adjoint element by a linear combination of projections and thus A is generated by a countable collection of projections $\{p_k\}_{k=0}^{\infty}$.

Define a sequence of self adjoint elements $x_n = \sum_{k=0}^{n} 4^{-k} p_k$, and let $x = \sum_{k=0}^{\infty} 4^{-k} p_k$. We denote by $A_0 = \{x\}''$. Define a continuous function $f : [-1, 2] \to \mathbb{R}$ such that $f(t) = 1$ if $t \in [1 - \frac{1}{3}, 1 + \frac{1}{3}]$ and $f(t) = 0$ if $t \leq \frac{1}{3}$, then we have that $f(x_n) = p_0$ for every n and hence by continuity of continuous functional calculus we have $p_0 = f(x) \in A_0$. The same argument shows that $p_1 = f(4(x - p_0)) \in A_0$ and by induction it follows easily that $p_k \in A_0$ for all $k \geq 0$, thus $A_0 = A$. ∎

Theorem 3.8.9. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra and φ a normal faithful state on A. Then there is there exists a probability space (X, μ) and an isomorphism $\theta : A \to L^\infty(X, \mu)$, such that $\varphi(a) = \int \theta(a)\, \mathrm{d}\mu$ for all $a \in A$.*

Moreover, (X, μ) may be taken to be of one of the following forms:

(*i*) *(K, ν), where K is countable;*

(*ii*) *$(K, \nu) \times ([0, c_0], \lambda)$ where K is countable, $0 < c_0 < 1$, and λ is Lebesgue measure;*

(*iii*) *$([0, 1], \lambda)$ where λ is Lebesgue measure.*

Proof. Since A is separable we have from Lemma 3.8.8 that as a von Neumann algebra A is generated by a single self-adjoint element $x \in A$.

We define $K = \{a \in \sigma(x) \mid 1_{\{a\}}(x) \neq 0\}$. Since the projections corresponding to elements in K are pairwise orthogonal it follows that K is countable. Further, if we denote by $p_K = \sum_{a \in K} 1_{\{a\}}$ then we have that $Ap_K \cong \ell^\infty K$, and restricting φ to $c_0(K)$ under this isomorphism gives a positive measure on K which is taken to $\varphi_{|Ap_K}$ under this isomorphism.

Thus, by considering $(1 - p_K)A$ we may assume that $p_K = 0$, and it is enough to show that in this case there exists an isomorphism $\theta : A \to L^\infty([0, 1], \lambda)$, such that $\varphi(a) = \int \theta(a)\, \mathrm{d}\lambda$ for all $a \in A$.

Thus, we suppose that $\sigma(x)$ has no isolated points. We may then inductively define a sequence of partitions $\{A_k^n\}_{k=1}^{2^n}$ of $\sigma(x)$ such that $A_k^n = A_{2k-1}^{n+1} \cup A_{2k}^{n+1}$, and A_k^n has non-empty interior, for all $n > 0$, $1 \leq k \leq 2^n$. If we then consider the elements $y_n = \sum_{k=1}^{\infty} \frac{k}{2^n} 1_{A_k}(x)$ then we have that $y_n \to y$ where $0 \leq y \leq 1$, $\{x\}'' = \{y\}''$ and every dyadic rational is contained in the spectrum of y (since the space of invertible operators is open in the norm topology), hence $\sigma(y) = [0, 1]$.

By Theorem 3.8.6 there exists an isomorphism $\theta_0 : \{y\}'' \to L^\infty([0, 1], \mu)$ for some Radon measure μ on $[0, 1]$ which has full support, no atoms and such that $\varphi(a) = \int \theta_0(a)\, \mathrm{d}\mu$ for all $a \in A$. If we define the function $\theta : [0, 1] \to [0, 1]$ by $\theta(t) = \mu([0, t])$ then θ gives a continuous bijection of $[0, 1]$, and we have $\theta_* \mu = \lambda$, since both are Radon probability measures such that for intervals $[a, b]$ we have $\theta_* \mu([a, b]) = \mu([\theta^{-1}(a), \theta^{-1}(b)]) = \lambda([a, b])$. The map $\theta^* : L^\infty([0, 1], \lambda) \to L^\infty([0, 1], \mu)$ given by $\theta^*(f) = f \circ \theta^{-1}$ is then easily seen to be the desired $*$-isomorphism. ∎

3.9 Standard probability spaces

A topological space X is a **Polish space** if X is homeomorphic to a separable complete metric space. A σ-algebra $(X, \mathcal{B})$ is a **standard Borel space** if $(X, \mathcal{B})$ is isomorphic to the σ-algebra of Borel subsets of a Polish space. A measure space (X, μ) is a **standard measure space** if it is σ-finite, and its underlying σ-algebra is a standard Borel space, and a **standard probability space** if it is also a probability space.

Lemma 3.9.1. *Let X be a Polish space with topology $\mathcal{T}$, and suppose $A \subset X$ closed, then $\mathcal{T} \cup \{A\}$ is again a Polish topology.*

Proof. Since X is Polish there exists a complete metric d on X such that (X, d) gives the topology $\mathcal{T}$ on X. Replacing d with $\frac{d(x,y)}{1+d(x,y)}$ we may assume that $\mathrm{diam}_d(X) \leq 1$.

Since $A \subset X$ is closed, d restricts to a complete metric on A, and from Proposition **??** we have a complete metric d_1 on A^c, which satisfies $\mathrm{diam}_{d_1}(A^c) \leq 1$, and gives the topological structure to A^c. We may then define a metric on X by

$$\tilde{d}(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in A, \\ d_1(x, y) & \text{if } x, y \in A^c, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\tilde{d}$ is a complete metric on X, and the corresponding topology is $\mathcal{T} \cup \{A\}$. ∎

Lemma 3.9.2. *Let X be a Polish space with topology $\mathcal{T}$, and suppose that $\{\mathcal{T}_n\}_{n=1}^{\infty}$ is a sequence of Polish topologies on X such that $\mathcal{T} \subset \mathcal{T}_n$ for each $n \in \mathbb{N}$, then the topology $\mathcal{T}_\infty$ generated by $\cup_{n\in\mathbb{N}} \mathcal{T}_n$ is again a Polish topology on X. Moreover, if $\mathcal{B}(\mathcal{T}_n) = \mathcal{B}(\mathcal{T})$ for all $n \in \mathbb{N}$, then $\mathcal{B}(\mathcal{T}_\infty) = \mathcal{B}(\mathcal{T})$.*

Proof. Let $X_n = X$ for $n \in \mathbb{N}$. Consider the map $\varphi : X \to \prod_{n=1}^{\infty} X_n$ given by $\varphi(x) = (x, x, \ldots)$. Then φ gives a homeomorphism between $(X, \mathcal{T}_\infty)$ and $\varphi(X) \subset \prod_{n=1}^{\infty} X_n$. Thus, to show that $(X, \mathcal{T}_\infty)$ is Polish, it is enough to show that $\varphi(X) \subset \prod_{n=1}^{\infty} X_n$ is closed. Suppose $(x_n) \notin \varphi(X)$, then for some $i < j$ we have $x_i \neq x_j$. We let U and V be disjoint open sets in $\mathcal{T}$ (hence also in $\mathcal{T}_i$ and $\mathcal{T}_j$) so that $x_i \in U$ and $x_j \in V$, then

$$(x_n) \in \pi_i^{-1}(U) \cap \pi_j^{-1}(V) \subset \varphi(X)^c.$$

Since Polish spaces are separable, given any set $\mathcal{G}$ which generates the topology, we have that any open set is a countable union of finite intersections in $\mathcal{G}$. Thus, if $\mathcal{G}$ is in a given σ-algebra $\mathcal{M}$, then this σ-algebra contains all Borel sets.

If $\mathcal{T}_n \subset \mathcal{B}(\mathcal{T})$, then $\cup_{n\in\mathbb{N}} \mathcal{T} \subset \mathcal{B}(\mathcal{T})$ and this generates the Polish topology $\mathcal{T}_\infty$. Thus, from the remark above we have that $\mathcal{B}(\mathcal{T}_\infty) \subset \mathcal{B}(\mathcal{T})$. ∎

Theorem 3.9.3. *Let X be a Polish space, and $\{E_n\}_{n\in\mathbb{N}}$ a countable collection of Borel subsets, then there exists a finer Polish topology on X with the same Borel structure, such that for each $n \in \mathbb{N}$, E_n is clopen in this new topology.*

Proof. We first consider the case of a single Borel subset $E \subset X$. We let $\mathcal{A}$ denote the set of subsets which satisfy the conclusion of the theorem and we let $\mathcal{B}$ be the σ-algebra of Borel subsets of X.

Lemma 3.9.1 shows that $\mathcal{A}$ contains all closed subsets of (X, d). It is also clear that $\mathcal{A}$ is closed under taking complements. Thus, to conclude that $\mathcal{B} \subset \mathcal{A}$ it is then enough to show that $\mathcal{A}$ is closed under countable intersections. If $A_n \in \mathcal{A}$, and $\mathcal{T}_n$ are finer Polish topologies on X, with Borel structure $\mathcal{B}$, such that A_n is clopen in $\mathcal{T}_n$ for each $n \in \mathbb{N}$, then by Lemma 3.9.2 there is a finer Polish topology $\mathcal{T}_\infty$ which generates $\mathcal{B}$ and such that A_n is clopen in $\mathcal{T}_\infty$, for each $n \in \mathbb{N}$. We then have that $\cap_{n\in\mathbb{N}} A_n$ is closed in $\mathbb{T}_\infty$, and hence by Lemma 3.9.1 we have that $\cap_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Having established the result for a single Borel set E, we may then apply Lemma 3.9.1 to obtain the result for a sequence of Borel sets $\{E_n\}_{n\in\mathbb{N}}$. ■

Corollary 3.9.4. *Let $(X, \mathcal{B})$ be a standard Borel space, and $E \in \mathcal{B}$ a Borel subset, then $(E, \mathcal{B}|_E)$ is a standard Borel space.*

Proof. By the previous theorem we may assume X is Polish and $E \subset X$ is clopen, and hence Polish. We then have that $\mathcal{B}|_E$ is the associated Borel structure on E and hence $(E, \mathcal{B}|_E)$ is standard. ■

Corollary 3.9.5. *Let X be a standard Borel space, Y a Polish space, and $f : X \to Y$ a Borel map, then there exists a Polish topology on X which generates the same Borel structure and such that f is continuous with respect to this topology.*

Proof. Let $\{E_n\}$ be a countable basis for the topology on Y. By Theorem 3.9.3 there exists a Polish topology on X which generates the same Borel structure and such that $f^{-1}(E_n)$ is clopen for each $n \in \mathbb{N}$. Hence, in this topology f is continuous. ■

We let $\mathbb{N}^{<\mathbb{N}}$ denote the set of finite sequences of natural numbers, i.e., $\mathbb{N}^{<\mathbb{N}}$ consists of the empty set, together with the disjoint union of $\mathbb{N}^n$, for $n \in \mathbb{N}$. If $s = (s_1, \ldots, s_n) \in \mathbb{N}^{<\mathbb{N}}$, and $k \in \mathbb{N}$, we denote by $s\hat{\ }k$ the sequence $(s_1, \ldots, s_n, k)$. If $s \in \mathbb{N}^{\mathbb{N}}$, and $n \in \{0\} \cup \mathbb{N}$ then we denote by $s|n$ the sequence which consists of the first n entires of s. A **Souslin scheme** on a set X is a family of subsets $\{E_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$.

Lemma 3.9.6. *Let X be a Polish space, then there exists a Souslin scheme $\{E_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ consisting of Borel subsets such that the the following conditions are satisfied:*

(*i*) $E_\emptyset = X$.

(*ii*) *For each $s \in \mathbb{N}^{<\mathbb{N}}$, $E_s = \sqcup_{k\in\mathbb{N}} E_{s\hat{\ }k}$.*

(*iii*) *For each $s \in \mathbb{N}^{\mathbb{N}}$ the set $\cap_{n\in\mathbb{N}} \overline{E_{s|n}}$ consists of at most one element.*

(*iv*) *For each $s \in \mathbb{N}^{\mathbb{N}}$, $\cap_{n\in\mathbb{N}} \overline{E_{s|n}} = \{x\} \neq \emptyset$ if and only if $E_{s|n} \neq \emptyset$ for all $n \in \mathbb{N}$, and in this case for any sequence $x_n \in E_{s|n}$ we have $x_n \to x$.*

Proof. Let d be a complete metric on X which generates the Polish topology on X. By replacing d with $\frac{d}{1+d}$ we may assume that the diameter of X is at most 1. We will inductively construct $\{E_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ so that for $s \in \mathbb{N}^n$ the diameter of E_s is at most 2^{-n}. First, we set $E_\emptyset = X$. Now suppose E_s has been constructed for each $s \in \{\emptyset\} \cup_{n=1}^k \mathbb{N}^n$. If $s \in \mathbb{N}^k$, let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of E_s (note that any subspace of a separable metric space is again separable).

We define $E_{s\hat{\ }i} = E_s \cap (B_{2^{-k-1}}(x_i) \setminus \cup_{j<i} B_{2^{-k-1}}(x_j))$, where $B_r(x)$ denotes the open ball of radius r centered at x. It is then easy to see that for each $s \in \mathbb{N}^{<\mathbb{N}}$, we have $E_s = \sqcup_{k\in\mathbb{N}} E_{s\hat{\ }k}$. Moreover, for each $s \in \mathbb{N}^{\mathbb{N}}$, we have that $E_{s|n}$ has diameter at most 2^{-n}, hence $\cap_{n\in\mathbb{N}} \overline{E_{s|n}}$ contains at most one element. Finally, if $\overline{E_{s|n}} \neq \emptyset$ for all $n \in \mathbb{N}$, then as the diameter of $E_{s|n}$ converges to 0, it follows from completeness, that there exists $x \in \cap_{n\in\mathbb{N}} \overline{E_{s|n}}$, and for each sequence $x_n \in E_{s|n}$ we have $x_n \to x$. ∎

If X is a standard Borel space and $A, B \subset X$ are disjoint, then we say that A and B are **Borel separated** if there exists a Borel subset $E \subset X$ such that $A \subset E$, and $B \subset X \setminus E$.

Lemma 3.9.7. *Let X be a standard Borel space and suppose that $A = \cup_{n\in\mathbb{N}} A_n$, and $B = \cup_{m\in\mathbb{N}} B_m$, are such that A_n and B_m are Borel separated for each $n, m \in \mathbb{N}$, then A and B are Borel separated.*

Proof. Suppose $E_{n,m}$ is a Borel subset which separates A_n and B_m for each $n, m \in \mathbb{N}$. Then $E = \cup_{n\in\mathbb{N}} \cap_{m\in\mathbb{N}} E_{n,m}$ separates A and B. ∎

If X is a Polish space, a subset $E \subset X$ is **analytic** if there exits a Polish space Y and a continuous function $f : Y \to X$ such that $E = f(Y)$. Note that it follows from Corollary 3.9.5 that if $f : Y \to X$ is Borel then $f(Y)$ is analytic. In particular, it follows that all Borel sets are analytic. If X is a standard Borel space then a subset $E \subset X$ is **analytic** if it is analytic for some (and hence all) Polish topologies on X which give the Borel structure.

Theorem 3.9.8 (The Lusin Separation Theorem)**.** *Let X be a standard Borel space, and $A, B \subset X$ two disjoint analytic sets, then A and B are Borel separated.*

Proof. We may assume that X is a Polish space, and that there are Polish spaces Y_1, and Y_2, and continuous functions $f_i : Y_i \to X$ such that $A = f_1(Y_1)$ and $B = f_2(Y_2)$.

Let $\{E_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ (resp. $\{F_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$) be a Souslin scheme for Y_1 (resp. Y_2) which satisfies the conditions in Lemma 3.9.6. If A and B are not Borel separated then by Lemma 3.9.7 we may recursively define sequences $\{s_n\}_n, \{r_n\}_n \in \mathbb{N}^{\mathbb{N}}$ such that $f_1(E_{s|n})$ and $f_2(F_{r|n})$ are not Borel separated for each $n \in \mathbb{N}$. In particular, we have that $E_{s|n}$ and $F_{r|n}$ are non-empty for each $n \in \mathbb{N}$, hence there exists $a \in Y_1$, $b \in Y_2$ such that $\cap_{n\in\mathbb{N}} \overline{E_{s|n}} = \{a\}$, $\cap_{n\in\mathbb{N}} \overline{F_{r|n}} = \{b\}$.

If $V, W \subset X$ are disjoint open subsets with $f_1(a) \in V$, and $f_2(b) \in W$, then by continuity of f_i, for large enough n we have $f_1(E_{x|n}) \subset V$, and $f_2(F_{y|n}) \subset W$. Hence V separates $E_{x|n}$ from $F_{y|n}$ for large enough n, a contradiction. ∎

Corollary 3.9.9. *If X is a standard Borel space then a subset $E \subset X$ is Borel if and only if both E and $X \setminus E$ are analytic.*

Corollary 3.9.10. *let X be a standard Borel space, and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint analytic subsets, then there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of disjoint Borel subsets such that $A_n \subset E_n$ for each $n \in \mathbb{N}$.*

Proof. It is easy to see that the countable union of analytic sets is analytic. Hence, by Lusin's separation theorem we may inductively define a sequence of Borel subsets $\{E_n\}_{n\in\mathbb{N}}$ such that $A_n \subset E_n$, while $(\cup_{k>n} A_k) \cup (\cup_{k<n} E_k) \subset X \setminus A_n$. ∎

Theorem 3.9.11 (Lusin-Souslin)**.** *Let X and Y be standard Borel spaces, and $f : X \to Y$ an injective Borel map, then $f(X)$ is Borel, and f implements an isomorphism of standard Borel spaces between X and $f(X)$.*

Proof. We first show that $f(X)$ is Borel. By Corollary 3.9.5 we may assume that X and Y are Polish spaces and f is continuous. Let $\{E_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ be a Souslin scheme for X which satisfies the conditions of Lemma 3.9.6. Then $\{f(E_s)\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ gives a Souslin scheme of analytic sets for Y, and since f is injective it follows that for each $s \in \mathbb{N}^n$ we have that $\{f(E_{s\hat{}k})\}_{k\in\mathbb{N}}$ are pairwise disjoint. Thus, by Corollary 3.9.10 there exist pairwise disjoint Borel subsets $\{Y_{s\hat{}k}\}_{k\in\mathbb{N}}$ such that $f(E_{s\hat{}k}) \subset Y_{s\hat{}k}$ for each $k \in \mathbb{N}$.

We inductively define a new Souslin scheme $\{C_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ for Y by setting $C_\emptyset = Y$, and $C_{s\hat{}k} = C_s \cap \overline{f(E_{s\hat{}k})} \cap Y_{s\hat{}k}$ for all $s \in \mathbb{N}^{<\mathbb{N}}$, and $k \in \mathbb{N}$. Then for each $s \in \mathbb{N}^{<\mathbb{N}}$ we have that C_s is Borel, and also

$$f(E_s) \subset C_s \subset \overline{f(E_s)}.$$

We claim that $f(X) = \cap_{k\in\mathbb{N}} \cup_{s\in\mathbb{N}^k} C_s$, from which it then follows that $f(X)$ is Borel.

If $y \in f(X)$, then let $x \in X$ be such that $f(x) = y$. There exists $s \in \mathbb{N}^{\mathbb{N}}$ such that $x \in \cap_{k\in\mathbb{N}} E_{s|k}$, and hence $y \in \cap_{k\in\mathbb{N}} f(E_{s|k})$. Thus, $y \in \cap_{k\in\mathbb{N}} C_{s|k} \subset \cap_{k\in\mathbb{N}} \cup_{s\in\mathbb{N}^k} C_s$. Conversely, if $y \in \cap_{k\in\mathbb{N}} \cup_{s\in\mathbb{N}^k} C_s$, then there exists $s \in \mathbb{N}^{\mathbb{N}}$ such that $y \in C_{s|k} \subset \overline{f(E_{s|k})}$ for each $k \in \mathbb{N}$. Hence $E_{s|k} \neq \emptyset$ for each $k \in \mathbb{N}$ and thus $\cap_{k\in\mathbb{N}} \overline{E_{s|k}} = \{x\}$ for some $x \in X$. We must then have that $f(x) = y$, since if this were not the case there would exists an open neighborhood U of $f(x)$ such that $y \notin \overline{U}$. By continuity of f we would then have that $f(E_{s|k}) \subset U$ for large enough k, and hence $y \in \cap_{k\in\mathbb{N}} \overline{f(E_{s|k})} \subset \overline{U}$, a contradiction.

Having established that $f(X)$ is Borel, the rest of the theorem follows easily. We have that f gives a bijection from X to $f(X)$ which is Borel, and if $E \subset X$ is Borel, then from Corollary 3.9.4 and the argument above we have that $f(E)$ is again Borel. Thus, f^{-1} is a Borel map. ∎

Corollary 3.9.12. *Suppose X and Y are standard Borel spaces such that there exists injective Borel maps $f : X \to Y$, and $g : Y \to X$, then X and Y are isomorphic.*

Proof. Suppose $f : X \to Y$, and $g : Y \to X$ are injective Borel maps. From Theorem 3.9.11 we have that f and g are Borel isomorphisms onto their image and hence we may apply an argument used for the Cantor-Bernstein theorem. Specifically, if we set $B = \cup_{n\in\mathbb{N}}(f \circ g)^n(Y \setminus f(X))$, and we set $A = X \setminus g(B)$, then we have $g(B) = X \setminus A$, and

$$f(A) = f(X) \setminus (f \circ g)(B) = Y \setminus ((Y \setminus f(X)) \cup (f \circ g)(B)) = Y \setminus B.$$

Hence if we define $\theta : X \to Y$ by $\theta(x) = f(x)$, if $x \in A$, and $\theta(x) = g^{-1}(x)$, if $x \in Y \setminus A = g(B)$, then we have that theta is a bijective Borel map whose inverse is also Borel. ∎

Theorem 3.9.13 (Kuratowski)**.** *Any two uncountable standard Borel spaces are isomorphic. In particular, two standard Borel spaces X and Y are isomorphic if and only if they have the same cardinality.*

Proof. Let X be an uncountable standard Borel space, we'll show that X is isomorphic as Borel spaces to the Polish space $\mathcal{C} = 2^{\mathbb{N}}$. Note that by Corollary 3.9.12 it is enough to show that there exist injective Borel maps $f : X \to \mathcal{C}$, and $g : \mathcal{C} \to X$.

To construct f, fix a metric d on X such that d gives the Borel structure to X and such that the diameter of X is at most 1. Let $\{x_n\}$ be a countable dense subset of (X, d), and define $f_0 : X \to [0, 1]^{\mathbb{N}}$ by $(f_0(x))(n) = d(x, x_n)$. The function f_0, is clearly injective and continuous, thus to construct f it is enough to construct an injective Borel map from $[0, 1]^{\mathbb{N}}$ to $\mathcal{C}$, and since $\mathcal{C}^{\mathbb{N}}$ is homeomorphic to $\mathcal{C}$, it is then enough to construct an injective Borel map from $[0, 1]$ to $\mathcal{C}$, and this is easily done. For example, if $y \in [0, 1)$ then we may consider its dyadic expansion $y = \sum_{k=1}^{\infty} b_k 2^{-k}$, where in the case when y is a dyadic rational we take the expansion such that b_k is eventually 0. Then it is easy to see that $[0, 1) \ni y \mapsto \{b_k\}_k \in \mathcal{C}$ gives an injective function which is continuous except at the countable family of dyadic rational, hence is Borel. We may then extend this map to $[0, 1]$ by sending 1 to $(1, 1, 1, \cdots) \in \mathcal{C}$.

To construct g, we again endow X with a compatible metric d such that X has diameter at most 1. We let $Z \subset X$ denote the subset of X consisting of all points x such that every neighborhood of x is uncountable. Then $X \setminus Z$ has a countable dense subset, and each point in this subset has a neighborhood with only countably many points, hence it follows that $X \setminus Z$ is countable and so Z is uncountable. By induction on n we may define sets F_s for $s \in \{0, 1\}^n$, with the following properties:

(i) $F_\emptyset = Z$,

(ii) For each $s \in \{0, 1\}^n$ we have that $F_{s\hat{}0}$ and $F_{s\hat{}1}$ are disjoint subsets of F_s.

(iii) For each $s \in \{0, 1\}^n$ we have that F_s is a closed ball of diameter at most 2^{-n}.

Thus, for each $s \in 2^{\mathbb{N}}$ we have that $\{F_{s|n}\}_{n\in\mathbb{N}}$ is a decreasing sequence of closed balls with diameter tending to 0. Since (X, d) is complete there then exists a unique element $g(s) \in \cap_{n\in\mathbb{N}} F_{s|n}$. It is then easy to see that $g : 2^{\mathbb{N}} \to X$ is an injective function which is continuous and hence also Borel. ∎

If (X, μ) and (Y, ν) are σ-finite measure spaces and $\pi : X \to Y$ is a measurable map such that $\pi_*\mu \prec \nu$, then we obtain a unital $*$-homomorphism $\pi^* : L^\infty(Y, \nu) \to L^\infty(X, \mu)$ given by $\pi^*(f) = f \circ \pi$. Note that π^* is well defined since $\pi^*\mu \prec \nu$. Note also that if (Y, ν) is standard, and if $\tilde{\pi} : X \to Y$ were another such map, then we would have $\pi^* = \tilde{\pi}^*$ if and only if $\pi(x) = \tilde{\pi}(x)$ for almost every $x \in X$.

We also have that π^* is normal. Indeed, the predual of $L^\infty(X, \mu)$ may naturally be identified with $M(X, \mu)$ the set of finite measures η on X such that $\eta \prec \mu$. The push forward of π then defines a bounded linear map $\pi_* : M(X, \mu) \to M(Y, \nu)$, and it is then easy to see that π^* is the dual map to π_*.

When the measure spaces are standard every normal endomorphism arrises in this way.

Theorem 3.9.14 (von Neumann)**.** *Suppose (X, μ) and (Y, ν) are σ-finite measure spaces such that Y is standard, and suppose that $\alpha : L^\infty(Y, \nu) \to L^\infty(X, \mu)$ is a normal unital $*$-homomorphism, then there exists a measurable map $\pi : X \to Y$ such that $\pi_*\mu \prec \nu$, and such that $\alpha = \pi^*$. Moreover, if X and Y are both standard and have the same cardinality and if α is an isomorphism then π can be chosen to be bijective.*

Proof. If Y is countable then the result is easy and so we only consider the case when Y is uncountable. Also, by replacing μ and ν with equivalent measures, we may assume that (X, μ) and (Y, ν) are probability spaces.

By Theorem 3.9.13 we may assume that Y is a separable compact Hausdorff space. We then have that $C(Y)$ is also separable and hence there exists a countable $\mathbb{Q}[i]$-algebra $A_0 \subset C(Y)$ which is dense in $C(Y)$. If for each $f \in A_0$ we chose a measurable function on X which realizes $\alpha(f)$, then as α is a unital $*$-homomorphism and since A_0 is countable it follows that for almost every $x \in X$ the functional $A_0 \ni f \mapsto \alpha(f)(x)$ extends to a continuous unital $*$-homomorphism on $C(Y)$. Thus, for almost every $x \in X$, there exists a unique point $\pi(x) \in Y$ such that $\alpha(f)(x) = f(\pi(x))$ for all $f \in A_0$. We let $E \subset X$ denote a conull set of points for which this is the case. We then fix $y_0 \in Y$ and set $\pi(x) = y_0$ for $x \in X \setminus E$.

If $K \subset Y$ is closed, then let $A_n = \{f \in A_0 \mid |f(y)| \leq 1/n \text{ for all } y \in K\}$. If $x \in E$ then since $A \subset C(Y)$ is dense we have that $\pi(x) \in K$ if and only if $|f(\pi(x))| \leq 1/n$ for all $f \in A_n$, and $n \in \mathbb{N}$. Hence $E \cap \pi^{-1}(K) = E \cap \cap_{n\in\mathbb{N}} \cap_{f\in A_n} \alpha(f)^{-1}(B(1/n))$ is measurable, where $B(1/n) \subset \mathbb{C}$ denotes the closed ball with center 0 and radius $1/n$. We therefore have that π is measurable and since A_0 is dense in $C(Y)$, we have that $\int f \, d\pi_*\mu = \int \alpha(f) \, d\mu$ for all $f \in C(Y)$. If $F \subset Y$ is measurable such that $\pi_*\mu(F) > 0$, then there exist $f_n \in C(Y)$ uniformly bounded so that $f_n \to 1_F$ almost everywhere with respect to $\pi_*\mu$. We then have $\pi_*\mu(F) = \lim_{n\to\infty} \int \alpha(f_n) \, d\mu = \int \alpha(1_F) \, d\mu$ by the bounded convergence theorem. Thus $\alpha(1_F) \neq 0$ and hence $\nu(F) \neq 0$. Therefore, $\pi_*\mu \prec \nu$, and since A_0 is weakly dense in $L^\infty(Y, \nu)$ we have $\pi^* = \alpha$.

If X is also standard and uncountable, then there exists a countable collection of Borel sets $\{E_n\}_{n\in\mathbb{N}}$ which separates points. If α is a $*$-isomorphism then there exist $F_n \subset Y$ measurable sets such that $\mu(E_n \Delta \pi^{-1}(F_n))$. We set $X_0 = \cup_{n\in\mathbb{N}}(E_n \Delta \pi^{-1}(F_n))$ then we have $\mu(X_0) = 0$, and $\pi^{-1}(F_n)$ separates points on $X \setminus X_0$. Hence π is injective when restricted to $X \setminus X_0$. Since X is uncountable, it has an uncountable Borel set with measure 0, and hence we may assume that X_0 is uncountable.

Similarly, there exists an uncountable Borel set $Y_0 \subset Y$ and an injective Borel map $\tilde{\pi} : Y \setminus Y_0 \to X$ such that $\nu(Y_0) = 0$, and $\tilde{\pi}^* = \alpha^{-1}$. By again removing a null set we may assume that the range of π agrees with the domain of $\tilde{\pi}$ and vice versa. We then have that $(\tilde{\pi} \circ \pi)^* = \mathrm{id}$, $(\pi \circ \tilde{\pi})^* = \mathrm{id}$ an hence $\tilde{\pi} \circ \pi$ and $\pi \circ \tilde{\pi}$ must agree almost everywhere with the identity map. Thus, by enlarging X_0 and Y_0 with sets of measure 0 we may assume that $\pi^{-1} = \tilde{\pi}$ on $Y \setminus Y_0$. Since X_0 and Y_0 are uncountable there exists a Borel bijection between them and hence we can extend π from $X \setminus X_0$ to a Borel bijection between X and Y such that $\pi^* = \alpha$. ■

Corollary 3.9.15. *Let (X, μ) be a standard probability space, then (X, μ) is isomorphic to a probability space of one of the following forms:*

(*i*) (K, ν) *where K is countable;*

(*ii*) $(K \sqcup [0, 1], \nu)$*, where K is countable, and* $\nu([0, 1]) = 0$.

(*iii*) $(K, \nu_0) \times ([0, c_0], \lambda)$ *where K is countable, $0 < c_0 < 1$, and λ is Lebesgue measure;*

(*iv*) $([0, 1], \lambda)$ *where λ is Lebesgue measure.*

Proof. Since (X, μ) is a standard probability space, then by Theorem 3.8.9 there exists a probability space (Y, ν) of one of the forms above, and a normal isomorphism $\theta : L^\infty(X, \mu) \to L^\infty(Y, \nu)$, such that $\int f \, d\mu = \int \theta(f) \, d\nu$, for all $f \in L^\infty(X, \mu)$. The result then follows easily from the previous theorem. ■

Chapter 4

Unbounded operators

4.1 Definitions and examples

Let $\mathcal{H}$, and $\mathcal{K}$ Hilbert spaces. An **linear operator** $T : \mathcal{H} \to \mathcal{K}$ consists of a linear subspace $D(T) \subset \mathcal{H}$ together with a linear map from $D(T)$ to $\mathcal{K}$ (which will also be denoted by T). A linear operator $T : \mathcal{H} \to \mathcal{K}$ is bounded if there exists $K \geq 0$ such that $\|T\xi\| \leq K\|\xi\|$ for all $\xi \in D(T)$.

The **graph** of T is the subspace

$$\mathcal{G}(T) = \{\xi \oplus T\xi \mid \xi \in D(T)\} \subset \mathcal{H} \oplus \mathcal{K},$$

T is said to be **closed** if its graph $\mathcal{G}(T)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{K}$, and T is said to be **closable** if there exists an unbounded closed operator $S : \mathcal{H} \to \mathcal{K}$ such that $\overline{\mathcal{G}(T)} = \mathcal{G}(S)$. If T is closable we denote the operator S by $\overline{T}$ and call it the **closure** of T. A linear operator T is **densely defined** if $D(T)$ is a dense subspace. We denote by $\mathcal{C}(\mathcal{H}, \mathcal{K})$ the set of closed, densely defined linear operators from $\mathcal{H}$ to $\mathcal{K}$, and we also write $\mathcal{C}(\mathcal{H})$ for $\mathcal{C}(\mathcal{H}, \mathcal{H})$. Note that we may consider $\mathcal{B}(\mathcal{H}, \mathcal{K}) \subset \mathcal{C}(\mathcal{H}, \mathcal{K})$.

If $T, S : \mathcal{H} \to \mathcal{K}$ are two linear operators, then we say that S is an **extension** of T and write $S \sqsubseteq T$ if $D(S) \subset D(T)$ and $T_{|D(S)} = S$. Also, if $T : \mathcal{H} \to \mathcal{K}$, and $S : \mathcal{K} \to \mathcal{L}$ are linear operators, then the composition $ST : \mathcal{H} \to \mathcal{L}$ is the linear operator with domain

$$D(ST) = \{\xi \in D(T) \mid T\xi \in D(S)\},$$

defined by $(ST)(\xi) = S(T(\xi))$, for all $\xi \in D(ST)$. We may similarly define addition of linear operators as

$$D(S + T) = D(S) \cap D(T),$$

and $(S + T)\xi = S\xi + T\xi$, for all $\xi \in D(S + T)$. Even if S and T are both densely defined this need not be the case for ST or $S + T$. Both composition and addition are associative operations, and we still have the right distributive property $(R+S)T = (RT) + (ST)$, although note that in general we only have $T(R+S) \sqsupseteq (TR) + (TS)$.

If $S \in \mathcal{C}(\mathcal{H})$, and $T \in \mathcal{B}(\mathcal{H})$ then ST is still closed, although it may not be densely defined. Similarly, TS will be densely defined, although it may not be closed. If T also has a bounded inverse, then both ST and TS will be closed and densely defined.

If $T : \mathcal{H} \to \mathcal{K}$ is a densely defined linear operator, and $\eta \in \mathcal{K}$ such that the linear functional $\xi \mapsto \langle T\xi, \eta \rangle$ is bounded on $D(T)$, then by the Riesz representation theorem there exists a unique vector $T^*\eta \in \mathcal{H}$ such that for all $\xi \in D(T)$ we have

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle.$$

We denote by $D(T^*)$ the linear subspace of all vectors η such that $\xi \mapsto \langle T\xi, \eta \rangle$ is bounded, and we define the linear operator $\eta \mapsto T^*\eta$ to be the **adjoint** of T. Note that T^* is only defined for operators T which are densely defined.

A densely defined operator $T : \mathcal{H} \to \mathcal{H}$ is **symmetric** if $T \sqsubseteq T^*$, and is **self-adjoint** if $T = T^*$.

Example 4.1.1. Let $A = (a_{i,j}) \in M_{\mathbb{N}}(\mathbb{C})$ be a matrix, for each $n \in \mathbb{N}$ we consider the finite rank operator $T_n = \sum_{i,j \leq n} a_{i,j} \delta_i \otimes \overline{\delta_j}$, so that we may think of T_n as changing the entries of A to 0 whenever $i > n$, or $j > n$.

We set $D = \{\xi \in \ell^2\mathbb{N} \mid \lim_{n\to\infty} T_n\xi \text{ exists.}\}$, and we define $T_A : D \to \ell^2\mathbb{N}$ by $T_A\xi = \lim_{n\to\infty} T_n\xi$.

Suppose now that for each $j \in \mathbb{N}$ we have $\{a_{i,j}\}_i \in \ell^2\mathbb{N}$. Then we have $\mathbb{C}\mathbb{N} \subset D$ and so T_A is densely defined. If $\eta \in D(T_A^*)$ then it is easy to see that if we denote by P_n the projection onto the span of $\{\delta_i\}_{i \leq n}$, then we have $P_n T_A^* \eta = T_n^* \eta$, hence $\eta \in D(T_{A^*})$ where A^* is the Hermitian transpose of the matrix A. It is also easy to see that $D(T_{A^*}) \subset D(T_A^*)$, and so $T_A^* = T_{A^*}$.

In particular, if $\{a_{i,j}\}_i \in \ell^2\mathbb{N}$, for every $j \in \mathbb{N}$, and if $\{a_{i,j}\}_j \in \ell^2\mathbb{N}$, for every $i \in \mathbb{N}$, then $T_A \in \mathcal{C}(\ell^2 N)$.

Example 4.1.2. Let (X, μ) be a σ-finite measure space and $f \in \mathcal{M}(X, \mu)$ a measurable function. We define the linear operator $M_f : L^2(X,\mu) \to L^2(X,\mu)$ by setting $D(M_f) = \{g \in L^2(X,\mu) \mid fg \in L^2(X,\mu)\}$, and $M_f(g) = fg$ for $g \in D(M_f)$. It's easy to see that each M_f is a closed operator, and we have $M_{\overline{f}} = M_f^*$. Also, if $f, g \in M(X,\mu)$ then we have $M_{f+g} \sqsupseteq M_f + M_g$, and $M_{fg} \sqsupseteq M_f M_g$.

Example 4.1.3. Let $D \subset L^2[0,1]$ denote the space of absolutely continuous functions $f : [0,1] \to \mathbb{C}$, such that $f(0) = f(1) = 0$, and $f' \in L^2[0,1]$. Then D is dense in $L^2[0,1]$, and we may consider the densely defined operator $T : L^2[0,1] \to L^2[0,1]$ with domain D, given by $T(f) = if'$. Note that the constant functions are orthogonal to the range of T. Moreover, if $g \in L^2[0,1]$ is such that $\int_0^1 g = 0$, then setting $G(x) = \int_0^x g(t)dt$ we have that $G \in D$ and $\langle TG, g \rangle = \|g\|_2^2$. Thus, if $g \in L^2[0,1]$ is any function which is orthogonal to the range of T then we have that g agrees almost everywhere with the constant function $\int_0^1 g$, i.e., $R(T)^\perp$ equals the constant functions.

If $g \in D(T^*)$, and $h = T^*g$, then set $H(x) = \int_0^x h(t)\,\mathrm{d}t$. For every $f \in D$, integration by parts gives

$$i\int_0^1 f'\overline{g} = \langle Tf, g\rangle = \langle f, h\rangle = \int_0^1 f\overline{H'} = -\int_0^1 f'\overline{H}.$$

Thus, $\langle f', H - ig\rangle = 0$ for all $f \in D$, so that $H - ig \in R(T)^\perp$, and so $H - ig$ is a constant function. In particular, we see that g is absolutely continuous, and $g' = ih \in L^2[0,1]$. Conversely, if $g : [0,1] \to \mathbb{C}$ is absolutely continuous and $g' \in L^2[0,1]$ then it is equally easy to see that $g \in D(T^*)$, and $T^*g = ig'$.

In particular, this shows that T is symmetric, but not self-adjoint. Note that the range of T^* is dense, and so the same argument above shows that if we take $g \in D(T^{**})$, $h = T^{**}g$, and $H(x) = \int_0^x h(t)dt$, then we have $H - ig \in R(T^*)^\perp = \{0\}$. Thus, $g = iH$ is absolutely continuous, $T^{**}g = h = ig'$, and we also have $g(1) = H(1) = \int_0^1 h(t)\,\mathrm{d}t = \langle 1, T^*g\rangle = \langle T(1), g\rangle = 0 = g(0)$. Thus, we conclude that $T^{**} = T$ (We'll see in Proposition 4.1.6 below that this implies that T is closed).

If we consider instead the space $\tilde{D}$ consisting of all absolutely continuous functions $f : [0,1] \to \mathbb{C}$, such that $f(0) = f(1)$, and if we define $S : \tilde{D} \to L^2[0,1]$ by $S(f) = if'$, then a similar argument shows that S is self-adjoint. Thus, we have the following sequence of extensions:

$$T^{**} = T \sqsubseteq S = S^* \sqsubseteq T^*.$$

Lemma 4.1.4. *Let $T : \mathcal{H} \to \mathcal{K}$ be a densely defined operator, and denote by $J : \mathcal{H} \oplus \mathcal{K} \to \mathcal{K} \oplus \mathcal{H}$ the isometry defined by $J(\xi \oplus \eta) = -\eta \oplus \xi$. Then we have $\mathcal{G}(T^*) = J(\mathcal{G}(T))^\perp$.*

Proof. If $\eta, \zeta \in \mathcal{K}$, the $\eta \oplus \zeta \in J(\mathcal{G}(T))^\perp$ if and only if for all $\xi \in D(T)$ we have

$$0 = \langle -T\xi \oplus \xi, \eta \oplus \zeta\rangle = \langle \xi, \zeta\rangle - \langle T\xi, \eta\rangle.$$

Which, since $\mathcal{H} = \overline{D(T)}$, is also if and only if $\eta \in D(T^*)$ and $\zeta = T^*\eta$. ■

Corollary 4.1.5. *For any densely defined operator $T : \mathcal{H} \to \mathcal{K}$, the operator T^* is closed. In particular, self-adjoint operators are closed, and symmetric operators are closable.*

Proposition 4.1.6. *A densely defined operator $T : \mathcal{H} \to \mathcal{K}$ is closable if and only if T^* is densely defined, and if this is the case then we have $\overline{T} = (T^*)^*$.*

Proof. Suppose first that T^* is densely defined. Then by Lemma 4.1.4 we have

$$\mathcal{G}((T^*)^*) = -J^*(J(\mathcal{G}(T))^\perp)^\perp = (\mathcal{G}(T)^\perp)^\perp = \overline{\mathcal{G}(T)},$$

hence T is closable and $(T^*)^* = \overline{T}$.

Conversely, if T is closable then take $\zeta \in D(T^*)^\perp$.

For all $\eta \in D(T^*)$ we have

$$0 = \langle \zeta, \eta\rangle = \langle 0 \oplus \zeta, -T^*\eta \oplus \eta\rangle,$$

and hence $0 \oplus \zeta \in (-J^*\mathcal{G}(T^*))^\perp = \overline{\mathcal{G}(T)}$. Since T is closable we then have $\zeta = 0$. ∎

We leave the proof of the following lemma to the reader.

Lemma 4.1.7. *Suppose $T : \mathcal{H} \to \mathcal{K}$, and $R, S : \mathcal{K} \to \mathcal{L}$ are densely defined operators such that ST (resp. $R+S$) is also densely defined, then $T^*S^* \sqsubseteq (ST)^*$ (resp. $R^* + S^* \sqsubseteq (R+S)^*$).*

4.1.1 The spectrum of a linear operator

Let $T : \mathcal{H} \to \mathcal{K}$ be an injective linear operator. The **inverse** of T is the linear operator $T^{-1} : \mathcal{K} \to \mathcal{H}$ with domain $D(T^{-1}) = R(T)$, such that $T^{-1}(T\xi) = \xi$, for all $\xi \in D(T^{-1})$.

The **resolvent** set of an operator $T : \mathcal{H} \to \mathcal{H}$ is

$$\rho(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is injective and } (T-\lambda)^{-1} \in \mathcal{B}(\mathcal{H})\}.$$

The **spectrum** of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

If $\sigma \in \mathcal{U}(\mathcal{H} \oplus \mathcal{K})$ is given by $\sigma(\xi \oplus \eta) = \eta \oplus \xi$, and if $T : \mathcal{H} \to \mathcal{K}$ is injective then we have that $\mathcal{G}(T^{-1}) = \sigma(\mathcal{G}(T))$. Hence, if $T : \mathcal{H} \to \mathcal{H}$ is not closed then $\sigma(T) = \mathbb{C}$. Also, note that if $T \in \mathcal{C}(\mathcal{H})$ then by the closed graph theorem shows that $\lambda \in \rho(T)$ if and only if $T - \lambda$ gives a bijection between $D(T)$ and $\mathcal{H}$.

Lemma 4.1.8. *Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ be injective with dense range, then $(T^*)^{-1} = (T^{-1})^*$. In particular, for $T \in \mathcal{C}(\mathcal{H})$ we have $\sigma(T^*) = \{\overline{z} \mid z \in \sigma(T)\}$.*

Proof. If we consider the unitary operators J, and σ from above then we have

$$\begin{aligned}\mathcal{G}((T^*)^{-1}) &= \sigma(\mathcal{G}(T^*)) = \sigma J(\mathcal{G}(T))^\perp \\ &= J^*(\sigma\mathcal{G}(T))^\perp = J^*(\mathcal{G}(T^{-1}))^\perp = \mathcal{G}((T^{-1})^*).\end{aligned}$$ ∎

Lemma 4.1.9. *If $T \in \mathcal{C}(\mathcal{H})$, then $\sigma(T)$ is a closed subset of $\mathbb{C}$.*

Proof. We will show that $\rho(T)$ is open by showing that whenever $\lambda \in \rho(T)$ with $|\alpha - \lambda| < \|(T-\lambda)^{-1}\|^{-1}$, then $\alpha \in \rho(T)$. Thus, suppose $\lambda \in \rho(T)$ and $\alpha \in \mathbb{C}$ such that $|\lambda - \alpha| < \|(T-\lambda)^{-1}\|^{-1}$. Then for all $\xi \in \mathcal{H}$ we have

$$\|\xi - (T-\alpha)(T-\lambda)^{-1}\xi\| = \|(\alpha - \lambda)(T-\lambda)^{-1}\xi\| < \|\xi\|.$$

Hence, by Lemma 1.1.1, $S = (T-\alpha)(T-\lambda)^{-1}$, is bounded, everywhere defined operator with a bounded everywhere defined inverse $S^{-1} \in \mathcal{B}(\mathcal{H})$. We then have $(T-\lambda)^{-1}S^{-1} \in \mathcal{B}(\mathcal{H})$, and it's easy to see that $(T-\lambda)^{-1}S^{-1} = (T-\alpha)^{-1}$. ∎

Note that an unbounded operator may have empty spectrum. Indeed, if $S \in \mathcal{B}(\mathcal{H})$ has a densely defined inverse, then for each $\lambda \in \sigma(S^{-1}) \setminus \{0\}$ we have $(S-\lambda^{-1})\lambda(\lambda - S)^{-1}S^{-1} = S(\lambda - S^{-1})(\lambda - S)^{-1}S^{-1} = \text{id}$. Hence $\sigma(S^{-1}) \setminus \{0\} \subset (\sigma(S) \setminus \{0\})^{-1}$. Thus, it is enough to find a bounded operator $S \in \mathcal{B}(\mathcal{H})$ such that S is injective but not surjective, and has dense range with $\sigma(S) = \{0\}$. For example, the compact operator $S \in \mathcal{B}(\ell^2\mathbb{Z})$ given by $(S\delta_n) = \frac{1}{|n|+1}\delta_{n+1}$ is injective with dense range, but is not surjective, and $\|S^{2n}\| \leq 1/n!$, so that $r(S) = 0$ and hence $\sigma(S) = \{0\}$.

4.1.2 Quadratic forms

A **quadratic form** $q : \mathcal{H} \to \mathbb{C}$ on a Hilbert space $\mathcal{H}$ consists of a linear subspace $D(q) \subset \mathcal{H}$, together with a sesquilinear form $q : D(q) \times D(q) \to \mathbb{C}$. We say that q is **densely defined** if $D(q)$ is dense. If $\xi \in D(q)$ then we write $q(\xi)$ for $q(\xi, \xi)$; note that we have the polarization identity $q(\xi, \eta) = \frac{1}{4} \sum_{k=0}^{3} i^k q(\xi + i^k \eta)$, and in general, a function $q : D \to \mathcal{H}$ defines a sesquilinear from through the polarization identity if and only if it satisfies the parallelogram identity $q(\xi + \eta) + q(\xi - \eta) = 2q(\xi) + 2q(\eta)$ for all $\xi, \eta \in D(q)$. A quadratic form q is **non-negative definite** if $q(\xi) \geq 0$ for all $\xi \in \mathcal{H}$.

If q is a non-negative definite quadratic form and we denote by $\mathcal{H}_q$ the separation and completion of $D(q)$ with respect to q, then we may consider the identity map $I : D(q) \to \mathcal{H}_q$, and note that for $\xi, \eta \in D(q)$ we have $\langle \xi, \eta \rangle_q := \langle \xi, \eta \rangle + q(\xi, \eta)$ coincides with the inner-product coming from the graph of I. The quadratic form q is **closed** if I is closed, i.e., $(D(q), \langle \cdot, \cdot \rangle_q)$ is complete. We'll say that q is **closable** if I is closable, and in this case we denote by $\overline{q}$ the closed quadratic form given by $D(\overline{q}) = D(\overline{I})$, and $\overline{q}(\xi, \eta) = \langle \overline{I}\xi, \overline{I}\eta \rangle$.

Theorem 4.1.10. *Let $q : \mathcal{H} \to [0, \infty)$ be a non-negative definite quadratic form, then the following conditions are equivalent:*

- (i) *q is closed.*
- (ii) *There exists a Hilbert space $\mathcal{K}$, and a closed linear operator $T : \mathcal{H} \to \mathcal{K}$ with $D(T) = D(q)$ such that $q(\xi, \eta) = \langle T\xi, T\eta \rangle$ for all $\xi, \eta \in D(T)$.*
- (iii) *q is lower semi-continuous, i.e., for any sequence $\xi_n \in D(q)$, such that $\xi_n \to \xi$, and $\liminf_{n\to\infty} q(\xi_n) < \infty$, we have $\xi \in D(q)$ and $q(\xi) \leq \liminf_{n\to\infty} q(\xi_n)$.*

Proof. The implication (i) $\implies$ (ii) follows from the discussion preceding the theorem. For (ii) $\implies$ (iii) suppose that $T : \mathcal{H} \to \mathcal{K}$ is a closed linear operator such that $D(T) = D(q)$, and $q(\xi, \eta) = \langle T\xi, T\eta \rangle$ for all $\xi, \eta \in D(T)$. If $\xi_n \in D(T)$ is a sequence such that $\xi_n \to \xi \in \mathcal{H}$, and $K = \liminf_{n\to\infty} \|T\xi_n\|^2 < \infty$, then by taking a subsequence we may assume that $K = \lim_{n\to\infty} \|T\xi_n\|^2$, and $T\xi_n \to \eta$ weakly for some $\eta \in \mathcal{H}$. Taking convex combinations we may then find a sequence ξ'_n such that $\xi'_n \to \xi \in \mathcal{H}$, $T\xi'_n \to \eta$ strongly, and $\|\eta\| = \lim_{n\to\infty} \|T\xi'_n\|^2 \leq K$. Since T is closed we then have $\xi \in D(T)$, and $T\xi = \eta$, so that $\|T\xi\|^2 \leq K$.

We show (iii) $\implies$ (i) by contraposition, so suppose that $\mathcal{H}_q$ is the separation and completion of $D(q)$ with respect to q, and that $I : D(q) \to \mathcal{H}_q$ is not closed. If I were closable, then there would exist a sequence $\xi_n \in D(q)$ such that $\xi_n \to \eta \in \mathcal{H}$, and $I(\xi_n)$ is Cauchy, but $\eta \notin D(q)$. However, if $I(\xi_n)$ is Cauchy then in particular we have that $q(\xi_n)$ is bounded, hence this sequence would show that (iii) does not hold.

Thus, we may assume that I is not closable, so that there exists a sequence $\xi_n \in D(q)$ such that $\|\xi_n\| \to 0$, and $I(\xi_n) \to \eta \neq 0$. Since, $D(q)$ is dense in $\mathcal{H}_q$ there exists $\eta_0 \in D(q)$ such that the square distance from η_0 to η in $\mathcal{H}_q$ is less than $q(\eta_0)$. We then have that $\eta_0 - \xi_n \to \eta_0 \in \mathcal{H}$, and by the triangle inequality, $\lim_{n\to\infty} q(\eta_0 - \xi_n) < q(\eta_0)$. Thus, the sequence $\eta_0 - \xi_n$ shows that (iii) does not hold in this case also. ∎

Note that it follows from the previous theorem that a if q_1 and q_2 are closed quadratic forms, then $q_1 + q_2$ is again a closed quadratic form and we have $D(q_1 + q_2) = D(q_1) \cap D(q_2)$.

Corollary 4.1.11. *Let $q_n : \mathcal{H} \to [0, \infty)$ be a sequence of closed non-negative definite quadratic forms, and assume that this sequence is increasing, i.e., $q_n(\xi)$ is an increasing sequence for all $\xi \in \cap_{n\in\mathbb{N}} D(q_n)$. Then there exists a closed quadratic form $q : \mathcal{H} \to [0, \infty)$ with domain*

$$D(q) = \{\xi \in \cap_{n\in\mathbb{N}} D(q_n) \mid \lim_{n\to\infty} q_n(\xi) < \infty\}$$

such that $q(\xi) = \lim_{n\to\infty} q_n(\xi)$, for all $\xi \in D(q)$.

Proof. If we define q as above then note that since each q_n satisfies the parallelogram identity then so does q, and hence q has a unique sesquilinear extension on $D(q)$. That q is closed follows easily from condition (iii) of Theorem 4.1.10. ∎

4.2 Symmetric operators and extensions

Lemma 4.2.1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a densely defined operator, then T is symmetric if and only if $\langle T\xi, \xi\rangle \in \mathbb{R}$, for all $\xi \in D(T)$.*

Proof. If T is symmetric then for all $\xi \in D(T)$ we have $\langle T\xi, \xi\rangle = \langle \xi, T\xi\rangle = \overline{\langle T\xi, \xi\rangle}$. Conversely, if $\langle T\xi, \xi\rangle = \langle \xi, T\xi\rangle$ for all $\xi \in D(T)$, then the polarization identity shows that $D(T) \subset D(T^*)$ and $T^*\xi = T\xi$ for all $\xi \in D(T)$. ∎

Proposition 4.2.2. *Let $T \in \mathcal{C}(\mathcal{H})$ be a symmetric operator, then for all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, we have $\ker(T - \lambda) = \{0\}$, and $R(T - \lambda)$ is closed.*

Proof. Fix $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, and set $\lambda = \alpha + i\beta$. For $\xi \in D(T)$ we have

$$\begin{aligned}\|(T-\lambda)\xi\|^2 &= \|(T-\alpha)\xi\|^2 + \|\beta\xi\|^2 - 2\mathrm{Re}(\langle (T-\alpha)\xi, i\beta\xi\rangle) \\ &= \|(T-\alpha)\xi\|^2 + \|\beta\xi\|^2 \geq \beta^2\|\xi\|^2. \qquad (4.1)\end{aligned}$$

Thus, $\ker(T - \lambda) = \{0\}$, and if $\xi_n \in D(T)$ such that $(T - \lambda)\xi_n$ is Cauchy, then so is ξ_n, and hence $\xi_n \to \eta$ for some $\eta \in \mathcal{H}$. Since T is closed we have $\eta \in D(T)$ and $(T - \lambda)\eta = \lim_{n\to\infty}(T - \lambda)\xi_n$. Hence, $R(T - \lambda)$ is closed. ∎

Lemma 4.2.3. *Let $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{H}$ be two closed subspaces such that $\mathcal{K}_1 \cap \mathcal{K}_2^{\perp} = \{0\}$, then $\dim \mathcal{K}_1 \leq \dim \mathcal{K}_2$.*

Proof. Let P_i be the orthogonal projection onto $\mathcal{K}_i$. Then by hypothesis we have that P_2 is injective when viewed as an operator from $\mathcal{K}_1$ to $\mathcal{K}_2$, hence if we let v be the partial isometry in the polar decomposition of $P_{2|\mathcal{K}_1}$ then v is an isometry and so $\dim \mathcal{K}_1 \leq \dim \mathcal{K}_2$. ∎

Theorem 4.2.4. *If* $T \in \mathcal{C}(\mathcal{H})$ *is symmetric, then* $\dim \ker(T^* - \lambda)$ *is a constant function for* $\operatorname{Im} \lambda > 0$*, and for* $\operatorname{Im} \lambda < 0$.

Proof. Note that the result will follow easily if we show that for all $\lambda, \alpha \in \mathbb{C}$ such that $|\alpha - \lambda| < |\operatorname{Im} \lambda|/2$, then we have $\dim \ker(T^* - \lambda) = \dim \ker(T^* - \alpha)$. And this in turn follows easily if we show that for all $\lambda, \alpha \in \mathbb{C}$ such that $|\alpha - \lambda| < |\operatorname{Im} \lambda|$, then we have $\dim \ker(T^* - \alpha) \leq \dim \ker(T^* - \alpha)$.

Towards this end, suppose we have such $\alpha, \lambda \in \mathbb{C}$. If $\xi \in \ker(T^* - \alpha) \cap (\ker(T^* - \lambda))^\perp$ such that $\|\xi\| = 1$, then since $R(T - \overline{\lambda})$ is closed we have $\xi \in (\ker(T^* - \lambda))^\perp = R(T - \overline{\lambda})$ and so $\xi = (T - \overline{\lambda})\eta$ for some $\eta \in D(T)$. Since, $\xi \in \ker(T^* - \alpha)$ we then have

$$0 = \langle (T^* - \alpha)\xi, \eta \rangle = \langle \xi, (T - \overline{\lambda})\eta \rangle + \langle \xi, \overline{\lambda - \alpha}\eta \rangle = \|\xi\|^2 + (\lambda - \alpha)\langle \xi, \eta \rangle.$$

Hence, $1 = \|\xi\|^2 = |\lambda - \alpha||\langle \xi, \eta \rangle| < |\operatorname{Im} \lambda| \|\eta\|$. However, by (4.1) we have $|\operatorname{Im} \lambda|^2 \|\eta\|^2 \leq \|(T - \overline{\lambda})\eta\|^2 = 1$, which gives a contradiction.

Thus, we conclude that $\ker(T^* - \alpha) \cap (\ker(T^* - \lambda))^\perp = \{0\}$, and hence $\dim \ker(T^* - \alpha) \leq \dim \ker(T^* - \lambda)$ by Lemma 4.2.3. ∎

Corollary 4.2.5. *If* $T \in \mathcal{C}(\mathcal{H})$ *is symmetric, then one of the following occurs:*

(*i*) $\sigma(T) = \mathbb{C}$.

(*ii*) $\sigma(T) = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \geq 0\}$.

(*iii*) $\sigma(T) = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \leq 0\}$.

(*iv*) $\sigma(T) \subset \mathbb{R}$.

Proof. For $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda \neq 0$ then by (4.1) we have that $T - \lambda$ is injective with closed range. Thus, $\lambda \in \rho(T)$ if and only if $T - \lambda$ is surjective, or equivalently, if $T^* - \overline{\lambda}$ is injective. By the previous theorem if $T^* - \overline{\lambda}$ is injective for some λ with $\operatorname{Im} \lambda > 0$, then $T^* - \overline{\lambda}$ is injective for all λ with $\operatorname{Im} \lambda > 0$. Hence, either $\sigma(T) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \leq 0\}$ or $\{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda > 0\} \subset \sigma(T)$.

Since $\sigma(T)$ is closed, it is then easy to see that only one of the four possibilities can occur. ∎

Theorem 4.2.6. *If* $T \in \mathcal{C}(\mathcal{H})$ *is symmetric, then the following are equivalent:*

(*i*) *T is self-adjoint.*

(*ii*) $\ker(T^* - i) = \ker(T^* + i) = \{0\}$.

(*iii*) $\sigma(T) \subset \mathbb{R}$.

Proof. (i) $\implies$ (ii) follows from Proposition 4.2.2, while (ii) $\Leftrightarrow$ (iii) follows from the previous corollary. To see that (ii) $\implies$ (i) suppose that $\ker(T^* - i) = \ker(T^* + i) = \{0\}$. Then by Proposition 4.2.2 we have that $R(T + i) = \ker(T^* - i)^\perp = \mathcal{H}$. Thus, $T + i$ is the only injective extension of $T + i$. Since $T^* + i$ is an injective extension of $T + i$ we conclude that $T^* + i = T + i$ and hence $T^* = T$. ∎

The subspaces $\mathcal{L}_+ = \ker(T^* - i) = R(T+i)^\perp$ and $\mathcal{L}_- = \ker(T^* + i) = R(T-i)^\perp$ are the **deficiency subspaces** of the symmetric operator $T \in \mathcal{C}(\mathcal{H})$, and $n_\pm = \dim \mathcal{L}_\pm$ is the **deficiency indices**.

4.2.1 The Cayley transform

Recall from Section 3.6 that the Cayley transform $t \mapsto (t-i)(t+i)^{-1}$ and its inverse $t \mapsto i(1+t)(1-t)^{-1}$ give a bijection between self-adjoint operators $x = x^* \in \mathcal{B}(\mathcal{H})$ and unitary operators $u \in \mathcal{U}(\mathcal{H})$ such that $1 \notin \sigma(u)$. Here, we extend this correspondence to the setting of unbounded operators.

If $T \in C(\mathcal{H})$ is symmetric with deficiency subspaces $\mathcal{L}_\pm$, then the **Cayley transform** of T is the operator $U : \mathcal{H} \to \mathcal{H}$ given by $U_{|\mathcal{L}_+} = 0$, and

$$U\xi = (T-i)(T+i)^{-1}\xi$$

for all $\xi \in \mathcal{L}_+^\perp = R(T+i)$. If $\eta \in D(T)$ then by (4.1) we have that $\|(T+i)\eta\|^2 = \|T\eta\|^2 + \|\eta\|^2 = \|(T-i)\eta\|^2$, hence it follows that U is a partial isometry with initial space $\mathcal{L}_+^\perp$ and final space $\mathcal{L}_-^\perp$. Moreover, if $\xi \in D(T)$ then $(1-U)(T+i)\xi = (T+i)\xi - (T-i)\xi = 2i\xi$. Since $R(T+i) = \mathcal{L}_+^\perp$ it follows that $(1-U)(\mathcal{L}_+^\perp) = D(T)$ is dense.

Conversely, if $U \in \mathcal{B}(\mathcal{H})$ is a partial isometry with $(1-U)(U^*U\mathcal{H})$ dense, then we also have that $(1-U)$ is injective. Indeed, if $\xi \in \ker(1-U)$ then $\|\xi\| = \|U\xi\|$ so that $\xi \in UU^*\mathcal{H}$. Hence, $\xi = U^*U\xi = U^*\xi$ and so $\xi \in \ker(1 - U^*) = R(1-U)^\perp = \{0\}$.

We define the **inverse Cayley transform** of U to be the densely defined operator with domain $D(T) = (1-U)(U^*U\mathcal{H})$ given by

$$T = i(1+U)(1-U)^{-1}.$$

Note that T is densely defined, and

$$\mathcal{G}(T) = \{(1-U)\xi \oplus i(1+U)\xi \mid \xi \in U^*U\mathcal{H}\}.$$

If $\xi_n \in U^*U\mathcal{H}$ such that $(1-U)\xi_n \oplus i(1+U)\xi_n$ is Cauchy, then both $(1-U)\xi_n$ and $(1+U)\xi_n$ is Cauchy and hence so is ξ_n. Thus, $\xi_n \to \xi$ for some $\xi \in U^*U\mathcal{H}$, and we have $(1-U)\xi_n \oplus i(1+U)\xi_n \to (1-U)\xi \oplus i(1+U)\xi \in \mathcal{G}(T)$. Hence, T is a closed operator.

Note also that for all $\xi, \zeta \in U^*U\mathcal{H}$ we have

$$\begin{aligned}
&\langle (1-U)\xi \oplus i(1+U)\xi, -i(1+U)\zeta \oplus (1-U)\zeta \rangle \\
&= i\langle (1-U)\xi, (1+U)\zeta \rangle + i\langle (1+U)\xi, (1-U)\zeta \rangle \\
&= 2i\langle \xi, \zeta \rangle - 2i\langle U\xi, U\zeta \rangle = 0
\end{aligned}$$

Thus, by Lemma 4.1.4 we have $\mathcal{G}(T) \subset J(\mathcal{G}(T))^\perp = \mathcal{G}(T^*)$, and hence T is symmetric.

Theorem 4.2.7. *The Cayley transform and its inverse give a bijective correspondence between densely defined closed symmetric operators $T \in \mathcal{C}(\mathcal{H})$, and partial isometries $U \in \mathcal{B}(\mathcal{H})$ such that $(1-U)(U^*U\mathcal{H})$ is dense. Moreover, self-adjoint operators correspond to unitary operators.*

Also, if $S, T \in \mathcal{C}(\mathcal{H})$ are symmetric, and $U, V \in \mathcal{B}(\mathcal{H})$ their respective Cayley transforms then we have $S \sqsubseteq T$ if and only if $U^*U\mathcal{H} \subset V^*V\mathcal{H}$ and $V\xi = U\xi$ for all $\xi \in U^*U\mathcal{H}$.

Proof. We've already seen above that the Cayley transform of a densely defined closed symmetric operator T is a partial isometry U with $(1-U)(U^*U\mathcal{H})$ dense. And conversely, the inverse Cayley transform of a partial isometry U with $(1-U)(U^*U\mathcal{H})$ dense, is a densely defined closed symmetric operator. Moreover, it is easy to see from construction that these are inverse operations.

We also see from construction that the deficiency subspaces of T are $\mathcal{L}_+ = \ker(U)$ and $\mathcal{L}_- = \ker(U^*)$ respectively. By Theorem 4.2.6 T is self-adjoint if and only if $\mathcal{L}_+ = \mathcal{L}_- = \{0\}$, which is if and only if U is a unitary.

Suppose now that $S, T \in \mathcal{C}(\mathcal{H})$ are symmetric and $U, V \in \mathcal{B}(\mathcal{H})$ are the corresponding Cayley transforms. If $S \sqsubseteq T$ then for all $\xi \in D(S) \subset D(T)$ we have $(S+i)\xi = (T+i)\xi$ and hence

$$U(S+i)\xi = (S-i)\xi = (T-i)\xi = V(S+i)\xi.$$

Therefore, $U^*U\mathcal{H} = R(S+i) \subset R(T+i) = V^*V\mathcal{H}$ and $V\xi = U\xi$ for all $\xi \in U^*U\mathcal{H}$. Conversely, if $U^*U\mathcal{H} \subset V^*V\mathcal{H}$ and $V\xi = U\xi$ for all $\xi \in U^*U\mathcal{H}$, then

$$D(S) = R((1-U)(U^*U)) = R((1-V)(U^*U)) \subset R((1-V)(V^*V)) = D(T),$$

and for all $\xi \in U^*U\mathcal{H}$ we have

$$S(1-U)\xi = i(1+U)\xi = i(1+V)\xi = T(1-V)\xi = T(1-U)\xi,$$

hence $S \sqsubseteq T$. ■

The previous theorem in particular shows us that if $T \in \mathcal{C}(\mathcal{H})$ is a symmetric operator, and U its Cayley transform, then symmetric extensions of T are in bijective correspondence with partial isometries which extend U. Since the latter are in bijective correspondence with partial isometries from $(UU^*\mathcal{H})^\perp$ to $(U^*U\mathcal{H})^\perp$, simply translating this via the inverse Cayley transform gives the following, whose details we leave to the reader.

Theorem 4.2.8. *Let $T \in \mathcal{C}(\mathcal{H})$ be a symmetric operator, and $\mathcal{L}_\pm$ its deficiency spaces. For each partial isometry $W : \mathcal{L}_+ \to \mathcal{L}_-$, denote the operator T_W by*

$$D(T_W) = \{\xi + \eta + W\eta \mid \xi \in D(T), \eta \in W^*W(\mathcal{L}_+)\},$$

and

$$T_W(\xi + \eta + W\eta) = T\xi + i\eta - iW\eta.$$

Then T_W is a symmetric extension of T with

$$\mathcal{G}(T_W^*) = \mathcal{G}(T_W) + (\mathcal{L}_+ \ominus W^*W(\mathcal{L}_+)) + (\mathcal{L}_- \ominus WW^*(\mathcal{L}_-)).$$

Moreover, every symmetric extension arises in this way, and T_W is self-adjoint if and only if W is unitary.

Corollary 4.2.9. *If $T \in \mathcal{C}(\mathcal{H})$ is symmetric, then T has a self-adjoint extension if and only if $n_+ = n_-$.*

Exercise 4.2.10. show that for any pair $(n_+, n_-) \in (\mathbb{N} \cup \{0\} \cup \{\infty\})^2$ there exists a densely defined closed symmetric operator $T \in \mathcal{C}(\ell^2\mathbb{N})$ such that n_+ and n_- are the deficiency indices for T.

4.3 Functional calculus for normal operators

If $T : \mathcal{H} \to \mathcal{K}$ is a closed operator, then a subspace $D \subset D(T)$ is a **core** for T if $\mathcal{G}(T) = \overline{\mathcal{G}(T_{|D})}$.

4.3.1 Positive operators

Theorem 4.3.1. *Suppose $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, then*

(*i*) *$D(T^*T)$ is a core for T.*

(*ii*) *T^*T is self-adjoint.*

(*iii*) *$\sigma(T^*T) \subset [0, \infty)$.*

Proof. We start by showing that $-1 \in \rho(T^*T)$. Since $\mathcal{K} \oplus \mathcal{H} = J\mathcal{G}(T) + \mathcal{G}(T^*)$, if $\xi \in \mathcal{H}$ then there exists $\eta \in \mathcal{H}$, $\zeta \in \mathcal{K}$ such that

$$0 \oplus \xi = -T\eta \oplus \eta + \zeta \oplus T^*\zeta.$$

Hence, $\zeta = T\eta$ and $\xi = \eta + T^*\zeta = (1 + T^*T)\eta$, showing that $(1 + T^*T)$ is onto.

If $\xi \in D(T^*T)$ then

$$\|\xi + T^*T\xi\|^2 = \|\xi\|^2 + 2\|T\xi\|^2 + \|T^*T\xi\|^2.$$

Hence, we see that $1+T^*T$ is injective. This also shows that if $\xi_n \in D(T^*T)$ is a sequence which is Cauchy in the graph norm of $1+T^*T$, then we must have that $\{\xi_n\}_n$, $\{T\xi_n\}_n$, and $\{T^*T\xi_n\}$ are all Cauchy. Since T and T^* are closed it then follows easily that $1 + T^*T$ is also closed. Thus, $(1 + T^*T)^{-1}$ is an everywhere defined closed operator and hence is bounded, showing that $-1 \in \rho(T^*T)$.

To see that $D(T^*T)$ is a core for T consider $\xi \oplus T\xi \in \mathcal{G}(T)$ such that $\xi \oplus T\xi \perp \{\eta \oplus T\eta \mid \eta \in D(T^*T)\}$. Then for all $\eta \in D(T^*T)$ we have

$$0 = \langle \xi \oplus T\xi, \eta \oplus T\eta \rangle = \langle \xi, \eta \rangle + \langle T\xi, T\eta \rangle = \langle \xi, (1 + T^*T)\eta \rangle.$$

Since $(1 + T^*T)$ is onto, this shows that $\xi = 0$.

In particular, T^*T is densely defined and we have $-1 \in \rho(T^*T)$. Note that by multiplying by scalars we see that $(-\infty, 0) \subset \rho(T)$. If $\xi = (1 + T^*T)\eta$ for $\eta \in D(T^*T)$ then we have

$$\langle (1 + T^*T)^{-1}\xi, \xi\rangle = \langle \eta, (1 + T^*T)\eta\rangle = \|\eta\|^2 + \|T\eta\|^2 \geq 0.$$

Thus $(1+T^*T)^{-1} \geq 0$ and hence it follows from Lemma 4.1.8 that $1+T^*T$ and hence also T^*T is self-adjoint. By Theorem 4.2.6 this shows that $\sigma(T^*T) \subset \mathbb{R}$, and hence $\sigma(T^*T) \subset [0, \infty)$. ■

An operator $T \in \mathcal{C}(\mathcal{H})$ is **positive** if $T = S^*S$ for some densely defined closed operator $S : \mathcal{H} \to \mathcal{H}$.

4.3.2 Borel functional calculus

Suppose K is a locally compact Hausdorff space, and E is a spectral measure on K relative to $\mathcal{H}$. We let $B(K)$ denote the space of Borel functions on K. For each $f \in B(K)$ we define a linear operator $T = \int f\,\mathrm{d}E$ by setting

$$D(T) = \{\xi \in \mathcal{H} \mid \eta \mapsto \int f\,\mathrm{d}E_{\xi,\eta} \text{ is bounded.}\},$$

and letting $T\xi$ be the unique vector such that $\int f\,\mathrm{d}E_{\xi,\eta} = \langle T\xi, \eta\rangle$, for all $\eta \in \mathcal{H}$.

If $B \subset K$ is any Borel set such that $f_{|B}$ is bounded, then for all $\xi, \eta \in \mathcal{H}$ we have that $1_B E_{\xi,\eta} = E_{E(B)\xi,\eta}$ and hence $|\int f\,\mathrm{d}E_{E(B)\xi,\eta}| = |\int f_{|B}\,\mathrm{d}E_{\xi,\eta}| \leq \|f_{|B}\|_\infty \|\xi\|\|\eta\|$, and so $E(B)\mathcal{H} \subset D(T)$. Taking $B_n = \{x \in K \mid |f(x)| \leq n\}$ we then have that $\cup_{n\in\mathbb{N}} E(B_n)\mathcal{H} \subset D(T)$ and this is dense since $E(B_n)$ converges strongly to 1. Thus T is densely defined.

If $S = \int \overline{f}\,\mathrm{d}E$, then for all $\xi \in D(T)$ and $\eta \in D(S)$ we have

$$\langle T\xi, \eta\rangle = \int f\,\mathrm{d}E_{\xi,\eta} = \overline{\int \overline{f}\,\mathrm{d}E_{\eta,\xi}} = \overline{\langle S\eta, \xi\rangle} = \langle \xi, S\eta\rangle.$$

A similar argument shows that $D(T^*) \subset D(S)$, so that in fact we have $S = T^*$ and $T^* = S$. In particular, T is a closed operator, and is self-adjoint if f is real valued. It is equally easy to see that $T^*T = TT^* = \int |f|^2\,\mathrm{d}E$.

It is easy to see that if $f, g \in B(K)$ then $\int f\,\mathrm{d}E + \int g\,\mathrm{d}E \sqsubseteq \int (f + g)\,\mathrm{d}E$, and $(\int f\,\mathrm{d}E)(\int g\,\mathrm{d}E) \sqsubseteq \int fg\,\mathrm{d}E$, and in both cases the domains on the left are cores for the operators on the right. In fact, if $f_1, \ldots, f_n \in B(K)$ is any finite collection of Borel functions, then we have that $\cap_{i=1}^n D(\int f_n\,\mathrm{d}E)$ is a common core for each operator $\int f_i\,\mathrm{d}E$. In particular, on the set of all operators of the form $\int f\,\mathrm{d}E$ we may consider the operations $\hat{+}$, and $\hat{\circ}$ given by $S\hat{+}T = \overline{S + T}$, and $S\hat{\circ}T = \overline{S \circ T}$, and under these operations we have that $f \mapsto \int f\,\mathrm{d}E$ is a unital $*$-homomorphism from $B(K)$ into $\mathcal{C}(\mathcal{H})$.

We also note that $\sigma(\int f\,\mathrm{d}E)$ is contained in the closure of the range of f, for each $f \in B(K)$.

An operator $T \in \mathcal{C}(\mathcal{H})$ is **normal** if $T^*T = TT^*$. Note that equality here implies also $D(T^*T) = D(TT^*)$. We would like to associate a spectral measure for each normal operator as we did for bounded normal operators. However, our approach for bounded operators, Theorem 3.7.5, does not immediately apply since we used there that a bounded normal operator generated an abelian C^*-algebra. Our approach therefore will be to reduce the problem to the case of bounded operators.

Lemma 4.3.2. *Suppose $T \in \mathcal{C}(\mathcal{H})$, then $R = T(1 + T^*T)^{-1}$ and $S = (1 + T^*T)^{-1}$ are bounded contractions. If T is normal then we have $SR = RS$.*

Proof. If $\xi \in \mathcal{H}$, fix $\eta \in D(T^*T)$ such that $(1 + T^*T)\eta = \xi$. Then

$$\|\xi\|^2 = \|(1+T^*T)\eta\|^2 = \|\eta\|^2 + 2\|T\eta\|^2 + \|T^*T\eta\|^2 \geq \|\eta\|^2 = \|(1+T^*T)^{-1}\xi\|^2.$$

Hence $\|S\| \leq 1$. Similarly, $\|\xi\|^2 \geq \|T\eta\|^2 = \|R\xi\|^2$, hence also $\|R\| \leq 1$.

Suppose now that T is normal and $\xi \in D(T)$. Since $\eta \in D(T^*T)$ and $\xi = (1 + T^*T)\eta \in D(T)$ we have that $T\eta \in D(TT^*) = D(T^*T)$. Hence, $T\xi = T(1 + T^*T)\eta = (1 + TT^*)T\eta = (1 + T^*T)T\eta$. Thus, $ST\xi = TS\xi$ for all $\xi \in D(T)$.

Suppose now that $\xi \in \mathcal{H}$ is arbitrary. Since $\eta \in D(T^*T) \subset D(T)$, we have $SR\xi = ST\eta = TS\eta = RS\xi$. ■

Theorem 4.3.3. *Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator, then $\sigma(T) \neq \emptyset$ and there exists a unique spectral measure E for $\sigma(T)$ relative to $\mathcal{H}$ such that*

$$T = \int t \, \mathrm{d}E(t).$$

Proof. Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator. For each $n \in \mathbb{N}$ we denote by $P_n = 1_{(\frac{1}{n+1}, \frac{1}{n}]}(S)$, where $S = (1+T^*T)^{-1}$. Notice that since S is a positive contraction which is injective, we have that P_n are pairwise orthogonal projections and $\sum_{n \in \mathbb{N}} P_n = 1$, where the convergence of the sum is in the strong operator topology. Note, also that if $\mathcal{H}_n = R(P_n)$ then we have $S\mathcal{H}_n = \mathcal{H}_n$ and restricting S to $\mathcal{H}_n$ we have that $\frac{1}{n+1} \leq S_{|\mathcal{H}_n} \leq \frac{1}{n}$. In particular, we have that $\mathcal{H}_n \subset R(S) = D(T^*T)$, $(1 + T^*T)$ maps $\mathcal{H}_n$ onto itself for each $n \in \mathbb{N}$, and $\sigma((1 + T^*T)_{|\mathcal{H}_n}) \subset \{\lambda \in \mathbb{C} \mid n \leq |\lambda| \leq n + 1\}$.

By Lemma 4.3.2 $R = T(1 + T^*T)^{-1}$ commutes with S and since S is self-adjoint we then have that R commutes with any of the spectral projections P_n. Since we've already established that $(1 + T^*T)$ give a bijection on $\mathcal{H}_n$ it then follows that $T\mathcal{H}_n \subset \mathcal{H}_n$ for all $n \in \mathbb{N}$. Note that since T is normal, by symmetry we also have that $T^*\mathcal{H}_n \subset \mathcal{H}_n$ for all $n \in \mathbb{N}$. Hence, restricting to $\mathcal{H}_n$ we have $(TP_n)^*(TP_n) = P_n(T^*T)P_n = P_n(TT^*)P_n = (TP_n)(TP_n)^*$ for all $n \in \mathbb{N}$.

Let $I = \{n \in \mathbb{N} \mid P_n \neq 0\}$, and note that $I \neq \emptyset$ since $\sum_{n \in I} P_n = 1$. For $n \in I$, restricting to $\mathcal{H}_n$, we have that TP_n is a bounded normal operator with spectrum $\sigma(TP_n) \subset \{\lambda \in \mathbb{C} \mid n - 1 \leq |\lambda| \leq n\}$. Let E_n denote the unique spectral measure on $\sigma(TP_n)$ so that $T_{|\mathcal{H}_n} = \int t \, \mathrm{d}E_n(t)$.

We let E be the spectral measure on $\overline{\cup_{n\in I}\sigma(TP_n)} = \cup_{n\in I}\sigma(TP_n)$ which is given by $E(F) = \sum_{n\in I} E_n(F)$ for each Borel subset $F \subset \cup_{n\in I}\sigma(TP_n)$. Since the $E_n(F)$ are pairwise orthogonal it is easy to see that E is indeed a spectral measure. We set $\tilde{T}$ to be the operator $\tilde{T} = \int t\,\mathrm{d}E(t)$.

We claim that $\tilde{T} = T$. To see this, first note that if $\xi \in \mathcal{H}_n$ then $\tilde{T}\xi = TP_n\xi = T\xi$. Hence, $\tilde{T}$ and T agree on $\mathcal{K}_0 = \cup_{n\in I}\mathcal{H}_n$. Since both operators are closed, and since $\mathcal{K}_0$ is clearly a core for $\tilde{T}$, to see that they are equal it is then enough to show that $\mathcal{K}_0$ is also core for T. If we suppose that $\xi \in D(T^*T)$, and write $\xi_n = P_n\xi$ for $n \in I$, then $\lim_{N\to\infty}\sum_{n\leq N}\xi_n = \xi$, and setting $\eta = (1 + T^*T)\xi$ we have

$$\begin{aligned}\sum_{n\in I}\|T\xi_n\|^2 &= \sum_{n\in I}\langle T^*T\xi_n, \xi_n\rangle \\ &= -\|\xi\|^2 + \sum_{n\in I}\langle (1+T^*T)\xi_n, \xi_n\rangle \\ &\leq \|\xi\|\|\eta\| < \infty.\end{aligned}$$

Since T is closed we therefore have $\lim_{N\to\infty} T(\sum_{n\leq N}\xi_n) = T\xi$. Thus, $\overline{\mathcal{G}(T_{|\mathcal{K}_0})} = \overline{\mathcal{G}(T_{|D(T^*T)})} = \mathcal{G}(T)$.

Since $\sigma(T) = \sigma(\tilde{T}) = \cup_{n\in I}\sigma(TP_n)$, this completes the existence part of the proof. For the uniqueness part, if $\tilde{E}$ is a spectral measure on $\sigma(T)$ such that $T = \int t\,\mathrm{d}\tilde{E}(t)$ then by uniqueness of the spectral measure for bounded normal operators it follows that for every $F \subset \sigma(T)$ Borel, and $n \in I$, we have $P_nE(F) = P_n\tilde{E}(F)$, and hence $E = \tilde{E}$. ■

If $T = \int t\,\mathrm{d}E(t)$ as above, then for any $f \in B(\sigma(T))$ we define $f(T)$ to be the operator $f(T) = \int f(t)\,\mathrm{d}E(t)$.

Corollary 4.3.4. *Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator. Then for any $*$-polynomial $p \in \mathbb{C}[t, t^*]$ we have that $p(T)$ is densely defined and closable, and in fact $D(p(T))$ is a core for T.*

Proposition 4.3.5. *Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator, and consider the abelian von Neumann algebra $W^*(T) = \{f(T) \mid f \in B_\infty(\sigma(T))\}'' \subset \mathcal{B}(\mathcal{H})$. If $u \in \mathcal{U}(\mathcal{H})$, then $u \in \mathcal{U}(W^*(T)')$ if and only if $uTu^* = T$.*

Proof. Suppose that $u \in \mathcal{U}(\mathcal{H})$ and $T \in \mathcal{C}(\mathcal{H})$ is normal. We let E be the spectral measure on $\sigma(T)$ such that $T = \int t\,\mathrm{d}E(t)$, and consider the spectral measure $\tilde{E}$ given by $\tilde{E}(F) = uE(F)u^*$ for all $F \subset \sigma(T)$ Borel. We then clearly have $uTu^* = \int t\,\mathrm{d}\tilde{E}(t)$ from which the result follows easily. ■

If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $T : \mathcal{H} \to \mathcal{H}$ is a linear operator, then we say that T is **affiliated with** M and write $T\eta M$ if $uTu^* = T$ for all $u \in \mathcal{U}(M')$, (note that this implies $uD(T) = D(T)$ for all $u \in U(M')$). The previous proposition shows that any normal linear operator is affiliated with an abelian von Neumann algebra.

Corollary 4.3.6. *If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $T \in \mathcal{C}(\mathcal{H})$ is normal, then $T \eta M$ if and only if $f(T) \in M$ for all $f \in B_\infty(\sigma(T))$.*

Proposition 4.3.7. *Suppose M is a von Neumann algebra and $T, S : \mathcal{H} \to \mathcal{H}$ are linear operators such that $T, S \eta M$. Then $TS, (T+S) \eta M$. Moreover, if T is densely defined then $T^* \eta M$, and if S is closable then $\overline{S} \eta M$.*

Proof. Since $T, S \eta M$, for all $u \in \mathcal{U}(M')$ we have

$$\begin{aligned} uD(TS) &= \{\xi \in \mathcal{H} \mid u^*\xi \in D(S), S(u^*\xi) \in D(T)\} \\ &= \{\xi \in \mathcal{H} \mid \xi \in D(S), u^*S\xi \in D(T)\} \\ &= \{\xi \in \mathcal{H} \mid \xi \in D(S), S\xi \in D(T)\} = D(TS). \end{aligned}$$

Also, for $\xi \in D(TS)$ we have $u^*TSu\xi = (u^*Tu)(u^*Su)\xi = TS\xi$, hence $TS \eta M$. The proof that $(T+S)\eta M$ is similar.

If T is densely defined, then for all $u \in \mathcal{U}(M')$ we have

$$\begin{aligned} uD(T^*) &= \{\xi \in \mathcal{H} \mid \eta \mapsto \langle T\eta, u^*\xi \rangle \text{ is bounded.}\} \\ &= \{\xi \in \mathcal{H} \mid \eta \mapsto \langle T(u\eta), \xi \rangle \text{ is bounded.}\} = D(T^*), \end{aligned}$$

and for $\xi \in D(T^*)$, and $\eta \in D(T)$ we have $\langle T\eta, u^*\xi \rangle = \langle Tu, \xi \rangle$, from which it follows that $T^*u^*\eta = u^*T^*\eta$, and hence $T^* \eta M$.

If S is closable, then in particular we have that $u\overline{D(S)} = \overline{D(S)}$ for all $u \in \mathcal{U}(M')$. Hence if p denotes the orthogonal projection onto $\overline{D(S)}$ then $p \in M'' = M$, and $S \eta pMp \subset \mathcal{B}(p\mathcal{H})$. Hence, we may assume that S is densely defined in which case we have $S \eta M \implies S^* \eta M \implies \overline{S} = S^{**} \eta M$. ∎

4.3.3 Polar decomposition

For $T \in \mathcal{C}(\mathcal{H})$ the **absolute value** of T is the positive operator $|T| = \sqrt{T^*T} \in \mathcal{C}(\mathcal{H})$.

Theorem 4.3.8 (Polar decomposition)**.** *Fix $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$. Then $D(|T|) = D(T)$, and there exists a unique partial isometry $v \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that* $\ker(v) = \ker(T) = \ker(|T|)$*, and $T = v|T|$.*

Proof. By Theorem 4.3.1 we have that $D(T^*T)$ is a core for both $|T|$ and T. We define the map $V_0 : \mathcal{G}(|T|_{|D(T^*T)}) \to \mathcal{G}(T)$ by $V_0(\xi \oplus |T|\xi) = \xi \oplus T\xi$. Since, for $\xi \in D(T^*T)$ we have $\|\xi\|^2 + \||T|\xi\|^2 = \|\xi\|^2 + \|T\xi\|^2$ this shows that V_0 is isometric, and since $D(T^*T)$ is a core for both $|T|$ and T we then have that V_0 extends to an isometry from $\mathcal{G}(|T|)$ onto $\mathcal{G}(T)$, and we have $D(|T|) = P_\mathcal{H}(\mathcal{G}(|T|)) = P_\mathcal{H}(V\mathcal{G}(|T|)) = P_\mathcal{H}(\mathcal{G}(T)) = D(T)$.

Moreover, this also shows that the map $v_0 : R(|T|) \to R(T)$ given by $v_0(|T|\xi) = T\xi$, is well defined and extends to a partial isometry $v \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\ker(v) = R(T)^\perp = \ker(T)$. From the definition of v we clearly have that $T = v|T|$. Uniqueness follows from the fact that any other partial isometry w which satisfies $T = w|T|$ must agree with v on $\overline{R(|T|)} = \ker(|T|)^\perp = \ker(T)^\perp$. ∎

Proposition 4.3.9. *If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $T \in \mathcal{C}(\mathcal{H})$ has polar decomposition $T = v|T|$, then $T \eta M$ if and only if $v \in M$ and $|T| \eta M$.*

Proof. If $T\eta M$, then $T^* T \eta M$ by Proposition 4.3.7. By Corollary 4.3.6 we then have that $|T|\eta M$. Hence, for any $u \in M'$ if $\xi \in R(|T|)$ say $\xi = |T|\eta$ for $\eta \in D(|T|) = D(T)$, then $uv\xi = uv|T|\eta = uT\eta = Tu\eta = v|T|u\eta = vu\xi$, hence $v \in M'' = M$.

Conversely, if $v \in M$ and $|T|\eta M$, then $T = (v|T|)\eta M$ by Proposition 4.3.7. ∎

4.3.4 The extended positive cone

If $S \in \mathcal{C}(\mathcal{H})$ is a positive operator, then we may consider the closed non-negative definite quadratic forms $q_S : \mathcal{H} \to [0, \infty)$ given by $D(q_S) = D(S^{1/2})$ and $q_S(\xi) = \|S^{1/2}\xi\|^2$. By Theorem 4.1.10 every densely defined non-negative definite closed quadratic form arrises in this way. Moreover, by Theorem 4.3.1 $D(S^{1/2})$ is a core for S and it then follows easily that the correspondence $S \mapsto q_S$ is unique. More generally, we see that there is a bijective correspondence between non-negative defnite closed quadratic forms and positive operators which are densely defined on a subspace of $\mathcal{H}$. If S, T are positive operators on densely defined subspaces of $\mathcal{H}$ then we define the **form sum** $S \dot{+} T$ to be the operator corresponding to $q_S + q_T$. I.e., $S \dot{+} T$ is uniquely defined by the relation

$$\langle (S \dot{+} T)^{1/2}\xi, (S \dot{+} T)^{1/2}\eta \rangle = \langle S^{1/2}\xi, S^{1/2}\eta \rangle + \langle T^{1/2}\xi, T^{1/2}\eta \rangle,$$

for all $\xi, \eta \in D((S \dot{+} T)^{1/2}) = D(S^{1/2}) \cap D(T^{1/2})$.

We also write $S \leq T$ if $q_S \leq q_T$, i.e., if $D(T^{1/2}) \subset D(S^{1/2})$ and $\|S^{1/2}\xi\|^2 \leq \|T^{1/2}\xi\|^2$ for all $\xi \in D(T^{1/2})$.

If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra then the **extended positive cone** $\hat{M}_+$ of M consists of all positive operators T which are densely defined on a subspace $\mathcal{K} \subset \mathcal{H}$, such that $P_\mathcal{K} \in M$ and T is affiliated with $P_\mathcal{K} M P_\mathcal{K}$. Then $\hat{M}_+$ is a cone where addition is given by form addition.

Suppose $A \in \hat{M}_+$ is densely defined on $\mathcal{K} \subset \mathcal{H}$, and we consider the spectral decomposition

$$A = \int_0^\infty t \, dE(t).$$

For each $\varphi \in M_*^+$ we set $m_A(\varphi) = \infty$ if $\varphi(1 - P_\mathcal{K}) \neq 0$, and otherwise we set

$$m_A(\varphi) = \int t \, d\varphi(E(t)) \in [0, \infty].$$

Then we have that

1. $m_A(\lambda\varphi) = \lambda m_A(\varphi), \varphi \in M_*^+, \lambda \geq 0$,
2. $m_A(\varphi + \psi) = m_A(\varphi) + m_A(\psi), \varphi, \psi \in M_*^+$,
3. $\sup_i m_A(\varphi_i) = m_A(\varphi)$, for any increasing net $\varphi_i \nearrow \varphi$ in M_*^+.

The third condition follows from the fact that $m_A(\varphi) = \lim_{n\to\infty} \int_0^n t d\varphi(E(t)) = \varphi(A_n)$ where $A_n = \int_0^n t dE(t)$.

In particular, for the vector states $\varphi_\xi(T) = \langle T\xi, \xi\rangle$, we have

$$m_A(\varphi_\xi) = \begin{cases} \|A^{1/2}\xi\|^2, & \text{if } \xi \in D(A^{1/2}), \\ \infty, & \text{otherwise.} \end{cases}$$

It therefore follows that $m_A = m_B$ only if $A = B$. Also, if $m : M_*^+ \to [0, \infty]$ satisfies the three conditions above, then we may define a closed non-negative definte quadratic form $q : \mathcal{H} \to [0, \infty)$ by $q(\xi) = m(\varphi_\xi)$. Moreover, we have that q is invariant under the unitary group of $\mathcal{U}(M')$ and it then follows easily that $m = m_A$ for some operator A as above. Thus, we may identify the extended positive cone with functions satisfying the three conditions above.

4.4 Semigroups and infinitesimal generators.

4.4.1 Contraction semigroups

A **one-parameter contraction semigroup** consists of a family of positive contractions $\{S_t\}_{t\geq 0}$ such that $S_0 = 1$, and $S_t S_s = S_{t+s}$, $s, t \geq 0$. The semigroup is **strongly continuous** if for all $t_0 \geq 0$ we have $\lim_{t\to t_0} S_t = S_{t_0}$, in the strong operator topology.

If $A \in \mathcal{C}(\mathcal{H})$ is a normal operator such that $\sigma(A) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$, then the operator $S_t = e^{-tA}$ is normal and satisfies $\sigma(S_t) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$. Hence, $\{S_t\}_{t\geq 0}$ defines a one-parameter contraction semigroup of normal operators. Moreover, for all $\xi \in \mathcal{H}$ and $t_0 > 0$, we have $\lim_{t\to t_0} \|(S_t - S_{t_0})\xi\|^2 = \lim_{t\to t_0} \int |e^{-ta} - e^{t_0 a}|^2 \mathrm{d}E_{\xi,\xi}(a) = 0$. Thus, the semigroup $\{S_t\}_{t\geq 0}$ is strongly continuous. The operator A is the **infinitesimal generator** of the semigroup $\{S_t\}_{t\geq 0}$.

Theorem 4.4.1 (Hille-Yosida)**.** *Let $\{S_t\}_{t\geq 0}$ be a strongly continuous one-parameter contraction semigroup of normal operators, then there exists a unique normal operator $A \in \mathcal{C}(\mathcal{H})$ satisfying $\sigma(A) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$ such that A is the infinitesimal generator of the semigroup.*

Proof. Let $\{S_t\}_{t\geq 0}$ be a strongly continuous one-parameter semigroup of normal contractions. We define the linear operator $A : \mathcal{H} \to \mathcal{H}$ by setting $A\xi = \lim_{t\to 0} \frac{1}{t}(1 - S_t)\xi$, where the domain of A is the subspace of all vectors ξ such that this limit exists. We claim that A is the unique densely defined normal operator which generates this semigroup.

To see that A is densely defined we introduce for each $\alpha > 0$ the operator $R_\alpha = \int_0^\infty e^{-\alpha t} S_t \mathrm{d}t$, where this integral is understood as a Riemann integral in the strong operator topology. This is a well defined bounded normal operator since each S_t is normal and $t \mapsto S_t$ is strong operator continuous and uniformly bounded. If $s, t > 0$ we have

$$\begin{aligned}\frac{1}{t}(1 - S_t)R_\alpha &= \frac{1}{t}\int_0^\infty (1 - S_t)e^{-\alpha r}S_r \mathrm{d}r \\ &= \frac{1}{t}\int_0^\infty e^{-\alpha r}S_r \mathrm{d}r - \frac{1}{t}\int_t^\infty e^{-\alpha(r-t)}S_r \mathrm{d}r \\ &= \frac{1}{t}\int_0^t e^{-\alpha r}S_r \mathrm{d}r + \int_t^\infty e^{-r\alpha}\frac{1 - e^{t\alpha}}{t}S_r \mathrm{d}r. \qquad (4.2)\end{aligned}$$

If $\alpha > 0$, and $\xi \in \mathcal{H}$, then taking a limit as t tends to 0 in Equation (4.2) gives

$$\lim_{t\to\infty}\frac{1}{t}(1 - S_t)R_\alpha \xi = \xi - \alpha R_\alpha \xi.$$

Thus, we see that $R_\alpha \mathcal{H} \subset D(A)$, and we have $AR_\alpha = 1 - \alpha R_\alpha$, or equivalently $(A + \alpha)R_\alpha = 1$, for all $\alpha > 0$.

If $\xi \in \mathcal{H}$, then since $\int_0^\infty \alpha e^{-\alpha t}\mathrm{d}t = 1$ for all $\alpha > 0$ it follows that

$$(\alpha R_\alpha - 1)\xi = \alpha \int_0^\infty e^{-\alpha t}(S_t - 1)\xi \mathrm{d}t. \qquad (4.3)$$

If we fix $\varepsilon > 0$ and take $\delta > 0$ such that $\|(S_t - 1)\xi\| < \varepsilon$ for all $0 < t \le \delta$, then using the triangle inequality in Equation (4.3) it follows that

$$\|(\alpha R_\alpha - 1)\xi\| \le \alpha \int_0^\delta e^{-\alpha t}\varepsilon \mathrm{d}t + \alpha \int_\delta^\infty e^{-\alpha t}2\|\xi\| \le \varepsilon + 2\|\xi\|e^{-\delta\alpha}.$$

As ε was arbitrary, and this holds for all $\alpha > 0$ it then follows that $\lim_{\alpha\to\infty}\alpha R_\alpha \xi = \xi$. Thus, A is densely defined since $R_\alpha \mathcal{H} \subset D(A)$, and αR_α converges strongly to 1 as α tends to ∞.

For all $\alpha, t > 0$ we have

$$\begin{aligned}\frac{1}{t}(1 - S_t)R_\alpha &= \frac{1}{t}(1 - S_t)\int_0^\infty e^{-\alpha r}S_r \mathrm{d}r \\ &= \int_0^\infty \frac{e^{-\alpha r}}{t}(S_r - S_{r+t})\mathrm{d}r \\ &= R_\alpha \frac{1}{t}(1 - S_t).\end{aligned}$$

Thus, for $\xi \in D(A)$ we have $\lim_{t\to 0}\frac{1}{t}(1 - S_t)R_\alpha \xi = \lim_{t\to 0} R_\alpha \frac{1}{t}(1 - S_t)\xi = R_\alpha A\xi$. Hence, we have that $R_\alpha D(A) \subset D(A)$, and $AR_\alpha = R_\alpha A$.

We then have that $A + \alpha$ and R_α commute, and since $A + \alpha$ is a left inverse for R_α it then follows that $A = R_\alpha^{-1} - \alpha$, for all $\alpha > 0$. In particular, since R_α is bounded and normal it then follows that A is a closed normal operator.

Finally, note that if $\xi \in D(A)$, then the function $r \mapsto S_r\xi$ is differentiable with derivative $-AS_r$, i.e., $-AS_r\xi = \lim_{h\to 0}\frac{1}{h}(S_{r+h} - S_r)\xi$. If we consider the semigroup $\tilde{S}_r = e^{-rA}$, then for all $\xi \in D(A)$ we similarly have $\frac{d}{dr}e^{-rA} =$

$-Ae^{-rA}$. If we now fix $t > 0$, and $\xi \in D(A)$, and we consider the function $[0,t] \ni r \mapsto S_r e^{-(t-r)A}\xi$, then the chain rule shows that this is differentiable and has derivative

$$\frac{d}{dr}(S_r e^{-(t-r)A}\xi) = AS_r e^{-(t-r)A}\xi - S_r A e^{-(t-r)A}\xi = 0.$$

Thus, this must be a constant function and if we consider the cases $s = 0$ and $s = t$ we have $e^{-tA}\xi = S_t\xi$. Since $D(A)$ is dense and these are bounded operators it then follows that $e^{-tA} = S_t$ for all $t \geq 0$. ∎

4.4.2 Stone's Theorem

A **one-parameter group of unitaries** consists of a family of unitary operators $\{u_t\}_{t\in\mathbb{R}}$ such that $u_t u_s = u_{t+s}$. The one-parameter group of unitaries is **strongly continuous** if for all $t_0 \in \mathbb{R}$ we have $\lim_{t\to t_0} u_t = u_{t_0}$ where the limit is in the strong operator topology. Note that by multiplying on the left by $u_{t_0}^*$ we see that strong continuity is equivalent to strong continuity for $t_0 = 0$. Also, since the strong and weak operator topologies agree on the space of unitaries we see that this is equivalent to $\lim_{t\to 0} u_t = u_0$ in the weak operator topology.

If $A \in \mathcal{C}(\mathcal{H})$, $A = A^*$, then we have $\sigma(A) \subset \mathbb{R}$ and hence $\sigma(e^{itA}) \subset \mathbb{T}$ for all $t \in \mathbb{R}$. Thus, $\{e^{itA}\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitaries.

Theorem 4.4.2 (Stone)**.** *Let $\{u_t\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter group of unitaries, then there exists a closed densely defined self-adjoint operator $A \in \mathcal{C}(\mathcal{H})$ which is the infinitesimal generator of $\{u_t\}_{t\in\mathbb{R}}$.*

Proof. Suppose that $\{u_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitaries. From the Hille-Yosida Theorem there exists a closed densely defined normal operator $A \in \mathcal{C}(\mathcal{H})$ such that $u_t = e^{itA}$, for all $t \geq 0$. Note that for $t < 0$ we have $u_t = u_{-t}^* = (e^{-itA})^* = e^{itA^*}$. Thus, it suffices to show that A is self-adjoint. To see this, note that for $\xi \in D(A^*A) = D(AA^*)$ we have

$$iA^*\xi = \lim_{t\to 0^+} \frac{1}{-t}(1 - u_t^*)\xi = \lim_{t\to 0^+} \frac{1}{t}(1 - u_t)u_t^*\xi = \lim_{t\to 0^+} \frac{1}{t}(1 - u_t)\xi = iA\xi.$$

Since $D(A^*A)$ is a core for A^* we then have $A \sqsubseteq A^*$. By symmetry we also have $A^* \sqsubseteq (A^*)^* = A$, hence $A = A^*$. ∎

Chapter 5

Types of von Neumann algebras

5.1 Projections

If M is a von Neumann algebra we denote by $\mathcal{P}(M)$ the space of projections. The following proposition we leave as an exercise.

Proposition 5.1.1. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $p \in \mathcal{P}(M)$ or $p \in \mathcal{P}(M')$ then pMp is a von Neumann subalgebra of $\mathcal{B}(p\mathcal{H})$.*

5.1.1 The projection lattice

If $K \subset \mathcal{H}$ is a subset, then we use the notation $[K]$ to denote the orthogonal projection onto the closure of $\mathrm{sp}\, K$.

Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose $\{p_\alpha\}_\alpha \subset \mathcal{P}(M)$ is a family of projections. The **infimum** of the family $\{p_\alpha\}_\alpha$ is $[\cap_\alpha p_\alpha \mathcal{H}]$ and is denoted by $\wedge_\alpha p_\alpha$. The **supremum** of the family is given by $\vee_\alpha p_\alpha = [\sum_\alpha p_\alpha \mathcal{H}] = 1 - \wedge_\alpha (1 - p_\alpha)$.

Given two projections $p, q \in \mathcal{P}(M)$ we say that q is **sub-equivalent** to p and write $p \preceq q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$. The projections $p, q \in \mathcal{P}(M)$ are **equivalent** and we write $p \sim q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. If $p \preceq q$ but $p \not\sim q$ then we write $p \prec q$.

Proposition 5.1.2. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then the relation $p \preceq q$ is a partial ordering on $\mathcal{P}(M)$, and the relation $p \sim q$ is an equivalence relation on $\mathcal{P}(M)$.*

Proof. To show that $\preceq$ is transitive, suppose that $p, q, r \in \mathcal{P}(M)$ such that $p = u^*u$, $uu^* \leq q = v^*v$, and $vv^* \leq r$. Since $uu^* \leq q$ we have

$$qu = q(uu^*)u = (uu^*)u = u.$$

Hence,

$$(vu)^*(vu) = u^*qu = u^*u = p$$

and

$$(vu)(vu)^* \leq vv^* \leq r.$$

The same argument shows that $\sim$ is transitive, and $\sim$ is clearly reflexive and symmetric. ■

Example 5.1.3. Consider a set X. Then each subset $S \subset X$ determines a closed subspace $\ell^2 S \subset \ell^2 X$ and hence a projection $[\ell^2 S] \in \mathcal{B}(\ell^2 X)$. Any bijection $f : S_1 \to S_2$ determines an isometry $v_f : \ell^2 S_1 \to \ell^2 S_2$ by the formula $v_f(\xi)(s) = \xi(f^{-1}(s))$. Thus, the two projections $[\ell^2 S_1]$ and $[\ell^2 S_2]$ are equivalent in $\mathcal{B}(\ell^2 X)$. Conversely, if $[\ell^2 S_1]$ and $[\ell^2 S_2]$ are equivalent then in particular the spaces have the same dimension and hence there exists a bijection between S_1 and S_2.

We also see similarly that $[\ell^2 S_1] \preceq [\ell^2 S_2]$ in $\mathcal{B}(\ell^2 X)$ if and only if there exists an injective function from S_1 to S_2.

More generally, for any Hilbert space $\mathcal{H}$ and any two projections $p, q \in \mathcal{B}(\mathcal{H})$ we have $p \preceq q$ in $\mathcal{B}(\mathcal{H})$ if and only if $\dim(p\mathcal{H}) \leq \dim(q\mathcal{H})$.

With the above example in mind we see that the following is a generalization of the Cantor-Bernstein-Schröeder theorem in set theory, and the proof is much the same.

Proposition 5.1.4. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, if $p, q \in \mathcal{P}(M)$ such that $p \preceq q$ and $q \preceq p$, then $p \sim q$.*

Proof. Suppose $vv^* \leq p = u^*u$, and $uu^* \leq q = v^*v$. Set $p_0 = p - vv^*$, $q_0 = up_0u^*$, and inductively define a pair of sequences of orthogonal projections $\{p_n\}_n, \{q_n\}_n$, as follows:

$$p_n = vq_{n-1}v^*, \qquad q_n = up_nu^*.$$

We also define the projections

$$p_\infty = p - \sum_{n=0}^{\infty} p_n, \qquad q_\infty = q - \sum_{n=0}^{\infty} q_n.$$

By construction we have $(up_n)^*(up_n) = p_n$, and $(up_n)(up_n)^* = q_n$, for every $n \geq 0$. Also, if we consider $v_k = v^*(p - \sum_{n=0}^{k} p_n)$ then it is easy to check that for $k \geq 0$ we have $vv^* = p - \sum_{n=0}^{k} p_n$ and $v^*v = q - \sum_{n=0}^{k-1} q_n$. Taking limits as $k \to \infty$ we see that $(vp_\infty)(vp_\infty)^* = p_\infty$ and $(vp_\infty)^*(vp_\infty) = q_\infty$. Hence, considering $w = u(\sum_{n=0}^{\infty} p_n) + v^*p_\infty$ we have $w^*w = p$ and $ww^* = q$. ■

Lemma 5.1.5. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, if $\{p_i\}_i$ and $\{q_i\}_i$ are two families of pairwise orthogonal projections in M such that $p_i \preceq q_i$, for each i, then we have $\sum_i p_i \preceq \sum_i q_i$.*

Proof. Suppose $u_i \in M$ such that $u_i^* u_i = p_i$ and $u_i u_i^* = r_i \leq q_i$. By orthogonality we have $u_i^* u_j = u_j u_i^* = 0$ for $i \neq j$, and hence $(\sum_i u_i)^* (\sum_i u_i) = \sum_i p_i$ while $(\sum_i u_i)(\sum_i u_i)^* = \sum_i r_i \leq \sum_i q_i$. ■

Lemma 5.1.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $x \in M$. Then $[x\mathcal{H}], [x^*\mathcal{H}] \in M$ and $[x\mathcal{H}] \sim [x^*\mathcal{H}]$.*

Proof. If we consider the polar decomposition $x = v|x|$. Then we see easily that $vv^* = [x\mathcal{H}]$, while $v^*v = [x^*\mathcal{H}]$. Since $v \in M$ the result folows. ■

Proposition 5.1.7 (Kaplansky's formula)**.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $p, q \in \mathcal{P}(M)$ then*

$$p \vee q - p \sim q - p \wedge q.$$

Proof. If we consider $x = (1-p)q$ then we have $\ker(x) = \ker(q) \oplus (q\mathcal{H} \cap p\mathcal{H})$, hence $[x^*\mathcal{H}] = 1 - (1 - q + q \wedge p) = q - q \wedge p$. By symmetry it then follows that

$$\begin{aligned} [x\mathcal{H}] &= (1-p) - (1-p) \wedge (1-q) \\ &= (1-p) - (1 - p \vee q) = p \vee q - p. \end{aligned}$$

The result then follows from Lemma 5.1.6 ■

If $x \in M$, the **central support** of x is the infimum of all central projections $z \in \mathcal{Z}(M)$ such that $zx = xz = x$. we denote the central support of x by $z(x)$. Two projections p, and q are **centrally orthogonal** if $z(p)z(q) = 0$.

Lemma 5.1.8. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p \in \mathcal{P}(M)$ be a projection, then the central support of p is*

$$z = \vee_{x \in M} [xp\mathcal{H}] = [Mp\mathcal{H}].$$

Proof. By considering $x = 1$ we see that $p \leq z$, and z is central since the range of z is clearly invariant to all operators in M. Thus, $z(p) \leq z$. We also have that the range of $z(p)$ is invariant to all operators in M and since $p \leq z(p)$ it follows that any operator in M maps the range of p into the range of $z(p)$. Thus $[xp\mathcal{H}] \leq z(p)$ for all $x \in M$ and hence $z \leq z(p)$. ■

Proposition 5.1.9. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p, q \in \mathcal{P}(M)$ be two projections, then the following are equivalent:*

(i) *p and q are centrally orthogonal.*

(ii) *$pMq = \{0\}$.*

(iii) *There does not exist nonzero projections $p_0 \leq p$, and $q_0 \leq q$ such that $p_0 \sim q_0$.*

Proof. The equivalence between (i) and (ii) follows easily from the previous lemma. Indeed, for all $x \in M$ we have $[pxq\mathcal{H}] \leq p \leq z(p)$, and by the previous lemma we have $[pxq\mathcal{H}] \leq z(q)$, thus if p and q are centrally orthogonal then we have $pxq = 0$ for all $x \in M$. Conversely, if $pMq = \{0\}$ then we have $pz(q) = 0$ and since $z(q)$ is central it follows that $z(p) \leq 1 - z(q)$.

To see that (ii) and (iii) are equivalence note that if $x \in M$ such that $[pxq\mathcal{H}] \neq 0$ then $[pxq\mathcal{H}] \leq p$ and $[pxq\mathcal{H}] \sim [qx^*p\mathcal{H}] \leq q$. Conversely, if $p_0 \leq p$ is a nonzero projection and $v \in M$ such that $v^*v = p$ and $vv^* \leq q$ then $v^* = pv^*q \in pMq \setminus \{0\}$. ■

Theorem 5.1.10 (The comparison theorem)**.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, if $p, q \in \mathcal{P}(M)$ then there exists a central projection $z \in \mathcal{P}(\mathcal{Z}(M))$ such that*

$$zp \preceq zq, \quad \text{and} \quad (1-z)q \preceq (1-z)p.$$

Proof. By Zorn's lemma there exists a maximal families $\{p_\alpha\}_{\alpha \in I}$ and $\{q_\alpha\}_{\alpha \in I}$ of pairwise orthogonal projections such that $p_0 = \sum_\alpha p_\alpha \leq p$, $q_0 = \sum_\alpha q_\alpha \leq q$, and $p_\alpha \sim q_\alpha$ for all $\alpha \in I$. If we let z_1 be the central support of $p - p_0$, and z_2 be the central support of $q - q_0$ then by Proposition 5.1.9 we have $z_1 z_2 = 0$, and hence $p - p_0 \leq z_1 \leq 1 - z_2$. Thus, $(p - p_0)z_2 = 0$ and since $p_0 \sim q_0$ implies $p_0 z_2 \sim q_0 z_2$ we have

$$pz_2 = p_0 z_2 \sim q_0 z_2 \leq q z_2,$$

while

$$q(1 - z_2) = q_0(1 - z_2) \sim p_0(1 - z_2) \leq p(1 - z_2).$$ ■

A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is a **factor** if it has trivial center, i.e., $\mathcal{Z}(M) = \mathbb{C}$.

Corollary 5.1.11. *If $M \subset \mathcal{B}(\mathcal{H})$ is a factor and $p, q \in \mathcal{P}(M)$ then exactly one of the following is true.*

$$p \prec q, \quad p \sim q, \quad q \prec p.$$

Proof. Since M is a factor the only central projections are 0, or 1. Hence, by the comparison theorem we have $p \preceq q$ or $q \preceq p$. If both occur then by Proposition 5.1.4 we have $p \sim q$. ■

5.2 Types of projections

Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A projection $p \in \mathcal{P}(M)$ is said to be

- **minimal** if $p \neq 0$, and the only subprojections are 0 and p, or equivalently if $\dim(pMp) = 1$.
- **abelian** if pMp is abelian.
- **finite** if $q \leq p$ and $q \sim p$ implies $q = p$.

- **semi-finite** if there are pairwise orthogonal finite projections $p_\alpha \in \mathcal{P}(M)$ such that $p = \sum_\alpha p_\alpha$.
- **purely infinite** if $p \neq 0$ and there is no non-zero finite projection $q \leq p$.
- **properly infinite** if $p \neq 0$ and zp is not-finite for any non-zero central projection $z \in \mathcal{P}(M)$.

We say that M is finite, semi-finite, purely infinite, or properly infinite depending on whether or not 1 has the corresponding property in M.

Note that we have the trivial implications:

$$\text{minimal} \implies \text{abelian} \implies \text{finite} \implies \text{semi}-\text{finite} \implies \text{not purely infinite},$$

and also

$$\text{purely infinite} \implies \text{properly infnite}.$$

Note also that M is finite if and only if the only isometries in M are unitary. Thus, $\mathcal{B}(\mathcal{H})$ is finite if and only if $\mathcal{H}$ is finite dimensional. When $\mathcal{H}$ is infinite dimensional we can fix an orthonormal basis $\{\xi_\alpha\}_\alpha$ and we have $1 = \sum_\alpha [\mathbb{C}\xi_\alpha]$, hence in this case $\mathcal{B}(\mathcal{H})$ is semi-finite. The same argument shows that for the von Neumann algebra $\mathcal{B}(\mathcal{H})$, every projection is semi-finite.

Lemma 5.2.1. *Let $\{p_\alpha\}_\alpha$ be a family of centrally orthogonal projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$. If each projection p_α is abelian (resp. finite) then $p = \sum_\alpha p_\alpha$ is also abelian (resp. finite).*

Proof. If each p_α is abelian then since they are centrally orthogonal for $\alpha \neq \beta$ and $x, y \in M$ we have $(p_\alpha x p y p_\beta) = 0$. Hence

$$(pxp)(pyp) = \sum_\alpha p_\alpha x p_\alpha y p_\alpha = (pyp)(pxp),$$

thus p is abelian.

If each p_α is finite and $u \in M$ such that $uu^* \leq u^*u = p$. Then for all α we have $z(p_\alpha)u^*uz(p_\alpha) = p$ and $uz(p_\alpha)u^* = z(p_\alpha)uu^* \leq p_\alpha$. Thus $uz(p_\alpha)u^* = p_\alpha$ for each α and hence

$$uu^* = uz(p)u^* = \sum_\alpha uz(p_\alpha)u^* = p. \qquad \blacksquare$$

Proposition 5.2.2. *Let p, q be non-zero projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ such that $p \preceq q$. If q is finite (resp. purely infinite), then p is also finite (resp. purely infinite).*

Proof. For the case when q is finite let us suppose that $p \sim q$ and let $v \in M$ be such that $v^*v = p$ and $vv^* = q$. Suppose $u \in M$ such that $u^*u = p$, and $uu^* \leq p$, then we have $(vuv^*)^*(vuv^*) = q$ and $(vuv^*)(vuv^*)^* \leq q$. Therefore $(vuv^*)(vuv^*)^* = q$ and hence $uu^* = p$.

Next suppose that $p \leq q$. If $u^*u = p$ and $uu^* \leq p$ then setting $w = u + (q - p)$ we have $w^*w = q$ and $ww^* \leq q$. Therefore, $uu^* + (q - p) = ww^* = q$ and hence

$uu^* = p$. In general, if $p \preceq q$ then there exists $q_0 \leq q$ such that $p \sim q_0 \leq q$ and the result follows.

Since projections are purely infinite when they have no non-zero finite subprojections, the purely infinite case follows from the finite case. ■

Proposition 5.2.3. *A projection $p \in \mathcal{P}(M)$ in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is semi-finite if and only if p is the supremum of finite projections. In particular, a supremum of semi-finite projections is again semi-finite.*

Proof. If p is semi-finite then is is a sum (and hence also a supremum) of a family of pairwise orthogonal finite projections. Conversely, if $p = \vee_\alpha p_\alpha$ where each p_α is finite. Then let $\{q_\beta\}_\beta$ be a maximal family of pairwise orthogonal finite subprojections of p. If $q_0 = p - \sum_\beta q_\beta \neq 0$ then there exists some p_α such that p_α and q_0 are not orthogonal, and hence not centrally orthogonal. By Proposition 5.1.9 there then exists a non-zero subprojection $\tilde{q}_0 \leq q_0$ such that $q_0 \preceq p_\alpha$ and hence is finite is finite by Proposition 5.2.2, contradicting maximality of the family $\{q_\beta\}_\beta$. ■

Corollary 5.2.4. *Let p be projection in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$, if p is semi-finite (resp. purely infinite) then the central support $z(p)$ is also semi-finite (resp. purely infinite).*

Proof. The central support is the supremum over all equivalent projections and hence from the previous corollary this proves the case when p is semi-finite. It follows from Propositions 5.1.9 and 5.2.2 that a non-zero projection is purely infinite if and only if it is centrally orthogonal to every semi-finite projection which then proves the corollary in this case. ■

Corollary 5.2.5. *Let p, q be non-zero projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ such that $p \preceq q$. If q is semi-finite then p is also semi-finite.*

Proof. By Corollary 5.2.4 it is enough to consider the case when $q \in \mathcal{Z}(M)$ in which case we have $p \leq q$. Let p_0 be the maximal semi-finite subprojection of p (i.e., p_0 is the supremum of all finite subprojections of p). Since q is semi-finite, it is the supremum of all its finite subprojections. Thus, since $z(p - p_0) \leq q = z(q)$, if $p - p_0$ were not zero then there would exist a non-zero finite subprojection of q which would be equivalent to a subprojection of $p - p_0$, contradicting our definition of p_0. Therefore we have that p is the supremum if its finite subprojections and hence is semi-finite. ■

Lemma 5.2.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a properly infinite von Neuman algebra, then there exists a projection $p \in \mathcal{P}(M)$ such that $p \sim 1 - p \sim 1$.*

Proof. By hypothesis there exists $u \in M$ such that $uu^* < u^*u = 1$. Set $p_0 = 1 - uu^*$, then $p_n = u^n p_0 (u^n)^*$ is a pairwise orthogonal family of equivalent projections. Let $\{q_\iota\}_\iota$ be a maximal family of pairwise orthogonal equivalent projections in M which extends the family $\{p_n\}_n$, and consider $q_0 = 1 - \sum_\iota q_\iota$.

By the comparison theorem there exists a central projection z such that $q_0 z \preceq q_{\iota_0} z$, and $q_{\iota_0}(1-z) \preceq q_0(1-z)$. If $z = 0$ then $q_{\iota_0} \preceq q_0$ contradicting the maximality of $\{q_\iota\}_\iota$, thus $z \neq 0$ and we have

$$z = q_0 z + \sum_\iota q_\iota z \preceq q_{\iota_0} z + \sum_{\iota \neq \iota_0} q_\iota z = \sum_\iota q_\iota z \leq z.$$

Thus, $z \sim \sum_\iota q_\iota z$ by the Cantor-Bernstein-Schröeder theorem for projections. By decomposing $\{q_\iota\}_\iota$ into two infinite sets we construct two projections p and $z - p$ such that $p \sim z - p \sim z$.

Consider $\{r_j\}_j$ a maximal family of centrally orthogonal projections such that $r_j \sim z(r_j) - r_j \sim z(r_j)$, then the argument above shows that $\sum_j z(r_j) = 1$ and hence setting $p = \sum_j r_j$ finishes the proof. ■

Proposition 5.2.7. *Let p, q be finite projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$, then $p \vee q$ is also finite.*

Proof. By Kaplansky's formula we have $p \vee q - p \sim q - p \wedge q \leq q$, and thus we may replace q by $p \vee q - p$ and assume that p and q are orthogonal. We will also assume that $p + q = 1$ by considering the von Neumann algebra $(p+q)M(p+q)$.

Let z_0 be the supremum of all central finite projections. By Lemma 5.2.1 z_0 is a finite projection and thus either $z_0 = 1$ in which case the proof is finished, or else by considering $(1 - z_0)p$ and $(1 - z_0)q$ we may assume $z_0 = 0$, i.e., we may assume that M is properly infinite.

Then by Lemma 5.2.6 there exists a projection $r \in \mathcal{P}(M)$ such that $r \sim 1 - r \sim 1$. By the comparison theorem there then exists $z \in \mathcal{P}(\mathcal{Z}(M))$ such that

$$z(p \wedge r) \preceq z(q \wedge (1-r)), \quad \text{and} \quad (1-z)(q \wedge (1-r)) \preceq (1-z)(p \wedge r).$$

Then $zr \sim z(1-r) \sim z$ and

$$z(p \wedge r) = zp \wedge zr \preceq z(1-r) \wedge zq.$$

Using Kaplansky's formula and Lemma 5.1.5 we then have

$$zr = z(r - r \wedge p) + z(r \wedge p) \preceq z(r \vee p - p) + z(q \wedge (1-r)) = zq, \tag{5.1}$$

which implies that $z \sim zr = 0$, since zq is finite.

Thus, $q \wedge (1-r) \preceq p \wedge r$, and replacing the roles of p with q, and of r with $1 - r$ in (5.1) we then have

$$\begin{aligned} 1 - r &= ((1-r) - (1-r) \wedge q) + ((1-r) \wedge q) \\ &\preceq ((1-r) \vee q - q) + (p \wedge r) = p, \end{aligned}$$

which gives a contradiction since p is finite. ■

Proposition 5.2.8. *Let p and q be finite equivalent projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$, then $1 - p$ and $1 - q$ are also equivalent. Thus, there exists a unitary operator $u \in M$ such that $upu^* = q$.*

Proof. By Proposition 5.2.7 we have that $p \vee q$ is finite, hence by considering $(p \vee q)M(p \vee q)$ we may assume that M is finite. By the comparison theorem there exists a central projection $z \in M$, and projections p_1, and q_1 such that

$$(1-p)z \sim q_1 \leq (1-q)z, \quad \text{and} \quad (1-q)(1-z) \sim p_1 \leq (1-p)(1-z).$$

Then,

$$z = (1-p)z + pz \sim q_1 + qz \leq (1-q)z + qz = z,$$

and

$$(1-z) = (1-q)(1-z) + q(1-z) \sim p_1 + p(1-z) \leq (1-z).$$

Since, z and $(1-z)$ are finite this implies that $q_1 = (1-q)z$ and $p_1 = (1-p)(1-z)$, and so $1-q \sim 1-p$. ■

A projection $p \in \mathcal{P}(M)$ is **countably decomposable** if every family of non-zero pairwise orthogonal subprojections is countable. A von Neumann algebra is countably decomposable if the identity projection is. Note that separable von Neumann algebras are always countably decomposable. Also, note that if p is countably decomposable and $q \leq p$ then so is q.

Proposition 5.2.9. *Let $p, q \in \mathcal{P}(M)$ be properly infinite projections and suppose that M is countably decomposable. If $z(p) \leq z(q)$ then $p \preceq q$.*

Proof. By the comparison theorem we may assume that $q \preceq p$, and hence we may assume $q \leq p$. By considering pMp we may also assume $p = 1$.

By Lemma 5.2.6 there exists a subprojection $q_0 \leq q$ such that $q_0 \sim q - q_0 \sim q$. Take $u \in M$ such that $u^*u = q$ and $uu^* = q - q_0$. Setting $q_n = u^n q_0 (u^n)^*$ we obtain a family of pairwise orthogonal equivalent projections. Let $\{r_n\}_n$ be a maximal family of pairwise orthogonal projections such that $r_n \preceq q$. Since M is countably decomposable we have that $\{r_n\}_n$ is countable and by maximality we have that $1 - \sum_n r_n$ and q are centrally orthogonal, thus $\sum_n r_n = 1$ since $z(q) = z(p) = 1$. Hence,

$$1 = \sum_n r_n \preceq \sum_{n=0}^{\infty} q_n \leq q,$$

and so $q \sim 1$ by the Cantor-Bernstein-Schröeder theorem for projections. ■

Corollary 5.2.10. *If $M \subset \mathcal{B}(\mathcal{H})$ is a countably decomposable factor, than any two infinite projections in M are equivalent.*

Exercise 5.2.11. Suppose that $M \subset \mathcal{B}(\mathcal{H})$ is a factor which is either finite, or is countably decomposable and purely infinite. Show that M is algebraically simple. I.e., the only two-sided ideals are $\{0\}$, or M.

Exercise 5.2.12. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and consider a non-zero vector $\xi \in \mathcal{H}$. Let p be the supremum of all projections q such that $q\xi = 0$. Show that $1 - p$ is countably decomposable.

Exercise 5.2.13. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and set $I = \{x \in M \mid [x\mathcal{H}]$ is countably decomposable.$\}$. Show that I is a norm closed 2-sided ideal and conclude that all finite factors are countably decomposable.

Exercise 5.2.14. Suppose that $M \subset \mathcal{B}(\mathcal{H})$ is a semi-finite factor which is not finite. Show that the set of elements whose support projection is finite forms a two-sided ideal $\mathbb{K}_0(M)$, whose closure $\mathbb{K}(M)$ does not equal M.

5.3 Type decomposition

A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is of **type** I if every non-zero projection has a non-zero abelian subprojection. M is of **type** II if it is semi-finite and has no non-zero abelian projections, if M is also finite then M is of type II_1, if M is properly infinite then M is of type II_∞. M is of **type** III if it has no non-zero finite projections.

Theorem 5.3.1. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then there exists unique projections $P_I, P_{II_1}, P_{II_\infty}$, and P_{III} in $\mathcal{Z}(M)$ such that MP_I, MP_{II_1}, MP_{II_∞}, and MP_{III} are of type I, II_1, II_∞, and III respectively, and such that $P_I + P_{II_1} + P_{II_\infty} + P_{III} = 1$.*

Proof. Let P_I be the supremum of all abelian projections in M. Since the family of abelian projections is closed under conjugation by unitaries it follows that $uP_Iu^* = P_I$ for all unitaries in M and since every operator is a linear combination of 4 unitaries it then follows that P_I is central. If $q \leq P_I$, $q \neq 0$, then since P_I is central it follows that for some abelian projection $r \in M$ we have that $z(r)z(q) \neq 0$ and hence by Proposition 5.1.9 there exists a non-zero subprojection $r_0 \leq r$ such that $r_0 \preceq q$, thus q has an abelian subprojection. Therefore, MP_I is type I, and $1 - P_I$ has no abelian subprojections.

Let P_{II_1} be the supremum of all finite central projections $p \in M$ such that $p \leq 1 - P_I$. Then MP_{II_1} is finite and $M(1 - P_I - P_{II_1})$ has no non-zero finite central subprojections.

Let P_{II_∞} be the supremum of all finite projections $p \in M$ such that $p \leq 1 - P_I - P_{II_1}$. Then since the family of finite projections is closed under conjugation by unitaries we again see that P_{II_∞} is central. By Proposition 5.2.3 we have that P_{II_∞} is semi-finite and has no non-zero finite central subprojections, hence MP_{II_∞} is type II_∞. By definition of P_{II_1} and P_{II_∞} we then have that there are no non-zero finite subprojections of $P_{III} = 1 - P_I - P_{II_1} - P_{II_\infty}$ and hence MP_{III} is type III.

To see that this decomposition is unique suppose that $Q_I + Q_{II_1} + Q_{II_\infty} + Q_{III} = 1$ gives another such decomposition. Since MQ_{III} is type III we have that Q_{III} is purely infinite, and from Proposition 5.2.2 it follows that $Q_{III}P_I = Q_{III}P_{II_1} = Q_{III}P_{II_\infty} = 0$, hence $Q_{III} \leq P_{III}$ and the same reasoning shows that $P_{III} \leq Q_{III}$.

Similarly, since $Q_{II_1} + Q_{II_\infty}$ has no abelian subprojections, we have that $(Q_{II_1} + Q_{II_\infty})P_I = 0$, hence $Q_{II_1} + Q_{II_\infty} \leq P_{II_1} + P_{II_\infty}$ and by symmetry $P_{II_1} + P_{II_\infty} \leq Q_{II_1} + Q_{II_\infty}$. Therefore, $P_I = Q_I$ as well.

Since Q_{II_1} is finite and central, and P_{II_∞} is properly infinite we have that $Q_{II_1}P_{II_\infty} = 0$. Similarly we have $P_{II_1}Q_{II_\infty} = 0$ and hence $P_{II_1} = Q_{II_1}$ and $P_{II_\infty} = Q_{II_\infty}$. ∎

Corollary 5.3.2. *A factor is either of type I, type II_1, type II_∞, or type III.*

Lemma 5.3.3. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $p \in \mathcal{P}(M)$, and $q \in \mathcal{P}(M')$. Then $(pMpq)' \cap \mathcal{B}(pq\mathcal{H}) = qM'qp$. In particular, if M is a factor then so is $pMpq$.*

Proof. It's sufficient to check in the cases when either $p = 1$ or $q = 1$. When $p = 1$ then clearly $qM'q \subset (Mq)' \cap \mathcal{B}(q\mathcal{H})$, and if $x \in (Mq)' \cap \mathcal{B}(q\mathcal{H})$ and $y \in M$ then

$$xy = x(qy + (1-q)y) = xqy = yqx = (yq + y(1-q))x = yx,$$

thus $(Mq)' \cap \mathcal{B}(q\mathcal{H}) \subset qM'q$.

Taking commutants and using von Neumann's double commutant theorem also gives $Mq = (qM'q)' \cap \mathcal{B}(q\mathcal{H})$. Replacing M with M' and q with p then gives the other case. ∎

Proposition 5.3.4. *Let M be a von Neumann algebra, and suppose $p \in \mathcal{P}(M)$, $q \in \mathcal{P}(M')$ are such that $pq \neq 0$, then*

1. *If M is type I (resp. has no non-zero abeliean projections) then $pMpq$ is also type I (resp. has no non-zero abelian projections).*
2. *If M is type semi-finite (resp. has no non-zero finite projections) then $pMpq$ is also semi-finite (resp. has no non-zero finite projections).*

Proof. If $z \in Z(M)$ is the central support of q in M' then the mapping $Mz \ni x \mapsto xq \in Mq$ is a $*$-isomorphism, we may therefore replace p with $pz \in Z(M)$ and assume that $q = 1$. As a projection in pMp is abelian if and only if it is abelian as a projection in M we see that (1) follows. Similarly, a projection in pMp is finite if and only if it is finite as a projection in M and hence (2) also follows. ∎

Theorem 5.3.5. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a cyclic vector $\xi \in \mathcal{H}$. Then for each $\eta \in \mathcal{H}$ there exist $x, y \in M$ with $x \geq 0$, and $\zeta \in \overline{x\mathcal{H}}$, such that $x\zeta = \xi$ and $y\zeta = \eta$.*

Proof. Fix $\eta \in \mathcal{H}$, and assume $\|\xi\|, \|\eta\| \leq 1$. Since ξ is cyclic there exists a sequence $x_n \in M$ such that $\|\eta - \sum_{n=1}^k x_n\xi\| \leq 4^{-k}$ for each $k \in \mathbb{N}$. Then $h_k^2 = 1 + \sum_{n=1}^k 4^n x_n^* x_n$ defines an increasing sequence in M. By Proposition 1.4.6 h_k^{-1} is then a decreasing sequence of positive elements and so by Lemma 3.7.1 must converge in the strong operator topology to a limit x.

For $k \in \mathbb{N}$ we have

$$\|h_k\xi\|^2 = \langle h_k^2\xi, \xi\rangle = \|\xi\|^2 + \sum_{n=1}^k 4^n\|x_n\xi\|^2 \leq 1 + 2\sum_{n=1}^k 4^{-n} < 2.$$

Therefore $\{h_k\xi\}_k$ is a bounded sequence and so must have a weak cluster point $\zeta \in \mathcal{H}$.

To see that $x\zeta = \xi$, fix $\xi_0 \in \mathcal{H}$, and $\varepsilon > 0$, and consider $k \in \mathbb{N}$ such that $\|\zeta - h_k\xi\| < \frac{\varepsilon}{2}$ and $\|(x - h_k^{-1})\xi_0\| < \frac{\varepsilon}{4}$.

Then

$$\begin{aligned} |\langle x\zeta - \xi, \xi_0\rangle| &< |\langle h_k\xi, x\xi_0\rangle - \langle \xi, \xi_0\rangle| + \frac{\varepsilon}{2} \\ &= |\langle h_k\xi, x\xi_0 - h_k^{-1}\xi_0\rangle| + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{4}\|h_k\xi\| + \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

so that $x\zeta = \xi$.

For $m > k$ we have

$$0 \leq h_m^{-1}4^k x_k^* x_k h_m^{-1} \leq h_m^{-1}(1 + \sum_{n=1}^{m} 4^n x_n^* x_n)h_m^{-1} = 1,$$

and since $h_m^{-1}4^k x_k^* x_k h_m^{-1}$ is strong operator convergent to $4^k x x_k^* x_k x$ we then have

$$\|x_k x\|^2 = \|x^* x_k^* x_k x\| \leq 4^{-k}.$$

Therefore, $\sum_{n=1}^{\infty} x_k x$ converges in norm to an operator y, and we have

$$y\zeta = \sum_{n=1}^{\infty} x_k x\zeta = \sum_{n=1}^{\infty} x_k\xi = \eta.$$

Since $\ker(y) \subset \ker(x)$ we can replace ζ by $[x\mathcal{H}]\zeta$ if needed. ∎

Proposition 5.3.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\xi, \eta \in \mathcal{H}$ are two cyclic vectors, then $[M'\xi] \sim [M'\eta]$.*

Proof. Let $\xi, \eta \in \mathcal{H}$ be cyclic vectors. By the previous theorem there exists $x, y \in M$ with $x \geq 0$, and $\zeta \in \overline{x\mathcal{H}}$ such that $x\zeta = \xi$ and $y\zeta = \eta$. Set $p = [M'\zeta]$.

Since $\zeta \in \overline{x\mathcal{H}}$, and $p\zeta = \zeta$, we then have $\zeta \in \overline{px\mathcal{H}}$. Hence $p \leq [M'px\mathcal{H}] \leq [px\mathcal{H}] \leq [p\mathcal{H}] = p$ and so $p = [px\mathcal{H}] \sim [xp\mathcal{H}] = [xM'\zeta] = [M'\xi]$.

On the other hand, we have $[M'\eta] = [yM'\zeta] = [yp\mathcal{H}] \sim [py\mathcal{H}] \leq p \sim [M'\xi]$. The result then follows by symmetry. ∎

Proposition 5.3.7. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose $\xi, \eta \in \mathcal{H}$. Then $[M'\xi] \preceq [M'\eta]$ in M if and only if $[M\xi] \preceq [M\eta]$ in M'.*

Proof. We will first show that $[M'\xi] \sim [M'\eta]$ in M if and only if $[M\xi] \sim [M\eta]$ in M'. For this, suppose $v \in M$ such that $v^*v = [M'\xi]$ and $vv^* = [M'\eta]$. Then $[M'v\xi] = [vM'\xi] = [M'\eta]$, and $[Mv\xi] \leq [M\xi] = [Mv^*v\xi] \leq [Mv\xi]$. Thus, replacing ξ with $v\xi$ we may assume that $[M'\xi] = [M'\eta]$.

Then the central support z of $[M\eta]$ is $[M'M\eta] = [MM'\eta] = [M'M\xi]$, and so by considering Mz we may assume that all projections $[M\xi]$, and $[M\eta]$ in M', and $p_0 = [M'\xi] = [M'\eta]$ in M, have central support equal to 1.

In particular then $x \mapsto xp_0$ gives an isomorphism from M' onto $M'p_0$, and so $[M\xi]$, and $[M\eta]$ are equivalent in M' if and only if $[M\xi]p_0$, and $[M\eta]p_0$ are equivalent in $M'p_0$. But $p_0\xi$ and $p_0\eta$ are cyclic vectors for $M'p_0$ and so by the previous proposition we have the equivalence.

For the general case, if $[M'\xi] \sim q \leq [M'\eta]$, then $q = [M'q\eta]$ and hence from above we have $[M\xi] \sim [Mq\eta] \leq [M\eta]$. ■

Lemma 5.3.8. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra and suppose that M has a cyclic and separating vector, then M' is finite.*

Proof. Suppose $\xi \in \mathcal{H}$ is a cyclic and separating vector, and that M' is not finite. If $q \in \mathcal{P}(M')$ is a maximal central finite projection, then $(1-q)\xi$ is a cyclic and separating vector for $M(1-q)$, and replacing M with $M(1-q)$ we will assume that M' is properly infinite.

If M were abelian then M would be maximal abelian since it has a cyclic and separating vector (see the remark before Theorem 3.8.5), contradicting the fact that M' is properly infinite. Thus, there exists a projection $p \in \mathcal{P}(M)$ such that $p < z(p) = 1$.

Let $q = [Mp\xi] \in M'$. Since M' has a separating vector it is countably decomposable (see Exercise 5.2.12), and hence by Proposition 5.2.9 we have $q \sim z(q) = [M'Mp\xi] = [MpM'\xi] = [Mp\mathcal{H}] = z(p) = 1$.

Thus $[Mp\xi] \sim 1 = [M\xi]$ in M' and so by Proposition 5.3.7 we have $p = [pM'\xi] = [M'p\xi] \sim [M'\xi] = 1$ in M, showing that M is not finite. ■

Theorem 5.3.9. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then*

1. *M is type I if and only if M' is type I;*
2. *M is type II if and only if M' is type II;*
3. *M is type III if and only if M' is type III.*

Proof. We only need to show one implication for each type as the other then follows from von Neumann's double commutant theorem.

By way of contradiction, suppose that $M \subset \mathcal{B}(\mathcal{H})$ is type I and M' is not type I. By Theorem 5.3.1 we may restrict to a corner and assume that M' has no non-zero abelian projection. If $p \in M$ is a non-zero abelian projection $e = [pMp\xi] \in M'$, and $q = [pMp'\xi] \in pMp$ we have that $qe\xi \in qe\mathcal{H}$ is a cyclic and separating vector for $qMqe$, which is abelian. Hence, $qMqe$ is maximal abelian so that $(qMqe)' = eM'eq$ has non-zero abelian projections which then contradicts Proposition 5.3.4. Taking commutants shows that if M has no non-zero abelian projections then M' also has no non-zero abelian projections.

The same argument (but replacing abelian projections with finite projections and using Lemma 5.3.8) shows that if M is semi-finite then so is M'. If M is type II, then it is semi-finite and has no non-zero abelian projection and hence from above we see that M' is also type II.

Similarly, if M has no non-zero finite projections then the argument above shows that M' also has no non-zero finite projections M. Hence if M is type III then so is M'. ■

5.4 Type I von Neumann algebras

For a cardinal number n, a von Neumann algebra M is **type** I_n, if 1 is a sum of n equivalent non-zero abelian projections. We use the terminology **type** I_∞ to describe a properly infinite type I von Neumann algebra.

Lemma 5.4.1. *If n and m are cardinal numbers such that a von Neumann algebra M is both type I_n and type I_m, then $n = m$.*

Proof. If n or m is finite this is clear. For infinite cardinals suppose $1 = \sum_{i \in I} p_i = \sum_{j \in J} q_j$ where $\{p_i\}$, and $\{q_j\}$ are each infinite pairwise orthogonal collections of abelian projections with central support 1.

First note that for each $\xi \in \mathcal{H}$, there are only countably many $j \in J$ such that $q_j \xi \neq 0$, hence for each fixed $i \in I$, there are only countably many $j \in J$ such that $[p_i q_j \mathcal{H}]\xi \neq 0$. If we denote by $z_{i,j}$ the central support of $[p_i q_j \mathcal{H}]$ then as $z_{i,j}[p_i q_j \mathcal{H}] = [p_i q_j \mathcal{H}]$, and as p_i is abelian we then have $[p_i q_j \mathcal{H}] \leq p_i z_{i,j} = [p_i M p_i q_j \mathcal{H}] \leq [p_i q_j p_i M p_i \mathcal{H}] \leq [p_i q_j \mathcal{H}]$. Thus, $z_{i,j} p_i = [p_i q_j \mathcal{H}]$, and so for each $i \in I$, and $\xi \in p_i \mathcal{H}$, there are only countably many $j \in J$ such that $z_{i,j}\xi \neq 0$. As $\{p_i\}$ is a pairwise orthogonal family we then have that for each $\xi \in \mathcal{H}$, and $i \in I$, there are only countably many $j \in J$ such that $z_{i,j}\xi \neq 0$.

Fix $\xi_0 \in \mathcal{H}$ a non-zero vector, and for each $i \in I$ set $J_i = \{j \in J \mid z_{i,j}\xi_0 \neq 0\}$. For each $j \in J$, we have $\vee_{i \in I} z_{i,j} = z(p_j) = 1$, and hence there exists some $i \in I$ such that $z_{i,j}\xi_0 \neq 0$. Hence we have $J = \cup_{i \in I} J_i$, and as each J_i is countable we then have $|J| \leq |I \times \mathbb{N}| = |I|$. By symmetry we also have $|I| \leq |J|$, and so $|I| = |J|$ by the Cantor-Bernstein-Schröeder theorem. ∎

Proposition 5.4.2. *Every type I von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is a direct sum of type I_n von Neumann algebras. Moreover, this decomposition is unique.*

Proof. If M is any type I von Neumann algebra and we fix q a non-zero abelian projection, then we may consider a maximal family $\{q_i\}_{i \in I}$ of pairwise orthogonal abelian projections such that $q_i \sim q$ for each $i \in I$. We then have that $\sum_{i \in I} q_i \leq z(q)$, and if $\sum_{i \in I} q_i z < z$ for some non-zero central subprojection $z \leq z(q)$, then there is a non-zero abelian subprojection $q_0 \leq z - \sum_{i \in I} q_i z$. Thus, if $\sum_{i \in I} q_i z < z$ for all non-zero central subprojections $z \leq z(q)$ then a maximality argument would produce an abelian subprojection $\tilde{q_0} \leq z(q) - \sum_{i \in I} q_i$ whose central support is $z(q)$. Buth then $q_0 \sim q$ contradicting maximality of the set $\{q_i\}_{i \in I}$. Hence, there exists some non-zero central subprojection $z \leq z(q)$ such that $\sum_{i \in I} q_i z = z$, and hence Mz is type $I_{|I|}$. Thus, every type I von Neumann algebra contains a type I_n direct summand for some cardinal number n.

For each cardinal number n, let $\{z_i\}_i$ be a maximal family of orthogonal central projections, such that Mz_i is type I_n for each i. If we set $p_n = \sum_i z_i$ then have that Mp_n is type I_n, and $M(1 - p_n)$ has no type I_n summand. By Lemma 5.4.1 we have that $\{p_n\}$ is a pairwise orthogonal family of projections and from the argument above, for a large enough cardinal number N, we have $1 = \sum_{n \leq N} p_n$. Uniqueness of this decomposition follows directly from Lemma 5.4.1. ∎

Theorem 5.4.3. *A type I factor M is $*$-isomorphic to $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$.*

Proof. By Zorn's Lemma there exists a maximal family of pairwise orthogonal minimal projection $\mathcal{F} \subset M$. If we denote by $P = \vee_{p \in \mathcal{F}} p$ then $1 - P$ can have no minimal subprojections and hence since M is a type I factor it follows that $P = 1$.

If $p, q \in \mathcal{F}$, then by minimality we cannot have $p \prec q$ or $q \prec q$, thus by Corollary 5.1.11 we must have that $p \sim q$. Hence, if we fix $p_0 \in \mathcal{F}$, then for any $q \in \mathcal{F}$ we can chose a partial isometry $v_q \in M$ such that $v_q^* v_q = p_0$, and $v_q v_q^* = q$. We may assume in addition that $v_{p_0} = p$.

For each $p, q \in \mathcal{F}$ we define the partial isometry $v_{p,q} = v_p v_q^*$. It is then easy to verify that $v_{p,p} = p$, $v_{p,q}^* = v_{q,p}$, and $v_{p,q} v_{q,r} = v_{p,r}$, for all $p, q, r \in \mathcal{F}$.

We may then define a map $\theta : \mathcal{B}(\ell^2 \mathcal{F}) \to M$ by $\theta(T) = \sum_{p,q \in \mathcal{F}} \langle T\delta_q, \delta_p \rangle v_{p,q}$, where the sum is taken in the SOT. The map θ is clearly unital, linear, adjoint preserving, and injective. From Parseval's identity we see that it is multiplicative as well. Indeed, for $T, S \in \mathcal{B}(\ell^2 \mathcal{F})$ we have

$$\begin{aligned}
\theta(T)\theta(S) &= \Big(\sum_{p,r \in \mathcal{F}} \langle T\delta_r, \delta_p \rangle v_{p,r}\Big)\Big(\sum_{r',q \in \mathcal{F}} \langle S\delta_q, \delta_{r'} \rangle v_{r',q}\Big) \\
&= \sum_{p,q,r \in \mathcal{F}} \langle \delta_r, T^*\delta_p \rangle \langle S\delta_q, \delta_r \rangle v_{p,q} \\
&= \sum_{p,q \in \mathcal{F}} \langle S\delta_q, T^*\delta_p \rangle v_{p,q} = \theta(TS).
\end{aligned}$$

The fact that θ is onto follows from the fact that for $x \in M$ we have $pxq \in \mathbb{C} v_{p,q}$, and $x = \sum_{p,q \in \mathcal{F}} pxq$. ∎

A similar classification for type I von Neumann algebras can be made, but we will delay this discussion until after we introduce tensor products of von Neumann algebras.

Chapter 6

Traces

6.1 Unique preduals

Lemma 6.1.1. *Let A be a C^*-algebra, and suppose that X is a Banach space such that $X^* \cong A$. Then $(A)_1 \cap A_+$ is $\sigma(A, X)$-compact.*

Proof. Note that by Alaoglu's theorem we have that $(A)_1$ is $\sigma(A, X)$-compact. Suppose $\{x_\alpha\}$ is a net of positive elements in $(A)_1$ which converge to $a + ib$ where a and b are self-adjoint.

If we fix $t \in \mathbb{R}$, then since $x_\alpha + it \in (A)_{\sqrt{1+t^2}}$ converges in the $\sigma(A, X)$-topology to $a + i(b + t)$ it follows from the $\sigma(A, X)$-compactness of $(A)_{\sqrt{1+t^2}}$ that $\|b + t\|^2 \leq \|a + i(b + t)\|^2 \leq 1 + t^2$ for all $t \in \mathbb{R}$. In particular, if $\lambda \in \sigma(b)$, then we have $|\lambda + t|^2 \leq \|b + t\|^2 \leq 1 + t^2$, for all $t \in \mathbb{R}$, which implies $\lambda = 0$. Hence, $b = 0$.

We then have $\|a\| \leq 1$, and since $\|1 - x_\alpha\| \leq 1$ and $1 - x_\alpha$ converges to $1 - a$ it follows that $\|1 - a\| \leq 1$, hence $a \geq 0$. Thus, $(A)_1 \cap A_+$ is $\sigma(A, X)$-compact. ■

Lemma 6.1.2. *Let A be a C^*-algebra, and suppose that X is a Banach space such that $X^* \cong A$. Then for $x \in A_+$, $x = 0$ if and only if $\varphi(x) = 0$ for every positive linear functional $\varphi \in X$.*

Proof. From Lemma 6.1.1 we have that $(A)_1 \cap A_+$ is $\sigma(A, X)$ compact, hence if $a \in A$, $a < 0$, by the Hahn-Banach separation theorem there exists $\varphi \in X$ such that $\varphi(a) < 0$, and $\varphi(b) \geq 0$ for all $b \in (A)_1 \cap A_+$. Then $\varphi \in X$ is a positive linear functional such that $\varphi(a) \neq 0$. ■

Lemma 6.1.3. *Let A be a C^*-algebra, and suppose that X is a Banach space such that $X^* \cong A$. Then every bounded increasing net $\{a_\alpha\}_\alpha \subset A_+$ $\sigma(A, X)$-converges to a least upper bound. Moreover, if a_α converges to a then $x^* a_\alpha x$ converges to $x^* a x$ for each $x \in A$.*

Proof. Suppose that $\{a_\alpha\}$ is a bounded increasing net of positive operators. By Lemma 6.1.1 $(A)_1 \cap A_+$ is $\sigma(A, X)$-compact, and so there exists a $\sigma(A, X)$-cluster point $a \geq 0$. Moreover, since for fixed α_0 we have that $a - a_{\alpha_0}$ is a cluster point of $\{a_\alpha - a_{\alpha_0}\}$ it follows that $a_{\alpha_0} \leq a$. Similarly, if b is an upper bound for $\{a_\alpha\}$ then $b - a$ is a cluster point of $\{b - a_\alpha\}$ and hence it follows that $a \leq b$. Therefore, a_α converges in the $\sigma(M, X)$-topology to a unique least upper bound.

Note that by taking an approximate identity for A, this in particular shows that A is unital.

If $x \in A$ is invertible then x^*ax is clearly the least upper bound for $\{x^*a_\alpha x\}$. In general, if b is the least upper bound of $\{x^*a_\alpha x\}$, then taking $\lambda \in \mathbb{R} \setminus \sigma(x)$, for any $\varphi \in X$ a positive linear functional we have

$$\begin{aligned}\lambda^2\varphi(a_\alpha) - \lambda\varphi(x^*a_\alpha) - \lambda\varphi(a_\alpha x) + \varphi(x^*a_\alpha x) &= \varphi((x-\lambda)^*a_\alpha(x-\lambda)) \\ &\to \varphi((x-\lambda)^*a(x-\lambda)).\end{aligned}$$

On the other hand, for $\beta > \alpha$ we have

$$\begin{aligned}|\varphi(x^*(a_\beta - a_\alpha)| &= |\varphi(x^*(a_\beta - a_\alpha)^{1/2}(a_\beta - a_\alpha)^{1/2})| \\ &\leq \varphi(x^*(a_\beta - a_\alpha)x)^{1/2}\varphi(a_\beta - a_\alpha)^{1/2}.\end{aligned}$$

And similarly,

$$|\varphi((a_\beta - a_\alpha)x) \leq \varphi(a_\beta - a_\alpha)^{1/2}\varphi(x^*(a_\beta - a_\alpha)x)^{1/2}.$$

Hence,

$$\begin{aligned}&\lambda^2\varphi(a_\alpha) - \lambda\varphi(x^*a_\alpha) - \lambda\varphi(a_\alpha x) + \varphi(x^*a_\alpha x) \\ &\to \lambda^2(a) - \lambda\varphi(x^*a) - \lambda\varphi(ax) + \varphi(b),\end{aligned}$$

And so $x^*ax = b$ by Lemma 6.1.2. ∎

Theorem 6.1.4 (Sakai)**.** *Let M be a von Neumann algebra, then M_* is the unique predual of M in the sense that if X is a Banach space, and $\theta : X^* \to M$ is an isomorphism, then $\theta^* : M^* \to X^{**}$ restricted to M_* defines an isomorphism from M_* onto X.*

Proof. By the Hahn-Banach Theorem, it is enough to show that under the identification $M \cong X^*$, we have $X \subset M_*$.

If $\varphi \in X \subset X^{**} = M^*$, then from Lemma 6.1.3 it follows that for any bounded increasing net $\{x_\alpha\}$ we have $\lim_\alpha \varphi(x_\alpha) = \varphi(\lim_\alpha x_\alpha)$, therefore $\varphi \in M_*$ by Corollary **??**. ∎

Proposition 6.1.5 (Kadison, Pedersen)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a C^*-algebra such that $1 \in A$, and the SOT-limit of any monotone increasing net in A is contained in A. Then A is a von Neumann algebra.*

Proof. As the span of projections in A'' is norm dense it is enough to show that A contains every projection in A''.

If $a \in A$, then taking $f_n : [0,\infty) \to [0,\infty)$ defined by $f_n(t) = nt$ for $0 \leq t \leq \frac{1}{n}$, and $f_n(t) = 1$ for $t > \frac{1}{n}$, we have that $f_n(aa^*)$ is increasing to $[a\mathcal{H}]$, and thus $[a\mathcal{H}] \in A$ for all $a \in A$.

If $p, q \in A$ are projections then $[(p+q)\mathcal{H}] = p \vee q$, thus A is closed under taking finite supremums of projections, and as an arbitrary supremum is an increasing net of finite supremums it follows that A is closed under taking arbitrary supremums (or infimums) of projections.

If $p \in \mathcal{P}(A'')$ then $p = \vee_{\xi \in \mathcal{H}} [A' p \xi]$ and hence it is enough to show that projections of the form $[A'\xi]$ are contained in A for each $\xi \in \mathcal{H}$. To show this it is enough to show that for $\eta \perp \overline{A'\xi}$ there exists a positive operator $y \in A$ such that $y\xi = \xi$, and $y\eta = 0$. Indeed, we would then have $[y\mathcal{H}] \geq [A'\xi]$ and $[y\mathcal{H}]\eta = 0$. Taking the infimum of such projections would then show that $[A'\xi] \in A$.

So suppose $\xi, \eta \in \mathcal{H}$ such that $\eta \perp \overline{A'\xi}$. By Kaplansky's density theorem there exists a sequence $a_n \in A_{s.a.} \cap (A)_1$ such that $\|\xi - a_n \xi\| \leq 1/n$, and $\|a_n \eta\| \leq 1/(n 2^n)$. By considering a_n^2 we may assume that $a_n \geq 0$.

For $n \leq m$ we define

$$y_{n,m} = (1 + \sum_{n \leq k \leq m} k a_k)^{-1} \sum_{n \leq k \leq m} k a_k.$$

Then $y_{n,m} \in A_+ \cap (A)_1$ and $y_{n,m} \leq \sum_{n \leq k \leq m} k a_k$. Thus, for $i \leq n$ we have

$$\langle y_{n,m} \eta, \eta \rangle \leq \sum_{n \leq k \leq m} 2^{-k} < 2^{-n+1}.$$

As $\sum_{n \leq k \leq m} k a_k \geq m a_m$ we have $y_{n,m} \geq (1 + m a_m)^{-1} m a_m$ (since $t/(1+t) = 1 - 1/(1+t)$ is an operator monotone function), hence

$$1 - y_{n,m} \leq (1 + m a_m)^{-1} \leq (1+m)^{-1}(1 + m(1 - a_m)).$$

Thus, for $n \leq m$ we have

$$\langle \xi - y_{n,m} \xi, \xi \rangle \leq 2/(1+m).$$

For fixed n the sequence $y_{n,m}$ is increasing and hence converges to an element $y_n \in A_+ \cap (A)_1$. Since $y_{n+1,m} \leq y_{n,m}$ for all m it follows that $y_{n+1} \leq y_n$, and hence y_n is decreasing and so converges to an element $y \in A_+ \cap (A)_1$. For each $n \in \mathbb{N}$ we have

$$\langle y_n \eta, \eta \rangle \leq 2^{-n+1} \text{ and } \langle \xi - y_n \xi, \xi \rangle \leq 0,$$

therefore $y\eta = 0$, and $y\xi = \xi$, completing the proof. ■

Theorem 6.1.6 (Sakai)**.** *Let A be a C^*-algebra such that there is a Banach space X, and an isomorphism $X^* \cong A$. Then A is isomorphic to a von Neumann algebra.*

Proof. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be the direct sum of all GNS-representations corresponding to states in X. Then by Lemma 6.1.2 π is faithful and so all that remains is to show that $\pi(A)$ is a von Neumann algebra. Let $\{a_\alpha\}$ be a bounded increasing net in A_+, and let $x \in \mathcal{B}(\mathcal{H})$ be the SOT limit of $\pi(a_\alpha)$.

From Lemma 6.1.3 we know that there also exists $a \in A$ such that a is the $\sigma(A, X)$-limit of $\{a_\alpha\}$.

For each state $\varphi \in X$ we have $\langle (x - \pi(a))1_\varphi, 1_\varphi \rangle = \lim_{\alpha \to \infty} \varphi(a_\alpha) - \varphi(a) = 0$, and similarly if $b \in A$, then $\langle (x - \pi(a))\pi(b)1_\varphi, \pi(b)1_\varphi \rangle = 0$. The polarization identity then gives $\langle (x - \pi(a))\pi(b)1_\varphi, \pi(c)1_\varphi \rangle = 0$ for all $b, c \in A$. As the net $\{a_\alpha\}$ is uniformly bounded, and as φ was arbitrary it then follows that $x = \pi(a)$. Proposition 6.1.5 then shows that $\pi(A)$ is a von Neumann algebra. ■

6.2 Standard representations

Theorem 6.2.1. *Let M be a von Neumann algebra, suppose that for $i \in \{1, 2\}$, $\pi_i : M \to \mathcal{B}(\mathcal{H}_i)$ is a normal faithful representation, and set $\mathcal{K} = \overline{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}$. Then the representations $\pi_i \otimes \mathrm{id} : M \to \mathcal{B}(\mathcal{H}_i \overline{\otimes} \mathcal{K})$ are unitarily equivalent.*

Proof. Let $\{\xi_\alpha\}_{\alpha \in I}$ be a maximal family of unit vectors in $\mathcal{H}_i \overline{\otimes} \mathcal{K}$ such that if P_α denotes the projection onto the closure of the subspace $(\pi_1 \otimes \mathrm{id})(M)\xi_\alpha$, then $\{P_\alpha\}_\alpha$ is a pairwise orthogonal family. Note that by maximality we have $\sum_\alpha P_\alpha = 1$.

By Proposition **??**, for any normal state $\varphi \in M_*$ there exists a unit vector $\xi \in \mathcal{H}_2 \overline{\otimes} \overline{\mathcal{H}_2}$ such that $\varphi(x) = \langle (\pi_2 \otimes \mathrm{id})(x)\xi, \xi \rangle$ for all $x \in N$. It then follows that there exists a family $\{\eta_\alpha\}_{\alpha \in I}$ of unit vectors in $\mathcal{H}_2 \overline{\otimes} \mathcal{K}$ such that the projections Q_α onto the closure of the subspaces $(\pi_2 \otimes \mathrm{id})(N)\xi_\alpha$ are pairwise orthogonal, and such that for each $\alpha \in I$, and $x \in N$ we have

$$\langle (\pi_1 \otimes \mathrm{id})(x)\xi_\alpha, \xi_\alpha \rangle = \langle (\pi_2 \otimes \mathrm{id})(x)\eta_\alpha, \eta_\alpha \rangle.$$

By uniqueness of the GNS-construction there then exists a family of partial isometries $\{V_\alpha\}_{\alpha \in I} \subset \mathcal{B}(\mathcal{H}_1 \overline{\otimes} \mathcal{K}, \mathcal{H}_2 \overline{\otimes} \mathcal{K})$ such that $V_\alpha^* V_\alpha = P_\alpha$, and $V_\alpha V_\alpha^* = Q_\alpha$. Setting $V = \sum_\alpha V_\alpha$ we then have that V is an isometry such that $V(\pi_1 \otimes \mathrm{id})(x) = (\pi_2 \otimes \mathrm{id})(x)V$ for all $x \in N$.

By symmetry, there also exists an isometry $W : \mathcal{H}_2 \overline{\otimes} \mathcal{K} \to \mathcal{H}_1 \overline{\otimes} \mathcal{K}$ such that $W(\pi_2 \otimes \mathrm{id})(x) = (\pi_1 \otimes \mathrm{id})(x)W$ for all $x \in N$. We then have $VV^*, VW \in (\pi_2 \otimes \mathrm{id})(N)'$, and $(VW)(VW)^* \leq VV^* \leq 1 = (VW)^*(VW)$, and so $VV^* \sim 1$ in $(\pi_2 \otimes \mathrm{id})(N)'$. Hence there exists an isometry $V_0 \in (\pi_2 \otimes \mathrm{id})(N)'$ such that $V_0 V_0^* = VV^*$.

Setting $U = V_0^* V$ we then have that U is a unitary operator such that $U(\pi_1 \otimes \mathrm{id})(x) = (\pi_2 \otimes \mathrm{id})(x)U$ for all $x \in N$. ■

Proposition 6.2.2. *Let M be a von Neumann algebra. Then M is countably decomposable if and only if M has a normal faithful state.*

Proof. If M has a normal faithful state φ, and $\{p_\alpha\}_{\alpha \in I}$ is a family of pairwise orthogonal projections such that $\sum_\alpha p_\alpha = 1$, then as $\varphi(\sum_\alpha p_\alpha) = \sum_\alpha \varphi(p_\alpha)$ it follows that $\varphi(p_\alpha) > 0$ for only countably many $\alpha \in I$. Faithfulness then implies that $p_\alpha \neq 0$ for only countably many $\alpha \in I$, and hence M is countably decomposable.

Conversely, suppose M is countably decomposable and by Zorn's lemma let $\{p_n\}$ be a maximal family of pairwise orthogonal projections, such that there exists a faithful normal state φ_n on $p_n M p_n$. Since M is countably decomposable we have that $\{p_n\}$ is countable and hence we will assume that the projections by indexed the natural numbers (the case when it is finite follows similarly). We must have $\sum_n p_n = 1$ since otherwise taking any normal state φ on $(1 - \sum_n p_n)M(1 - \sum_n p_n)$, we would have that φ is faithful on $s(\varphi)Ms(\varphi)$, contradicting the maximality of $\{p_n\}$.

If we then define $\psi(x) = \sum_{n\in\mathbb{N}} 2^{-n}\varphi_n(p_n x p_n)$, then it follows easily from Proposition **??** that ψ defines a normal state. Moreover, if $x \in M$, such that $\psi(x^*x) = 0$, then $\varphi_n(p_n x^* x p_n) = 0$ for each $n \in \mathbb{N}$ and hence $p_n x^* x p_n = 0$, and so $xp_n = 0$, for each $n \in \mathbb{N}$. Thus, ψ is faithful. ∎

If M is a countably decomposable von Neumann algebra then a normal faithful representation $\pi : M \to \mathcal{B}(\mathcal{H})$ is **standard**[1] if there exists a cyclic and separating vector.

Example 6.2.3. Let M be a countably decomposable von Neumann algebra and suppose that $\varphi \in M_*$ is a normal faithful state, then the GNS representation $M \subset \mathcal{B}(L^2(M, \varphi))$ is a standard representation.

Theorem 6.2.4. *Let M be a countably decomposable von Neumann algebra. Then all standard representations are unitarily equivalent.*

Proof. Suppose for $i \in \{1, 2\}$ we have a standard representation $\pi_i : M \to \mathcal{B}(\mathcal{H}_i)$. By Theorem 6.2.1 we may assume that there is a normal representation $\pi : M \to \mathcal{B}(\mathcal{K})$ and projections $p_i \in \pi(M)'$, $i = 1, 2$, such that $\pi_i = p_i\pi$. If $\xi_i \in p_i\mathcal{K}$ are cyclic and separating vectors for $p_i\pi(M)$, then in particular we have that ξ_i are separating for $\pi(M)$, and hence $[\pi(M)'\xi_1] = 1 = [\pi(M)'\xi_2]$, thus by Proposition 5.3.7 we have $p_1 = [\pi(M)\xi_1] \sim [\pi(M)\xi_2] = p_2$ in $\pi(M)'$, and hence π_1 and π_2 are unitarily equivalent. ∎

It will be convenient to give the notation $M \subset \mathcal{B}(L^2M)$ to the standard representation. This is only defined up to unitary conjugacy but for each normal faithful state φ we obtain a concrete realization as $M \subset \mathcal{B}(L^2(M, \varphi))$.

Corollary 6.2.5. *Let $M \subset \mathcal{B}(L^2M)$, and $N \subset \mathcal{B}(L^2N)$ be two countably decomposable von Neumann algebras. If $\theta : M \to N$ is an isomorphism, then there exists a unitary $U : L^2M \to L^2N$, such that $\theta(x) = UxU^*$ for all $x \in M$.*

6.3 The universal enveloping von Neumann algebra

If A is a C^*-algebra then we can consider the direct sum of all GNS-representations $\pi = \bigoplus_{\varphi\in S(A)} \pi_\varphi$, we call this representation the **universal $*$-representation**,

[1] Standard representations can be defined in general, but for simplicity we will only consider the case when M is countably decomposable.

and the von Neumann algebra $\pi(A)''$ generated by this representation is the **universal enveloping von Neumann algebra** of A. We will denote the universal enveloping von Neumann algebra by $\tilde{A}$.

Theorem 6.3.1. *Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation of a C^*-algebra A. Then there exists a unique linear map $\tilde{\pi} : A^{**} \to \pi(A)''$ with $\tilde{\pi} \circ i = \pi$ where i is the canonical embedding of A into A^{**}, such that $\tilde{\pi}$ takes the unit ball of A^{**} onto the unit ball of $\pi(A)''$ and is continuous with respect to the weak* and σ-weak topologies. Moreover, in the case when π is the universal $*$-representation, $\tilde{\pi}$ will be isometric, and a homeomorphism with respect to the weak* and σ-weak topologies.*

Proof. Set $M = \pi(A)''$. Then π induces a linear map from M^* to A^* and we will denote by π_* the restriction of this map to $M_* \subset M^*$. Taking the dual again we obtain the map $\tilde{\pi}$ from A^{**} into $(M_*)^* \cong M$.

This map is continuous by construction, and clearly satisfies $\tilde{\pi} \circ i = \pi$. Since homomorphisms of C^*-algebras are contractions, it follows that $\tilde{\pi}$ applied to the unit ball of A^{**} is compact and contains the image of unit ball of A under the map π as a dense subset. By Kaplansky's density theorem it then follows that $\tilde{\pi}$ applied to the unit ball of A^{**} is equal to the unit ball of M.

If π is the universal $*$-representation then $\pi_*(M_*)$ contains all positive linear functionals, and hence by Corollary 2.3.14 is equal to A^*. Thus, it follows that $\tilde{\pi}$ is injective and hence gives an isometry, and since A^{**} and M are locally compact with respect to the weak* and σ-weak topologies, it follows that $\tilde{\pi}$ is a homeomorphism. ∎

6.4 Traces on finite von Neumann algebras

Lemma 6.4.1. *Let M be a finite von Neumann algebra, and $p \in \mathcal{P}(M)$ a non-zero projection. If $\{p_i\}_{i\in I}$ is a family of pairwise orthogonal projection in M such that $p_i \sim p$ for each $i \in I$, then I is finite.*

Proof. If I were infinite then there would exist a proper subset $J \subset I$ with the same cardinality and we would then have that $\sum_{i\in I} p_i \sim \sum_{j\in J} p_j < \sum_{i\in I} p_i$, showing that $\sum_{i\in I} p_i$ is not finite, contradicting Proposition 5.2.2. ∎

Lemma 6.4.2. *Let M be a type II_1 von Neumann algebra. Then there exists a projection $p_{1/2} \in \mathcal{P}(M)$ such that $p_{1/2} \sim 1 - p_{1/2}$.*

Proof. Let $\{p_i, q_i\}$ be a maximal family of pairwise orthogonal projections, such that $p_i \sim q_i$ for each i. If $p_{1/2} = \sum_i p_i$, and $q = \sum_i q_i$ then $p_{1/2} \sim q$, and if $p_{1/2} + q \neq 1$, then taking $p_0, q_0 \leq 1 - (p_{1/2} + q)$ which are orthogonal, but not centrally orthogonal, (this would be possible since $(1-(p_{1/2}+q))M(1-(p_{1/2}+q))$ is not abelian), there would then exist equivalent subprojections of p_0, and q_0 contradicting maximality. Thus we have $q = 1 - p_{1/2}$. ∎

If M is a type II_1 von Neumann algebra, and if we set $p_1 = 1$, $p_0 = 0$, and $p_{1/2} \in \mathcal{P}(M)$ is as in the previous lemma, so that $p_{1/2} = v^*v$, and $p_1 - p_{1/2} = vv^*$

for some partial isometry $v \in M$, then $p_{1/2}Mp_{1/2}$ is also type II_1 and so we may iterate the previous lemma to produce $p_{1/4} \leq p_{1/2}$ such that $p_{1/4} \sim p_{1/2} - p_{1/4}$. If we set $p_{3/4} = p_{1/2} + vp_{1/4}v^*$, then we have $p_{1/4} \sim p_{(k+1)/4} - p_{k/4}$ for all $0 \leq k \leq 3$.

Proceeding by induction we may construct for each dyadic rational $r \in [0, 1]$, a projection $p_r \in \mathcal{P}(M)$ such that $p_r \leq p_s$ if $r \leq s$, and if $0 \leq r \leq s \leq 1$, and $0 \leq r' \leq s \leq 1$, such that $s - r = s' - r'$, then we have $p_s - p_r \sim p_{s'} - p_{r'}$.

Lemma 6.4.3. *Let M by a type II_1 von Neumann algebra, and let $\{p_r\}_r$ be as above. If $p \in \mathcal{P}(M)$, $p \neq 0$, then there exists a central projection $z \in \mathcal{P}(\mathcal{Z}(M))$ such that $pz \neq 0$, and $p_r z \preceq pz$ for some positive dyadic rational r.*

Proof. By considering $Mz(p)$ we may assume that $z(p) = 1$. If the above does not hold, then by the comparison theorem we would have $p \preceq p_r$ for every positive dyadic rational r. Thus, p would be equivalent to a subprojection of $p_{2^{-k}} - p_{2^{-(k+1)}}$ for every $k \in \mathbb{N}$, which would contradict Lemma 6.4.1. ∎

If M is a von Neumann algebra, then a projection $p \in \mathcal{P}(M)$ is **monic** if there exists a finite collection of pairwise orthogonal projections $\{p_1, p_2, \ldots, p_n\}$ such that $p_i \sim p$ for each $1 \leq i \leq n$, and $\sum_{i=1}^n p_i \in \mathcal{Z}(M)$. Note that in the type II_1 case, any of the projections $p_{1/2^k}$ as defined above, are monic.

Proposition 6.4.4. *If M is a finite von Neumann algebra, then every projection is the sum of pairwise orthogonal monic projections.*

Proof. By a maximality argument it is sufficient to show that every non-zero projection has a non-zero monic subprojection. Also, by restricting with central projections it suffices to consider separately the cases when M is type II, or type I_n, with $n < \infty$.

The type II case follows from Lemma 6.4.3, and the type I_n case follows by considering any non-zero abelian projection. ∎

If M is a von Neumann algebra, then a **center-valued state** is a linear map $\varphi : M \to \mathcal{Z}(M)$ such that $\varphi(x^*x) \geq 0$ for all $x \in M$, $\varphi_{|\mathcal{Z}(M)} = \text{id}$, and $\varphi(zx) = z\varphi(x)$ for all $x \in M$, $z \in \mathcal{Z}(M)$. We say that φ is faithful if $\varphi(x^*x) \neq 0$ whenever $x \neq 0$.

Lemma 6.4.5. *Let M be a von Neumann algebra, and $\varphi : M \to \mathcal{Z}(M)$ a center-valued state, then φ is bounded and $\|\varphi\| = 1$.*

Proof. This is exactly the same as the proof of Lemma 2.3.2. First note that φ is Hermitian since if y is self-adjoint we have $\varphi(y) = \varphi(y_+) - \varphi(y_-)$ is also self-adjoint, and in general, if $y = y_1 + iy_2$ where $y_i^* = y_i$ then we have $\varphi(y^*) = \varphi(y_1) - i\varphi(y_2) = \varphi(y)^*$.

Next, note that for all $y \in M$ we have $\varphi(\|y + y^*\| \pm (y + y^*)) \geq 0$, and so $|\varphi(y + y^*)| \leq \|y + y^*\|$, hence $\|\varphi(y)\| = \|\varphi(\frac{y+y^*}{2})\| \leq \|\frac{y+y^*}{2}\| \leq \|y\|$, showing $\|\varphi\| \leq 1 \leq \|\varphi\|$. ∎

Lemma 6.4.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then M has a normal center-valued state.*

Proof. The von Neumann algebra $\mathcal{Z}(M)'$ is type I, and hence has an abelian projection q with central support equal to 1. We then have $qMq \subset q\mathcal{Z}(M)'q = \mathcal{Z}(M)q$, and $\theta(z) = zq$ defines a normal isomorphism from $\mathcal{Z}(M)$ onto $\mathcal{Z}(M)q$. If we set $\varphi(x) = \theta^{-1}(qxq)$, for $x \in M$, then φ is a normal center-valued state. ∎

Lemma 6.4.7. *Let M be a von Neumann algebra, and $\tau : M \to \mathcal{Z}(M)$ a center-valued state. The following are equivalent:*

(i) $\tau(xy) = \tau(yx)$, *for all* $x, y \in M$.

(ii) $\tau(xx^*) = \tau(x^*x)$, *for all* $x \in M$.

(iii) $\tau(p) = \tau(q)$, *for all equivalent projections* $p, q \in \mathcal{P}(M)$.

Proof. The implication (i) $\implies$ (ii) is obvious, as is (ii) $\implies$ (iii). Suppose (iii) holds, then for all $p \in \mathcal{P}(M)$, and $u \in \mathcal{U}(M)$ we have $\tau(upu^*) = \tau(p)$. Since τ is bounded it then follows from functional calculus that $\tau(uxu^*) = \tau(x)$ for all $x = x^* \in M$, and $u \in \mathcal{U}(M)$. Considering the real and imaginary parts this then holds for all $x \in M$, and replacing x with xu it then follows that $\tau(ux) = \tau(xu)$ for all $x \in M$, $u \in \mathcal{U}(M)$. As every operator is a span of four unitaries this then shows (i). ∎

We say that τ is a **center-valued trace** if it satisfies the equivalent conditions of the previous lemma.

Lemma 6.4.8. *Let M be a finite von Neumann algebra. If $\varphi : M \to \mathcal{Z}(M)$ is a normal center-valued state, then for each $\varepsilon > 0$ there exists $p \in \mathcal{P}(M)$, such that $\varphi(p) \neq 0$, and for all $x \in pMp$ we have*

$$\varphi(xx^*) \leq (1 + \varepsilon)\varphi(x^*x).$$

Proof. Let $q_0 = 1 - \sum_i q_i$ where $\{q_i\}$ is a maximal family of pairwise orthogonal projections with $\varphi(q_i) = 0$. By normality we have $\varphi(q_0) = 1$, and φ is faithful when restricted to $q_0 M q_0$.

We let $\{e_i, f_i\}$ be a maximal family of projections such that $\{e_i\}$, and $\{f_i\}$ are each pairwise orthogonal, $e_i \sim f_i$ for each i, and $\varphi(e_i) > \varphi(f_i)$ for each i. If we set $e = q_0 - \sum_i e_i$, and $f = q_0 - \sum_i f_i$ then unless φ was already a trace we have $\varphi(f) > \varphi(e) \geq 0$, Hence, $f \neq 0$, and by Proposition 5.2.8 we have $e \sim f$, hence $e \neq 0$.

If we let μ be the smallest number such that $\varphi(\tilde{e}) \leq \mu\varphi(\tilde{f})$ whenever $\tilde{e} \leq e$, $\tilde{f} \leq f$, and $\tilde{e} \sim \tilde{f}$ then $\mu \neq 0$ since $\varphi(e) \neq 0$, and there exists $\tilde{e} \leq e$, $\tilde{f} \leq f$, such that $\tilde{e} \sim \tilde{f}$ and $(1+\varepsilon)\varphi(\tilde{e}) \not\leq \mu\varphi(\tilde{f})$, and thus cutting down by a suitable central projection we may assume $(1 + \varepsilon)\varphi(\tilde{e}) > \mu\varphi(\tilde{f})$.

If we now take $\{\hat{e}_i, \hat{f}_i\}$ a maximal family such that $\{\hat{e}_i\}$, and $\{\hat{f}_i\}$ are each pairwise orthogonal, $\hat{e}_i \leq \tilde{e}$, $\hat{f}_i \leq \tilde{f}_i$, $\hat{e}_i \sim \hat{f}_i$, and $(1 + \varepsilon)\varphi(\hat{e}_i) \leq \mu\varphi(\hat{f}_i)$, then $p = \tilde{e} - \sum_i \hat{e}_i$ is non-zero, and equivalent to $q = \tilde{f} - \sum_i \hat{f}_i$. Moreover, if

$p_1, p_2 \leq p$, such that $p_1 \sim p_2$, then there exists $r \leq q$ such that $r \sim p_1$, and hence

$$\varphi(p_1) \leq \mu\varphi(r) \leq (1+\varepsilon)\varphi(p_2).$$

Since φ is bounded and every positive operator can be approximated uniformly by the span of its spectral projections it then follows that

$$\varphi(ux^*xu) \leq (1+\varepsilon)\varphi(x^*x),$$

for each $x \in pMp$, and $u \in \mathcal{U}(pMp)$. If x^* has polar decomposition $x^* = v|x^*|$, then since pMp is finite we can extend v to a unitary $u^* \in \mathcal{U}(pMp)$ and from above this shows $\varphi(xx^*) \leq (1+\varepsilon)\varphi(x^*x)$. ■

Lemma 6.4.9. *Let M be a finite von Neumann algebra, and $\varepsilon > 0$. Then there is a normal center-valued state φ such that for all $x \in M$ we have*

$$\varphi(xx^*) \leq (1+\varepsilon)\varphi(x^*x).$$

Proof. We need only show existence of such a state on Mz for some non-zero central projection z as a maximality argument will then finish the proof.

By Lemma 6.4.6 there exists a normal center-valued state ψ, and so by Lemma 6.4.8 there exists some non-zero projection $p \in \mathcal{P}(M)$ such that $\psi(xx^*) \leq (1+\varepsilon)\psi(x^*x)$ for all $x \in pMp$, and by Proposition 6.4.4 we may assume that p is monic.

Let $\{p_1, p_2, \ldots, p_n\}$ be a finite family of pairwise orthogonal projections such that $p_i \sim p$, and $z_0 = \sum_i p_i \in \mathcal{Z}(M)$. Take $v_i \in M$ such that $v_i^* v_i = p_i$ and $v_i v_i^* = p$. For $x \in Mz_0$ we then set $\varphi_0(x) = \sum_{i=1}^n \psi(v_i x v_i^*)$. If $x \in Mz_0$ we have

$$\begin{aligned} 0 \leq \varphi_0(xx^*) = \varphi_0(xz_0x^*) &= \sum_{j=1}^n \varphi_0(xp_jx^*) \\ &= \sum_{i,j=1}^n \psi(v_i x v_j^* v_j x^* v_i^*) \leq (1+\varepsilon) \sum_{i,j=1}^n \psi(v_j x^* v_i^* v_i x v_j^*) \\ &= (1+\varepsilon) \sum_{i=1}^n \varphi_0(x^* p_i x) = (1+\varepsilon)\varphi_0(x^*x). \end{aligned}$$

For $\tilde{z} \in \mathcal{Z}(M)z_0$, and $x \in Mz_0$ we have $\varphi_0(\tilde{z}x) = \sum_{i=1}^n \psi(v_i \tilde{z} x v_i^*) = \tilde{z}\varphi_0(x)$.

In general, it may not be the case that $\varphi_0(z_0) = z_0$. However, we do have $\varphi_0(z_0) > 0$, and hence, taking a spectral projection z of the form $z = 1_{[\varepsilon,\infty)}(\varphi_0(z_0))$, then we have that $0 \neq z = y\varphi_0(z_0)$ for some $y \geq 0$, $y \in \mathcal{Z}(M)$. If we set $\varphi(x) = y\varphi_0(x)$, then we still have $0 \leq \varphi(xx^*) \leq (1+\varepsilon)\varphi(xx^*)$, for $x \in Mz$, and φ is then a center-valued state on Mz. ■

Theorem 6.4.10. *A von Neumann algebra M is finite if and only if there exists a normal center-valued trace. Moreover, any such normal trace is faithful and unique.*

Proof. If M is finite, then from Lemma 6.4.9, if $\{a_n\}$ is a strictly decreasing sequence of real number that converge to 1, there exists a sequence of normal center-valued states τ_n such that $\tau_n(xx^*) \leq a_n \tau_n(x^*x)$ for each $n \in \mathbb{N}$, and $x \in M$. We claim that if $1 \leq m < n$, then the function $a_m^2 \tau_m - \tau_n$ is a positive linear map. From this Lemma 6.4.5 would then imply that $\|a_m^2 \tau_m - \tau_n\| \leq a_m^2 - 1$.

To see that $a_m^2 \tau_m - \tau_n$ is positive it is enough to consider projections, and since this map is normal, by Proposition 6.4.4 it is then enough to consider monic projections. So let $p \sim p_1 \sim \cdots \sim p_k$ be non-zero projections such that $z = \sum_{i=1}^k p_i$ is a projection in $\mathcal{Z}(M)$. We then have $\tau_n(p) \leq a_n \tau_n(p_i)$, and $\tau_m(p) \leq a_m \tau_m(p_i)$ for each $1 \leq i \leq k$. Hence,

$$\begin{aligned} k\tau_n(p) &\leq a_n \sum_{i=1}^k \tau_n(p_i) = a_n \tau_n(z) = a_n z \\ &= a_n \tau_m(z) = a_n \sum_{i=1}^k \tau_m(p_i) \\ &\leq k a_n a_m \tau_m(p) \leq k a_m^2 \tau_m(p). \end{aligned}$$

Thus, we have shown that $\|a_m^2 \tau_m - \tau_n\| \leq a_m^2 - 1$, and hence it follows that there is a bounded linear map τ, such that $\|\tau - \tau_m\| \to 0$.

We then have $0 \leq \tau(xx^*) \leq \tau(x^*x)$ and so we must have equality for all $x \in M$. Considering the polar decomposition of x it then follows that $\tau(u^*yu) = \tau(y)$ for all $u \in \mathcal{U}(M)$, and $y \geq 0$, invertible. Taking linear combinations it then follows that $\tau(u^*yu) = \tau(y)$ for all $u \in \mathcal{U}(M)$, and $y \in M$, or equivalently that $\tau(yu) = \tau(uy)$ for all $u \in \mathcal{U}(M)$, $y \in M$. Since every element is a linear combination of unitaries we then have $\tau(xy) = \tau(yx)$ for all $x, y \in M$.

Clearly, $\tau_{|\mathcal{Z}(M)} = \text{id}$ and $\tau(zx) = z\tau(x)$ for all $z \in \mathcal{Z}(M)$, and $x \in M$. Thus, the only thing remaining to check is that τ is normal. If $\varphi \in M_*$, then $\|\varphi \circ \tau - \varphi \circ \tau_m\| \leq \|\varphi\| \|\tau - \tau_m\|$, and hence $\varphi \circ \tau$ since M_* is closed. Thus, τ is normal.

Now if we have any normal trace φ on M then for any monic projection, $p = \sum_{i=1}^n p_i = z \in \mathcal{Z}(M)$ where $p_i \sim p$, we have $z = \tau(\sum_{i=1}^n p_i) = n\tau(p)$, hence τ is non-zero, and uniquely determined on monic projections. Proposition 6.4.4 then shows that $\tau(p) > 0$ for any non-zero projection, and by normality it follows that τ is uniquely determined on all projections. By the spectral theorem we see that τ is uniquely determined on all self-adjoint elements, hence τ is unique. Also from the spectral theorem we see that $\tau(x) > 0$ for any non-zero positive operator $x \in M$, so that τ is faithful.

Conversely, if M is a general von Neumann algebra which has a faithful trace τ, and if $v \in M$ is an isometry, then we have $1 = \tau(v^*v) = \tau(vv^*)$, so that $\tau(1 - vv^*) = 0$, which shows that $vv^* = 1$ and hence M is finite. ∎

Proposition 6.4.11. *Let M be a finite von Neumann algebra with center-valued trace τ. If $p, q \in \mathcal{P}(M)$, then $p \preceq q$ if and only if $\tau(p) \leq \tau(q)$*

Proof. If $p = v^*v$, and $vv^* \leq q$ then $\tau(p) = \tau(vv^*) \leq \tau(q)$. Conversely, if $\tau(p) \leq \tau(q)$, then by the comparison theorem there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ such that $pz \preceq qz$ and $(1-z)q \preceq (1-z)p$. If $v^*v = (1-z)q$ and $vv^* \leq (1-z)p$ then we have $\tau(vv^*) \leq \tau((1-z)p) = (1-z)\tau(p) \leq \tau((1-z)q) = \tau(vv^*)$, and since τ is faithful we have $vv^* = (1-z)p$, hence $(1-z)q \sim (1-z)p$ and so $p \preceq q$. ∎

Proposition 6.4.12. *Let M be a finite von Neumann algebra with normal faithful trace τ, then conjugation $x \mapsto x^*$ extends to an anti-linear isometry $J : L^2(M,\tau) \to L^2(M,\tau)$, and we have $M' \cap \mathcal{B}(L^2(M,\tau)) = JMJ$.*

Proof. If $x, y \in M$ then $\langle J\hat{x}, J\hat{y}\rangle = \tau(yx^*) = \langle \hat{y}, \hat{x}\rangle$. Thus, for all $\xi, \eta \in L^2(M,\tau)$ we have $\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle$.

If $a \in M'$ and $z \in M$ we have

$$\langle JaJ\hat{1}, \hat{z}\rangle = \langle J\hat{z}, aJ\hat{1}\rangle = \langle a^* z^* \hat{1}, \hat{1}\rangle = \langle a^* \hat{1}, \hat{z}\rangle$$

Since M is dense in $L^2(M,\tau)$ we then have $JaJ\hat{1} = a^*\hat{1}$.

If we also have $b \in M'$ then

$$\langle bJaJ\hat{1}, \hat{z}\rangle = \langle a^* \widehat{z^*}, b^*\hat{1}\rangle = \langle \widehat{z^*}, aJbJ\hat{1}\rangle = \langle JaJb\hat{1}, \hat{z}\rangle,$$

Since $z \in M$ is arbitrary we then have $bJaJ\hat{1} = JaJb\hat{1}$. If $a, b, c \in M'$ then applying this last equation with either ac or c in place of a gives

$$(bJaJ)Jc\hat{1} = JacJb\hat{1} = (JaJb)Jc\hat{1}.$$

Since $JM'\hat{1}$ is dense in $L^2(M,\tau)$ it then follows that $b(JaJ) = (JaJ)b$. Thus, $JaJ \in M'' = M$ and so $JM'J \subset M \subset JM'J$.

Therefore $JM'J = M$ and taking commutants give $JMJ = M'$. ∎

6.4.1 A second proof for existence of the trace

We present here another proof for the existence of the trace due to Yeadon, which is based on using the Ryll-Nardzewski fixed point theorem.

Lemma 6.4.13. *Let M be a von Neumann algebra, suppose $q \in \mathcal{P}(M)$ and $\{p_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(M)$ pairwise orthogonal projections such that $\sum_{k=0}^n p_k \preceq q$ for each $n \in \mathbb{N}$, then $\sum_{n\in\mathbb{N}} p_n \preceq q$.*

Proof. First note that if q is properly infinite then from Lemma 5.2.6 there exists a sequence of pairwise orthogonal projections $\{q_n\}_{n\in\mathbb{N}}$ such that $q_n \leq q$ and $q_n \sim q$ for each $n \in \mathbb{N}$. It then follows that $\sum_{n\in\mathbb{N}} p_n \sim \sum_{n\in\mathbb{N}} q_n \leq q$. By restricting by a central projection it then suffices to consider the case when q is finite.

We suppose therefore that q is finite and we will construct a sequence of pairwise orthogonal projections $\{q_n\}_{n\in\mathbb{N}}$ such that $p_n \sim q_n \leq q$ for each $n \in \mathbb{N}$. We may assume $p_0 = 0$, and set $q_0 = 0$. Suppose now that $q_0, \ldots, q_{n-1}$ have been constructed. Since $\sum_{k=0}^n p_k \preceq q$ there exists $w \in M$ such that $\sum_{k=0}^n p_k = w^*w$ and $ww^* \leq q$. Since q is finite and

$$\sum_{k=0}^{n-1} q_k \sim \sum_{k=0}^{n-1} p_k \sim w \sum_{k=0}^{n-1} p_k w^* \leq q$$

it follows from Proposition 5.2.8 that

$$p_n \preceq q - w\sum_{k=0}^{n-1} p_k w^* \sim q - \sum_{k=0}^{n-1} q_k.$$

Thus, there exists a projection $q_n \leq q - \sum_{k=0}^{n-1} q_k$ such that $q_n \sim p_n$. ■

Lemma 6.4.14. *Let M be a finite von Neumann algebra and suppose $\{p_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(M)$ are pairwise orthogonal. If $\{q_n\}_{n\in\mathbb{M}} \subset \mathcal{P}(M)$ is such that $q_n \sim p_n$ for each $n \in \mathbb{N}$ then $q_n \to 0$ in the strong operator topology.*

Proof. We will show the stronger condition $\wedge_{n\in\mathbb{N}} (\vee_{k\geq n} q_k) = 0$.

For $n \leq m$ set $r_{n,m} = \vee_{n\leq k\leq m} q_k$, so that for $n < m$ we have $r_{n,m} = \sum_{j=1}^{m-n} r_{n,n+j} - r_{n,n+j-1}$. Set

$$r_n = \vee_{k\geq n} q_n = \sum_{j=1}^{\infty} r_{n,n+j} - r_{n,n+j-1}.$$

For $n, j \geq 1$, Kaplansky's formula gives

$$\begin{aligned} r_{n,n+j} - r_{n,n+j-1} &= (r_{n,n+j-1} \vee q_{n+j}) - r_{n,n+j-1} \\ &\sim q_{n+j} - (r_{n,n+j-1} \wedge q_{n+j}) \\ &\leq q_{n+j} \sim p_{n+j}. \end{aligned}$$

Hence we have

$$r_n = \sum_{j=1}^{\infty} r_{n,n+j} - r_{n,n+j-1} \preceq \sum_{j=1}^{\infty} p_{n+j}.$$

Since M is finite, Proposition 5.2.8 shows that for all $n \in \mathbb{N}$ we then have

$$1 - \sum_{j=1}^{\infty} p_{n+j} \prec 1 - r_n \leq 1 - \wedge_{n\in\mathbb{N}} r_n.$$

Lemma 6.4.13 then gives

$$1 = 1 - \wedge_{n\in\mathbb{N}} \left(\sum_{j=1}^{\infty} p_{n+j} \right) \leq 1 - \wedge_{n\in\mathbb{N}} r_n \leq 1.$$

As M is finite it then follows that $\wedge_{n\in\mathbb{N}} r_n = 0$. ■

Lemma 6.4.15. *Let M be a finite von Neumann algebra, fix $\varphi \in M_*$, and let $K = \{\varphi \circ \mathrm{Ad}(u) \mid u \in \mathcal{U}(M)\} \subset M_*$. Then K is precompact in the weak topology.*

Proof. We will view K as a subset of M^* and show that the weak*-closure of K is contained in $M_* \subset M^*$. Suppose therefore that ψ is in the the weak*-closure of K but ψ is not normal. It follows that there exist $c_0 > 0$ and a decreasing sequence of projections $\{e_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(M)$ such that $\wedge_{n\in\mathbb{N}} e_n = 0$ and $|\psi(e_n)| \geq c_0$, for all $n \in \mathbb{N}$. As ψ is in the weak closure of K, it follows that for each $n \in \mathbb{N}$ there exists $u_n \in \mathcal{U}(M)$ such that $|\varphi(u_n p_n u_n^*)| > c_0/2$. Since φ is normal, for each $n \in \mathbb{N}$ there exists $m(n) > n$ so that $|\varphi(u_n(p_n - p_{m(n)})u_n^*)| > c_0/3$.

If we inductively define $k_n \in \mathbb{N}$ by setting $k_0 = 0$, and $k_n = m(k_{n-1})$, then we have that $\{p_{k_n} - p_{k_{n+1}}\}_{n\in\mathbb{N}}$ are pairwise orthogonal and satisfy $|\varphi(u_{k_n}(p_{k_n} - p_{k_{n+1}})u_{k_n}^*)| > c_0/3$, for each $n \in \mathbb{N}$. However, by Lemma 6.4.14 $u_{k_n}(p_{k_n} - p_{k_{n+1}})u_{k_n}^* \to 0$ in the strong operator topology, which then contradicts the fact that φ is normal. ■

Theorem 6.4.16. *Let M be a finite von Neumann algebra, then M has a normal center-valued trace.*

Proof. For each normal linear functional $\zeta \in \mathcal{Z}(M)_*$ we may consider the space $\mathcal{K}_\zeta \subset M_*$ consisting of all normal linear functionals φ such that $\|\varphi\| = \|\zeta\|$ and $\varphi_{\mathcal{Z}(M)} = \zeta$. By Lemma 6.4.15 we may apply the Ryll-Nardzewski fixed point theorem to conclude that there exists some $\tau_\zeta \in \mathcal{K}_\zeta$ such that $\tau_\zeta(uxu^*) = \tau_\zeta(x)$ for all $x \in M$, $u \in \mathcal{U}(M)$. The same proof of uniqueness of the trace from Theorem 6.4.10 shows that τ_ζ is the unique linear functional in $\mathcal{K}_\zeta$ which satisfies $\tau_\zeta(uxu^*) = \tau_\zeta(x)$ for all $x \in M$, $u \in \mathcal{U}(M)$. Thus, we obtain a linear isometry $\tau_* : \mathcal{Z}(M)_* \to M_*$ by $\tau_*(\zeta) = \tau_\zeta$. The dual map $\tau : M \to \mathcal{Z}(M)$ is then easily seen to be a center-valued trace. ■

The following theorem gives a strengthening of Lemma 6.4.14, which is of independent interest.

Theorem 6.4.17. *Let M be a von Neumann algebra. Then M is finite if and only if the adjoint operation $M \ni x \mapsto x^* \in M$ is continuous in the σ-strong operator topology.*

Proof. We suppose first that M is not finite. Then there exists $u \in M$ such that $u^*u = 1$ and $uu^* < 1$. We then have that $u^n(u^n)^*$ is a decreasing sequence of projections and if we set $p = \wedge_{n\in\mathbb{N}} u^n(u^n)^*$ then we have $pu = up$. Setting $v = p^\perp u$ we then have $(v^n)(v^n)^* \to 0$ so that $(v^n)^* \to 0$ in the σ-strong operator topology. However, we have $(v^n)^*(v^n) = p^\perp \neq 0$ so that $v^n \not\to 0$ in the σ-strong operator topology.

Now suppose that M is finite and we have a net $\{x_i\}_i \subset M$ so that $x_i \to 0$ in the σ-strong operator topology, so that $x_i^* x_i \to 0$ in the σ-weak operator topology. If we consider the polar decomposition $x_i = v_i|x_i|$, then we may extend each v_i to a unitary u_i since M is finite. We then have $x_i x_i^* = u_i(x_i^* x_i)u_i^*$ and if $\varphi \in M_*$ then by Lemma 6.4.15 $\{\varphi \circ \mathrm{Ad}(u) \mid u \in \mathcal{U}(M)\}$ is weakly precompact and hence

$$\lim_{i\to\infty} \sup_{u\in\mathcal{U}(M)} \varphi(u x_i^* x_i u^*) = 0.$$

Taking $u = u_i$ then shows that we have $\varphi(x_i x_i^*) \to 0$. Hence, $x_i^* \to 0$ in the σ-strong operator topology. ■

6.5 Dixmier's property

Lemma 6.5.1. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $x = x^* \in M$, then there exists a unitary $u \in \mathcal{U}(M)$, and $y = y^* \in \mathcal{Z}(M)$ such that $\|\frac{1}{2}(x + u^*xu) - y\| \leq \frac{3}{4}\|x\|$.*

Moreover, if M is a countably decomposable type III factor and $\|x\| \in \sigma(x)$, we may take $u \in \mathcal{U}(M)$ such that $\|x\| \in \sigma(\frac{1}{2}(x + u^*xu))$.

Proof. We may assume that $\|x\| = 1$. Let $p = 1_{[0,\infty)}(x)$, and $q = 1 - p$. By the comparison theorem there exists $z \in \mathcal{P}(\mathcal{Z}(M))$, $q_1, q_2, p_1, p_2 \in \mathcal{P}(M)$ such that

$$zq \sim p_1 \leq p_1 + p_2 = zp, \quad \text{and} \quad (1 - z)p \sim q_1 \leq q_1 + q_2 = (1 - z)q.$$

Suppose $v, w \in M$ such that $v^*v = zq$, $vv^* = p_1$, $w^*w = (1-z)p$, and $ww^* = q_1$. Set $u = v + v^* + w + w^* + q_2 + p_2$. Then $u \in \mathcal{U}(M)$, and we have

$$\begin{aligned} &u^*p_1u = zq, && u^*zqu = p_1, && u^*p_2u = p_2; \\ &u^*q_1u = (1 - z)p, && u^*(1 - z)pu = q_1, && u^*q_2u = q_2. \end{aligned}$$

We then have $-zq \leq zx \leq zp = p_1 + p_2$, and conjugating by u gives $-p_1 \leq zu^*xu \leq zq + p_2$. Hence,

$$-\frac{1}{2}(zq + p_1) \leq \frac{1}{2}(zx + zu^*xu) \leq \frac{1}{2}(p_1 + zq) + p_2.$$

As $zq + p_1 + p_2 = z$ we then have

$$-\frac{1}{2}z \leq \frac{1}{2}(zx + zu^*xu) \leq z,$$

hence,

$$-\frac{3}{4}z \leq \frac{1}{2}(zx + zu^*xu) - \frac{1}{4}z \leq \frac{3}{4}z.$$

A similar argument shows

$$-\frac{3}{4}(1 - z) \leq \frac{1}{2}((1 - z)x + (1 - z)u^*xu) + \frac{1}{4}(1 - z) \leq \frac{3}{4}(1 - z).$$

Thus,

$$\|\frac{1}{2}(x + u^*xu) - \frac{1}{4}(2z - 1)\| \leq \frac{3}{4}.$$

In the case when M is a countably decomposable type III factor, and $\|x\| \in \sigma(x)$, we may find non-zero orthogonal spectral projections of x, $p_0, p, q \in \mathcal{P}(M)$ such that $\|xp_0\| = \|xp\| = \|x\|$, $xp_0, xp \geq 0$, and $1 = p_0 + p + q$. Since M is type III, p and q are orthogonal and equivalent, hence there exists a unitary $u_0 \in \mathcal{U}((p+q)M(p+q))$ such that $p = uqu^*$, and $q = upu^*$. If we set $u = p_0 + u_0$ then proceeding as above we see that $\|\frac{1}{2}(x + u^*xu) - \frac{1}{2}\|x\|\| \leq \frac{3}{4}\|x\|$. Moreover, we have $\|x\| \in \sigma(px) = \sigma(\frac{1}{2}(x + u^*xu)p) \subset \sigma(\frac{1}{2}(x + u^*xu))$. ■

Theorem 6.5.2 (Dixmier's property)**.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For all $x \in M$ denote by $\overline{K}(x)$ the norm closed convex hull of the unitary orbit of x. Then we have $\mathcal{Z}(M) \cap \overline{K}(x) \neq \emptyset$.*

Moreover, if M is a countably decomposable type III factor, $x = x^*$, and $\|x\| \in \sigma(x)$, then $\|x\| \in \overline{K}(x)$.

Proof. We denote by $\mathcal{K}$ the set of all maps α from M to M of the form $\alpha(y) = \sum_{i=1}^n \alpha_i u_i^* y u_i$, where $u_1, \ldots, u_n \in \mathcal{U}(M)$, $\alpha_1, \ldots, \alpha_n \geq 0$, and $\sum_{i=1}^n \alpha_i = 1$.

Suppose $x = a_0 + ib_0$ where a_0 and b_0 are self-adjoint. By iterating Lemma 6.5.1, there exists a sequence $\alpha_k \in \mathcal{K}$, and $y_k = y_k^* \in \mathcal{Z}(M)$ such that if we set $\tilde{y}_k = \sum_{i=1}^k y_i$, and $a_k = \alpha_k(a_{k-1})$ then

$$\|a_k - \tilde{y}_k\| = \|\alpha_k(a_{k-1} - \tilde{y}_{k-1}) - y_k\| \leq \left(\frac{3}{4}\right)^k \|a_0\|.$$

Hence, for any $\varepsilon > 0$ there exists $\alpha \in \mathcal{K}$, and $y \in \mathcal{Z}(M)$ such that $\|\alpha(a_0) - y\| < \varepsilon$. Similarly, there then exists $\beta \in \mathcal{K}$, and $z \in \mathcal{Z}(M)$ such that $\|\beta(\alpha(b_0)) - z\| < \varepsilon$, and note that we still have $\|\beta(\alpha(a_0)) - y\| = \|\beta(\alpha(a_0) - y)\| < \varepsilon$. Thus, we have $\|\beta \circ \alpha(x) - (y + iz)\| < 2\varepsilon$.

We can therefore take a sequence $\alpha_k \in \mathcal{K}$, and $z_k \in \mathcal{Z}(M)$ such that if we define $x_0 = x$, and $x_k = \alpha_k(x_{k-1})$ then

$$\|x_k - z_k\| < 1/2^k.$$

In particular, we have $\|x_{k+1} - x_k\| \leq \|\alpha_{k+1}(x_k - z_k) + (z_k - x_k)\| \leq 1/2^{k-1}$, and so the sequences $\{x_k\}$ and $\{z_k\}$ converge in norm to an element $z \in \mathcal{Z}(M) \cap \overline{K}(x)$.

In the case when M is a countably decomposable type III factor, $x = x^*$, and $\|x\| \in \sigma(x)$, then from Lemma 6.5.1 we may take the sequence $x_k = x_k^*$ above so that $\|x\| \in \sigma(x_k)$ for all k. From this it follows that $x_k \to \|x\|$, and so $\|x\| \in \overline{K}(x)$. ■

Corollary 6.5.3. *A von Neumann algebra M is finite if and only if $\mathcal{Z}(M) \cap \overline{K}(x)$ consists of a single point for each $x \in M$. Moreover, if M is finite then it has a unique center-valued trace τ, and we have $\mathcal{Z}(M) \cap \overline{K}(x) = \{\tau(x)\}$ for all $x \in M$.*

Proof. If M is finite, then for any center-valued trace τ we have that τ is constant on $\overline{K}(x)$, and so $\emptyset \neq \overline{K}(x) \cap \mathcal{Z}(M) \subset \{\tau(x)\}$. Since the trace τ was arbitrary, and since M has a trace by Theorem 6.4.10, it follows that the trace must be unique.

Conversely, if $\overline{K}(x) \cap \mathcal{Z}(M)$ consists of a single element $\tau(x)$ for each $x \in M$, then τ defines a center-valued state, and we have $\tau(u^* x u) = \tau(x)$ for each $u \in \mathcal{U}(M)$ and $x \in M$, hence τ is a trace and so M is finite by Theorem 6.4.10. ■

6.5.1 The fundamental group of a II_1 factor

Let M be a II_1 factor, and let τ be the unique trace on M. Then for all $n \in \mathbb{N}$ we have that $\mathbb{M}_n(M)$ is again a II_1 factor with unique trace given by $\tau_n([x_{i,j}]) =$

$\frac{1}{n}\sum_{i=1}^{n}\tau(x_{i,i})$. If $0 < t \leq n$ then we know from Proposition 6.4.11 that for any two projections $p, q \in \mathcal{P}(\mathbb{M}_n(M))$ with trace $\tau_n(p) = \tau_n(q) = t/n$ there is a unitary $u \in \mathcal{U}(\mathbb{M}_n(M))$ such that $upMpu^* = qMq$. Thus, up to isomorphism the factor pMp only depends on t. Note that this is also independent of n since any two matrix algebras over M can be embedded into a larger common matrix algebra. The **amplification** of M with parameter t is the factor M^t which is define as the II_1 factor pMp (which is unique up to isomorphism class).

Note that $M^1 \cong M$, and also $(M^t)^s \cong M^{ts}$ for all $t, s > 0$. The **fundamental group**[2] of M is $\mathcal{F}(M) = \{t > 0 \mid M^t \cong M\}$ which is easily seen for form a subgroup of the multiplicative group $\mathbb{R}_{>0}$. Note that if $\theta : M \to N$ is a $*$-isomorphism then $\mathcal{F}(M) = \mathcal{F}(N)$ and hence the fundamental group is an isomorphism class invariant of M.[3] Note that for all $t > 0$ we have $\mathcal{F}(M^t) = \mathcal{F}(M)$.

If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and we consider the conjugate Hilbert space $\overline{\mathcal{H}}$. Recall that to each operator $x \in M$ we may associate the operator $\overline{x} \in \mathcal{B}(\overline{\mathcal{H}})$ which is defined by $\overline{x}\overline{\xi} = \overline{x\xi}$. The **opposite von Neumann algebra** $M^{\circ} = \{\overline{x} \mid x \in M\} \subset \mathcal{B}(\overline{\mathcal{H}})$ is clearly a von Neumann algebra, and the map $x \mapsto \overline{x}$ defines an anti-linear isomorphism between M and M°. If one prefers to work with linear maps then consider $x^{\circ} = \overline{x^*}$, the map $x \mapsto x^{\circ}$ is then a normal linear isometry from M to M°, however, this is not an isomorphism but rather an **anti-isomorphism**, i.e., $(xy)^{\circ} = y^{\circ}x^{\circ}$ for all $x, y \in M$. It is clear that $\mathcal{F}(M^{\circ}) = \mathcal{F}(M)$, for all II_1 factors M.

6.6 Weights

By the Riesz representation theorem, if K is a compact Hausdorff space then states on $C(K)$ are in 1-1 correspondence with Radon probability measures on K. If we consider positive Radon measures which are not necessarily finite then this leads to the notion of a weight. Specifically, a **weight** on a C^*-algebra A is a map $\varphi : A_+ \to [0, \infty]$ satisfying the following conditions for all $x, y \in A_+$, $\lambda > 0$:

$$\varphi(x + y) = \varphi(x) + \varphi(y), \quad \varphi(\lambda x) = \lambda\varphi(x), \quad \varphi(0) = 0.$$

The weight is **faithful** if $\varphi(x) \neq 0$ for every non-zero $x \in A_+$, and φ is **tracial** if $\varphi(xx^*) = \varphi(x^*x)$, for all $x \in A$.

If A is a von Neumann algebra then a weight φ is **semi-finite** if

$$\mathfrak{p}_{\varphi} = \{x \in A_+ \mid \varphi(x) < \infty\}$$

generates A as a von Neumann algebra, and φ is **normal** if $\varphi(\sup x_i) = \sup \varphi(x_i)$, for every bounded increasing net $\{x_i\}$ in A_+.

Example 6.6.1. The trace Tr is a normal faithful semi-finite tracial weight on $\mathcal{B}(\mathcal{H})$.

[2] This has no relation to the better known notion in topology.

[3] Better terminology might be the fundamental subgroup, since it is an invariant of M not just as an abstract group but rather as a subgroup of $\mathbb{R}_{>0}$.

Lemma 6.6.2. *Let φ be a weight on a C^*-algebra A, then*

a) $\mathfrak{n}_\varphi = \{x \in A \mid x^*x \in \mathfrak{p}_\varphi\}$ *is a left ideal of A.*

b) $\mathfrak{m}_\varphi = \{\sum_{i=1}^n y_i^* x_i \mid x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{n}_\varphi\}$ *is a $*$-subalgebra of $\mathfrak{n}_\varphi$ which is equal to the span of $\mathfrak{p}_\varphi$.*

c) $\mathfrak{m}_\varphi \cap M_+ = \mathfrak{p}_+$ *and φ has a unique extension to a linear functional on $\mathfrak{m}_\varphi$.*

Proof. That $\mathfrak{n}_\varphi$ is a linear subspace follows from the inequality

$$(x+y)^*(x+y) \leq (x+y)^*(x+y) + (x-y)^*(x-y) = 2(x^*x + y^*y).$$

That $\mathfrak{n}_\varphi$ is a left ideal then follows from the inequality $(ax)^*(ax) \leq \|a\|x^*x$. Since $\mathfrak{n}_\varphi$ is a linear subspace we have that $\mathfrak{m}_\varphi$ is a $*$-subalgebra. It is easy to see that $\mathfrak{m}_\varphi$ contains $\mathfrak{p}_\varphi$, and the fact that it is equal to the span of $\mathfrak{p}_\varphi$ follows from the polarization identity

$$4y^*x = \sum_{k=0}^{3} i^k (x + i^k y)^*(x + i^k y).$$

If $x \in \mathfrak{m}_\varphi$ and we have

$$x = a_1 - a_2 + i(a_3 - a_4) = b_1 - b_2 + i(b_3 - b_4),$$

where $a_i, b_i \in \mathfrak{p}_\varphi$ for $1 \leq i \leq 4$, then we have

$$a_1 - a_2 = \frac{1}{2}(x + x^*) = b_1 - b_2$$

and hence

$$\varphi(a_1) + \varphi(b_2) = \varphi(a_1 + b_2) = \varphi(b_1 + a_2) = \varphi(b_1) + \varphi(a_2).$$

Therefore, $\varphi(a_1) - \varphi(a_2) = \varphi(b_1) - \varphi(b_2)$. We similarly have that $\varphi(a_3) - \varphi(a_4) = \varphi(b_3) - \varphi(b_4)$. Hence, if we set

$$\varphi(x) = \varphi(a_1) - \varphi(a_2) + i(\varphi(a_3) - \varphi(a_4))$$

then this gives a well-defined unique extension of φ from $\mathfrak{p}_\varphi$ to $\mathfrak{m}_\varphi$ that is also linear.

If $x \in \mathfrak{m}_\varphi \cap M_+$ then $x = a_1 - a_2$ with $a_1, a_2 \in \mathfrak{p}_\varphi$. Hence, $x \leq a_1 \in \mathfrak{p}_\varphi$ so that $x \in \mathfrak{p}_\varphi$. ■

We refer to $\mathfrak{m}_\varphi$ as the **domain of definition** for φ. If $\varphi(1) < \infty$, then we can linearly extend φ to all of A, and after rescaling obtain a state.

If we fix a weight φ on a C^*-algebra A, then the set

$$N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$$

is a left ideal of A contained in $\mathfrak{n}_\varphi$. We may then consider the quotient space $\mathfrak{n}_\varphi/N_\varphi$ with the quotient map $\eta_\varphi : \mathfrak{n}_\varphi \to \mathfrak{n}_\varphi/N_\varphi$. This then gives a positive definite sesqui-linear form on $\mathfrak{n}_\varphi/N_\varphi$ by the formula

$$\langle \eta_\varphi(x), \eta_\varphi(y) \rangle = \varphi(y^*x),$$

for $x, y \in \mathfrak{n}_\varphi$. We denote by $L^2(A, \varphi)$ the Hilbert space completion of this form. Since $\mathfrak{n}_\varphi$, and N_φ are both left ideals we obtain a A module structure on $\mathfrak{n}_\varphi/N_\varphi$ by left multiplication, which from the inequality $(ax)^*(ax) \leq \|a\|^2 x^*x$ extends to a representation $\pi_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$.

The following proposition follows as in the GNS-construction and so we leave it to the reader.

Proposition 6.6.3. *Let φ be a weight on a C^*-algebra A, then $\pi_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$ is a continuous $*$-representation, which is faithful if φ is faithful. Moreover, if A is a von Neumann algebra, and φ is normal, then so is the representation π_φ.*

We call the triple $(\pi_\varphi, L^2(A, \varphi), \eta_\varphi)$ the **semi-cyclic representation** of A. Note that if A is unital, and φ is a state on A, then the map $\eta_\varphi : A \to \mathcal{H}_\varphi$ is completely determined by the value $\eta_\varphi(1)$. And so in this case we can think of this triple as a representation, together with a cyclic vector.

If φ is a normal weight on M, then we may extend φ to the extended positive cone $\hat{M}_+$ by setting

$$\varphi(A) = \int t \, d\varphi(E(t)) \in [0, \infty],$$

where $A = \int t \, dE(t)$ is the spectral decomposition of A. We leave it to the reader to check that we have

1. $\varphi(\lambda A) = \lambda \varphi(A), A \in \hat{M}_+, \lambda \geq 0,$
2. $\varphi(A \dotplus B) = \varphi(A) + \varphi(B)$, $A, B \in \hat{M}_+$,
3. $\varphi(\sup_i A_i) = sup_i \varphi(A_i)$, for an increasing net $A_i \nearrow A$.

6.6.1 Traces on semi-finite von Neumann algebras

Theorem 6.6.4. *Let M be a semi-finite factor, then there exists a unique, up to scalar multiplication, normal semi-finite tracial weight* $\mathrm{Tr} : M_+ \to [0, \infty]$. *Moreover,* Tr *is faithful, and $p \in \mathcal{P}(M)$ is finite if and only if* $\mathrm{Tr}(p) < \infty$.

Proof. We have already constructed a tracial state on finite factors, thus for existence we need only consider the case when M is properly infinite. Let $p_0 \in M$ be a non-zero finite projection, then there exists an infinite family $\{p_n\}_{n \in I}$ of pairwise orthogonal projections, such that $\sum_n p_n = 1$, and $p_0 \sim p_n$ for each $n \in I$. Take $v_n \in M$ such that $v_n^* v_n = p_0$, and $v_n^* v_n = p_n$, and let τ_0 be the unique tracial state on $p_0 M p_0$. We define $\mathrm{Tr} : M_+ \to [0, \infty]$ by $\mathrm{Tr}(x) = \sum_{n \in I} \tau_0(v_n^* x v_n)$. Then Tr is a weight on M, which is normal since $x \mapsto \tau_0(v_n^* x v_n)$ is normal for each $n \in I$.

Note that we have $v_n^* M v_n \subset \mathfrak{m}_{\mathrm{Tr}}$ for all $n \in I$, and thus $\mathfrak{m}_{\mathrm{Tr}}$ is weakly dense in M, so that Tr is semi-finite. For each $x \in M$ we apply Fubini's theorem to obtain

$$\mathrm{Tr}(xx^*) = \sum_{n,m \in I} \tau_0(v_n^* x v_m v_m^* x^* v_n) = \sum_{n,m \in I} \tau_0(v_m x^* v_n v_n^* x v_m^*) = \mathrm{Tr}(xx^*).$$

If $p \in \mathcal{P}(M)$ is finite then so is $p_0 \vee p$, and there is a non-zero subprojection $q \leq p_0$ which is monic in $(p_0 \vee p)M(p_0 \vee p)$. We then have $0 < \mathrm{Tr}(q)$, and $\mathrm{Tr}(p_0 \vee p) < \infty$. Thus, Tr is faithful and is finite on finite projections. Conversely, if $p \in \mathcal{P}(M)$ is infinite then there is a subprojection $q \leq p$ such that $p \sim q \sim p - q$, hence $\mathrm{Tr}(p) = \mathrm{Tr}(q) = \mathrm{Tr}(p - q)$, and so $\mathrm{Tr}(p) = 2\,\mathrm{Tr}(p)$, and since Tr is faithful we must have $\mathrm{Tr}(p) = \infty$.

It only remains to prove uniqueness. So suppose M is a semi-finite factor and ω is a normal semi-finite tracial weight on M. Since ω is semi-finite, for each $x \in M_+$, $x \neq 0$, there exists $y \in \mathfrak{n}_\omega$ such that $x^{1/2} y \neq 0$. Hence, $x^{1/2} y y^* x^{1/2} \in \mathfrak{m}_\omega$ and so $\omega(x^{1/2} y y^* x^{1/2}) < \infty$. Taking a suitable non-zero spectral projection q of $x^{1/2} y y^* x^{1/2}$ there then exists some number $c > 0$ such that $cq \leq x^{1/2} y y^* x^{1/2} \leq \|y\|^2 x$.

From above it follows that q is a finite projection (since $\omega(q) < \infty$), and so $\omega(q) \neq 0$ since otherwise we would have $\omega(p) = 0$ for all finite projections, contradicting semi-finiteness. Hence, $\omega(x) \neq 0$ showing that ω is faithful, and it then follows from the argument above that $\omega(p) < \infty$ if and only if p is finite. In particular, if $p \in \mathcal{P}(M)$ is any non-zero finite projection then $\omega_{|pMp}$ defines a tracial positive linear functional and hence must be a scalar multiple of the trace on pMp. Since M is semi-finite, every projection is an increasing limit of finite projections we then obtain uniqueness of Tr up to a scalar multiple. ■

Proposition 6.6.5. *Let M be a countably decomposable semi-finite factor. If $p, q \in \mathcal{P}(M)$, then $p \preceq q$ if and only if* $\mathrm{Tr}(p) \leq \mathrm{Tr}(q)$.

Proof. If p and q are finite then $\mathrm{Tr}_{(p \vee q)M(p \vee q)}$ is a scalar multiple of the trace on $(p \vee q)M(p \vee q)$, hence the conclusion follows from Proposition 6.4.11. By Theorem 6.6.4 we have $\mathrm{Tr}(p) = \infty$ if and only if p is not finite, and so the result then follows from Corollary 5.2.10. ■

Chapter 7

Examples of von Neumann algebras

7.1 Group von Neumann algebras

7.1.1 Group representations

Let Γ be a discrete group. A (unitary) **representation** of Γ is a homomorphism $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$. The **trivial** representation of Γ on $\mathcal{H}$ is given by $\pi(g) = 1$, for all $g \in \Gamma$. The **left-regular** (resp. **right-regular**) representation of Γ is $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ (resp. $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$) given by $(\lambda_g \xi)(x) = \xi(g^{-1}x)$ (resp. $(\rho_g \xi)(x) = \xi(xg)$). If $\Lambda < \Gamma$ is a subgroup, then the representation $\pi : \Gamma \to \ell^2(\Gamma/\Lambda)$ given by $(\pi_g \xi)(x) = \xi(g^{-1}x)$ is a **quasi-regular** representation.

Two representations $\pi_i : \Gamma \to \mathcal{U}(\mathcal{H}_i)$, $i = 1, 2$, are **equivalent** if there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(g) = \pi_2(g)U$, for all $g \in \Gamma$. Note that the left and right-regular representations are seen to be equivalent by considering the unitary $U : \ell^2\Gamma \to \ell^2\Gamma$ given by $(U\xi)(x) = \xi(x^{-1})$.

Given a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ we define the **adjoint representation** $\overline{\pi} : \Gamma \to \mathcal{U}(\overline{\mathcal{H}})$ by setting $\overline{\pi}_g \overline{\xi} = \overline{\pi_g \xi}$. We then have the natural identification $\pi = \overline{\overline{\pi}}$.

Given a family of unitary representations $\pi_\iota : \Gamma \to \mathcal{U}(\mathcal{H}_\iota)$, with $\iota \in I$, the **direct-sum representation** is $\bigoplus_{\iota \in I} \pi_\iota : \Gamma \to \mathcal{U}(\bigoplus_{\iota \in I} \mathcal{H}_\iota)$ defined by

$$(\bigoplus_{\iota \in I} \pi_\iota)(g) = \bigoplus_{\iota \in I} (\pi_\iota(g)).$$

If I is finite, then the **tensor product representation** is given by the map $\bigotimes_{\iota \in I} \pi_\iota : G \to \mathcal{U}(\overline{\bigotimes}_{\iota \in I} \mathcal{H}_\iota)$ defined by

$$(\bigotimes_{\iota \in I} \pi_\iota)(g) = \bigotimes_{\iota \in I} (\pi_\iota(g)).$$

If we use the identification $\mathcal{H}\overline{\otimes}\overline{\mathcal{K}} \cong \text{HS}(\mathcal{H}, \mathcal{K})$, then for representations $\pi_1 : \Gamma \to \mathcal{U}(\mathcal{H})$, and $\pi_2 : \Gamma \to \mathcal{U}(\mathcal{K})$, the representation $\pi_1 \otimes \overline{\pi}_2$ is realized on $\text{HS}(\mathcal{H}, \mathcal{K})$ as $(\pi_1 \otimes \overline{\pi}_2)(g)(T) = \pi_1(g)T\pi_2(g^{-1})$.

Lemma 7.1.1 (Fell's Absorption Principle)**.** *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a discrete group Γ, and let $1_{\mathcal{H}}$ denote the trivial representation of Γ on $\mathcal{H}$. Then the representations $\lambda \otimes \pi$ and $\lambda \otimes 1_{\mathcal{H}}$ are equivalent.*

Proof. Consider the unitary $U \in \mathcal{U}(\ell^2\Gamma \otimes \mathcal{H})$ determined by $U(\delta_g \otimes \xi) = \delta_g \otimes \pi(g)\xi$, for all $g \in \Gamma$, $\xi \in \mathcal{H}$. Then for all $h, g \in \Gamma$, and $\xi \in \mathcal{H}$ we have

$$\begin{aligned} U^*(\lambda \otimes \pi)(h)U(\delta_g \otimes \xi) &= U^*(\lambda \otimes \pi)(h)(\delta_g \otimes \pi_g\xi) \\ &= U^*(\delta_{hg} \otimes \pi_h\pi_g\xi) \\ &= \delta_{hg} \otimes \pi_{(hg)^{-1}}\pi_h\pi_g\xi = (\lambda \otimes 1_{\mathcal{H}})(h)(\delta_g \otimes \xi). \end{aligned}$$

■

If $\xi, \eta \in \ell^2\Gamma$, the **convolution** of ξ with η is the function $\xi * \eta : \Gamma \to \mathbb{C}$ given by

$$(\xi * \eta)(x) = \sum_{g\in\Gamma} \xi(g)\eta(g^{-1}x) = \sum_{g\in\Gamma} \xi(xg^{-1})\eta(g).$$

Note that by the Cauchy-Schwarz inequality we have that $\xi * \eta \in \ell^\infty\Gamma$, and $\|\xi * \eta\|_\infty \leq \|\xi\|_2\|\eta\|_2$. If $\xi, \eta \in \ell^1\Gamma$ then we also have the estimate $\|\xi * \eta\|_1 \leq \|\xi\|_1\|\eta\|_1$. Also note that for $g \in \Gamma$ we have $\delta_g * \xi = \lambda_g\xi$, and $\xi * \delta_g = \rho_{g^{-1}}\xi$.

If $\xi \in \ell^2\Gamma$ we let $\overline{\xi}$ be the function defined by $\overline{\xi}(x) = \overline{\xi(x^{-1})}$. Also, it is easy to see that if $\xi, \eta, \zeta \in \ell^2\Gamma$, then $(\xi*\eta)*\zeta \in \ell^2\Gamma$ if and only if $\xi*(\eta*\zeta) \in \ell^2\Gamma$, and if both are in $\ell^2\Gamma$ then we have $(\xi*\eta)*\zeta = \xi*(\eta*\zeta)$. In particular $\ell^1\Gamma$ with the norm $\|\cdot\|_1$ forms a unital involutive Banach algebra which is the **convolution algebra** of Γ.

Given $\xi \in \ell^2\Gamma$ we set $D_\xi = \{\eta \in \ell^2\Gamma \mid \xi * \eta \in \ell^2\Gamma\}$. We then define the convolution operator $L_\xi : D_\xi \to \ell^2\Gamma$ by $L_\xi\eta = \xi * \eta$. We also set $D'_\xi = \{\eta \in \ell^2\Gamma \mid \eta * \xi \in \ell^2\Gamma\}$, and $R_\xi : D'_\xi \to \ell^2\Gamma$, by $R_\xi\eta = \eta * \xi$.

Lemma 7.1.2. *For each $\xi \in \ell^2\Gamma$, the operators L_ξ, and R_ξ have closed graph in $\ell^2\Gamma \oplus \ell^2\Gamma$.*

Proof. Let $\{\eta_n\} \subset \ell^2\Gamma$ be a sequence such that $\eta_n \to \eta \in \ell^2\Gamma$, and $L_\xi\eta_n \to \zeta \in \ell^2\Gamma$. Then for $x \in \Gamma$ we have $|\zeta(x)-(\xi*\eta)(x)| = \lim_{n\to\infty} |(\xi*\eta_n)(x)-(\xi*\eta)(x)| \leq \lim_{n\to\infty} \|\xi\|_2\|\eta_n - \eta\|_2 = 0$. Hence, $\xi*\eta = \zeta \in \ell^2\Gamma$, and so $\eta \in D_\xi$ and $L_\xi\eta = \zeta$. The proof for R_ξ is identical. ■

A **left-convolver** (resp. **right-convolver**) is a vector $\xi \in \ell^2\Gamma$ such that $\xi * \ell^2\Gamma \subset \ell^2\Gamma$ (resp. $\ell^2\Gamma * \xi \subset \ell^2\Gamma$). If ξ is a left-convolver then by the closed graph theorem we have that $L_\xi \in \mathcal{B}(\ell^2\Gamma)$, and similarly $R_\xi \in \mathcal{B}(\ell^2\Gamma)$ for ξ a right-convolver. Note that the space of left (resp. right) convolvers is a linear space which contains δ_g for each $g \in \Gamma$.

We set

$$L\Gamma = \{L_\xi \mid \xi \in \ell^2\Gamma \text{ is a left-convolver.}\} \subset \mathcal{B}(\ell^2\Gamma);$$
$$R\Gamma = \{R_\xi \mid \xi \in \ell^2\Gamma \text{ is a right-convolver.}\} \subset \mathcal{B}(\ell^2\Gamma).$$

If ξ is a left-convolver then it is easy to see that $\overline{\xi}$ is also a left-convolver and we have $L_{\overline{\xi}} = L_\xi^*$. Similarly, we have $R_{\overline{\xi}} = R_\xi^*$ for right-convolvers. Also, since convolution is associative we have $L_{\xi * \eta} = L_\xi L_\eta$, and $R_{\xi*\eta} = R_\eta R_\xi$. Hence, $L\Gamma$ and $R\Gamma$ are unital $*$-subalgebras of $\mathcal{B}(\ell^2\Gamma)$. We next show that actually $L\Gamma$ and $R\Gamma$ are von Neumann algebras.

Theorem 7.1.3. *Let Γ be a discrete group, then $L\Gamma$ and $R\Gamma$ are von Neumann algebras. Moreover, we have $L\Gamma = R\Gamma' = \rho(\Gamma)'$, and $R\Gamma = L\Gamma' = \lambda(\Gamma)'$.*

Proof. By von Neumann's double commutant theorem it is enough to show that $L\Gamma = R\Gamma' = \rho(\Gamma)'$. Note that we trivially have the inclusions $L\Gamma \subset R\Gamma' \subset \rho(\Gamma)'$ and so we need only show $\rho(\Gamma)' \subset L\Gamma$.

Suppose $T \in \rho(\Gamma)'$ and define $\xi = T\delta_e$. Then for all $g \in \Gamma$ we have

$$\xi * \delta_g = \rho_{g^{-1}} T\delta_e = T\rho_{g^{-1}}\delta_e = T\delta_g.$$

By linearity we then have $\xi*\eta = T\eta$ for all η in the dense subspace $\mathrm{sp}\{\delta_g \mid g \in \Gamma\}$. Hence it follows that ξ is a left-convolver and $T = L_\xi \in L\Gamma$. ■

The von Neumann aglebra $L\Gamma$ is the (left) **group von Neumann algebra** of Γ, and $R\Gamma$ is the right group von Neumann algebra of Γ. Note that since the left and right-regular representations are equivalent it follows that $L\Gamma \cong R\Gamma$.

Proposition 7.1.4. *Let Γ be a discrete group, then $\tau(x) = \langle x\delta_e, \delta_e\rangle$ defines a normal faithful trace on $L\Gamma$. In particular, $L\Gamma$ is a finite von Neumann algebra.*

Proof. If $\tau(x^*x) = 0$, where $x = L_\xi$, then $\|\xi\|^2 = \|L_\xi\delta_e\|^2 = \tau(x^*x) = 0$, hence $x = 0$, and so τ is faithful. As a vector state, τ is clearly normal, thus to check that it is a trace it is enough to check the tracial property on a weakly dense subalgebra, and by linearity it is then enough to show $\tau(\lambda_{ghg^{-1}}) = \tau(\lambda_h)$ for all $g, h \in \Gamma$. By a direct calculation we see $\tau(\lambda_{ghg^{-1}}) = \delta_e(ghg^{-1}) = \delta_e(h) = \tau(\lambda_h)$. ■

Example 7.1.5. If Γ is abelian then we may consider the dual group $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{T})$ which is a compact group when endowed with the topology of point-wise convergence. We consider this group endowed with a Haar measure μ normalized so that $\mu(\hat{\Gamma}) = 1$. The Fourier transform $\mathcal{F} : \ell^2\Gamma \to L^2\hat{\Gamma}$ is defined as $(\mathcal{F}\xi)(\chi) = \sum_{g\in\Gamma} \xi(g)\langle\chi, g\rangle$. The Fourier transform implements a unitary between $\ell^2\Gamma$ and $\ell^2\hat{\Gamma}$.

If $\xi \in \ell^2\Gamma$ is a (left) convolver, then we have $L_\xi = \mathcal{F}^{-1} M_{\mathcal{F}(\xi)} \mathcal{F}$, and hence we obtain a canonical isomorphism $L\Gamma \cong L^\infty\hat{\Gamma}$. Moreover, we have $\tau(L_\xi) = \int \mathcal{F}(\xi)\, d\mu$, for each $L_\xi \in L\Gamma$.

Recall that when (X, μ) was a probability space, we could view $L^\infty(X,\mu)$ both as a von Neumann subalgebra of $\mathcal{B}(L^2(X,\mu))$, and as a subspace of $L^2(X,\mu)$. When we wanted to make a distinction between the two embeddings we would write M_f to explicitly denote the multiplication operator by f. Similarly, we may view $L\Gamma$ both as a von Neumann subalgebra of $\mathcal{B}(\ell^2\Gamma)$, and as a subspace of $\ell^2\Gamma$ under the identification $L_\xi \mapsto \xi$. We will therefore not always be specific as to which identification we are taking and leave it to the reader to determine from the context.

In particular, if $x = \sum_{g\in\Gamma} \alpha_g \delta_g \in \ell^2\Gamma$ is a left-convolver, then we will often also write x or $\sum_{g\in\Gamma} \alpha_g u_g$ to denote the operator $L_x \in L\Gamma$. (We switch δ_g to u_g to emphasize that u_g is a unitary operator.) By analogy with the abelian case we call the set $\{\alpha_g\}_{g\in\Gamma}$ the **Fourier coefficients** of x. This convention is quite standard, however we should issue a warning at this point that the sum $\sum_{g\in\Gamma} \alpha_g u_g$ does not in general converge to L_x in any operator space topology (e.g., norm, weak, or strong). This is already the case for $L\mathbb{Z}$ in fact. Thus, writing $x = \sum_{g\in\Gamma} \alpha_g u_g$ should be considered as an abbreviation for writing $L_x = L_{\sum_{g\in\Gamma} \alpha_g \delta_g}$, and nothing more.

A discrete group Γ is said to be **i.c.c.**[1] if every non-trivial conjugacy class of Γ is infinite.[2]

Theorem 7.1.6. *Let Γ be a discrete group. Then $L\Gamma$ is a factor if and only if Γ is i.c.c.*

Proof. First suppose that $h \in \Gamma \setminus \{e\}$, such that $h^\Gamma = \{ghg^{-1} \mid g \in G\}$ is finite. Let $x = \sum_{k\in h^\Gamma} u_k$. Then $x \neq \mathbb{C}$, and for all $g \in G$ we have $u_g x u_g^* = \sum_{k\in h^\Gamma} u_{gkg^{-1}} = x$, hence $x \in \{u_g\}'_{g\in\Gamma} \cap L\Gamma = \mathcal{Z}(L\Gamma)$.

Conversely, suppose that Γ is i.c.c. and $x = \sum_{g\in\Gamma} \alpha_g u_g \in \mathcal{Z}(L\Gamma) \setminus \mathbb{C}$, then for all $h \in \Gamma$ we have $x = u_h x u_h^* = \sum_{g\in\Gamma} \alpha_g u_{hgh^{-1}} = \sum_{g\in\Gamma} \alpha_{h^{-1}gh} u_g$. Thus the Fourier coefficients for x are constant on conjugacy classes, and since $x \in L\Gamma \subset \ell^2\Gamma$ we have $\alpha_g = 0$ for all $g \neq e$, hence $x = \tau(x) \in \mathbb{C}$. ■

Examples of i.c.c. groups which can be verified directly include the symmetric group S_∞ of all finite permutations of $\mathbb{N}$, free groups $\mathbb{F}_n$ of rank $n \geq 2$, free products $\Gamma_1 * \Gamma_2$ when $|\Gamma_1|, |\Gamma_2| > 1$ and $|\Gamma_1| + |\Gamma_2| \geq 5$, projective special linear groups $PSL_n(\mathbb{Z})$, $n \geq 2$, groups without non-trivial finite index subgroups, and many others.

7.1.2 Group C^*-algebras

If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a representation of a discrete group Γ then note that we may extend π linearly to a representation of the convolution algebra $\ell^1\Gamma$, we will use the same notation π for this representation. The (full) **group C^*-algebra** $C^*\Gamma$, is defined as the C^*-algebra completion of $\ell^1\Gamma$ with respect to the norm

[1] Infinite conjugacy classes

[2] Note that the trivial group is i.c.c., but this is the only finite i.c.c. group.

$\|f\| = \sup_{\pi:\Gamma\to\mathcal{U}(\mathcal{H})} \|\pi(f)\|$. The **reduced group C^*-algebra** $C^*_r\Gamma$ is the C^*-algebra completion of $L^1\Gamma$ with respect to the left-regular representation. Note that we have a natural inclusions $C^*_r\Gamma \subset L\Gamma \subset \mathcal{B}(\ell^2\Gamma)$.

Note that by definition we have that any representation of Γ extends uniquely to a representation of $C^*\Gamma$, and conversely every representation of $C^*\Gamma$ arises in this way. Moreover, two representations of Γ are equivalent if and only if the representations are equivalent when extended to $C^*\Gamma$. Thus, $C^*\Gamma$ is a C^*-algebra which encodes the representation theory of Γ. Given a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, a vector $\xi \in \mathcal{H}$ is **cyclic** if it is cyclic for $\pi(C^*\Gamma)$, i.e., $\overline{\mathrm{sp}}\{\pi(g)\xi \mid g \in \Gamma\} = \mathcal{H}$.

A function $\varphi : \Gamma \to \mathbb{C}$ is of **positive type** if for all $g_1, \ldots, g_n \in \Gamma$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \varphi(g_j^{-1} g_i) \geq 0.$$

Note, that by considering $g_1 = e$, and $\alpha_1 = 1$ we have $\varphi(e) \geq 0$. Also, by considering $g_1 = g$, $g_2 = e$, and $|\alpha_1| = \alpha_2 = 1$ we see that $\overline{\alpha_1}\varphi(g^{-1}) + \alpha_1\varphi(g) \leq 2\varphi(e)$, and from this it follows easily that $\varphi(g^{-1}) = \overline{\varphi(g)}$, and $\varphi \in \ell^\infty\Gamma$ with $\|\varphi\|_\infty = \varphi(e)$. Thus, a simple calculation shows that positive type is equivalent to the conditions $\varphi \in \ell^\infty\Gamma$, and

$$\sum_{g\in\Gamma} (f * \overline{f})(g)\varphi(g) \geq 0,$$

for all $f \in \ell^1\Gamma$.

The same proof as in the GNS-construction allows us to construct a representation $\pi_\varphi : \Gamma \to \mathcal{H}$, and a cyclic vector $\xi \in \mathcal{H}$ such that $\varphi(g) = \langle \pi(g)\xi, \xi \rangle$ for each $g \in \Gamma$. In particular, we may then extend φ to a positive linear functional on $C^*\Gamma$. Conversely, if φ is a positive linear functional on $C^*\Gamma$, then restricted to $\ell^1\Gamma$ this again gives a positive linear functional and hence restricted to Γ gives a function of positive type. We have thus proved the following theorem.

Theorem 7.1.7. *Let Γ be a discrete group, and let $\varphi : \Gamma \to \mathbb{C}$, then the following conditions are equivalent:*

(*i*) *φ is of positive type.*

(*ii*) *φ extends to a positive linear functional on $C^*\Gamma$.*

(*iii*) *There exists a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and a cyclic vector $\xi \in \mathcal{H}$ such that $\varphi(g) = \langle \pi(g)\xi, \xi \rangle$ for each $g \in \Gamma$.*

Note that a multiplicative linear functional on $C^*\Gamma$ restricts to a homomorphism from Γ to $\mathbb{T}$. Conversely, any such homomorphism extends to a multiplicative linear functional on $C^*\Gamma$. Thus, for abelian groups we have the following.

Theorem 7.1.8. *Let A be a discrete abelian group and $\hat{A}$ its dual group, then the natural map from $\hat{A}$ to $\sigma(C^*A)$ is a homeomorphism. Equivalently, the natural embedding of A into $C(\hat{A})$ extends to an isomorphism $C^*A \cong C(\hat{A})$.*

7.1.3 Other von Neumann algebras generated by groups

Given a unitary representation of a discrete group $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ one can always consider the von Neumann algebra it generates $\pi(\Gamma)''$. Properties of the representation can sometimes be reflected in the von Neumann algebra $\pi(\Gamma)''$, here we will discuss a couple of these properties.

A representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **reducible** if $\mathcal{H}$ contains a non-trivial closed Γ-invariant subspace. Otherwise, the representation is **irreducible**. Note that if $\mathcal{K} \subset \mathcal{H}$ is a closed Γ-invariant subspace then $\mathcal{K}^\perp$ is also Γ-invariant. Indeed, if $\xi \in \mathcal{K}, \eta \in \mathcal{K}^\perp$, and $g \in \Gamma$ then $\langle \pi(g)\eta, \xi\rangle = \langle \eta, \pi(g^{-1})\xi\rangle = 0$. Thus, π would then decompose as $\pi_{|\mathcal{K}} \oplus \pi_{|\mathcal{K}^\perp}$.

Lemma 7.1.9 (Schur's Lemma)**.** *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ be two irreducible unitary representations of a discrete group Γ, if $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Γ-invariant then either $T = 0$, or else T is a scalar multiple of a unitary. In particular, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ has a non-zero Γ-invariant operator if and only if π and ρ are isomorphic.*

Proof. Let π and ρ be as above and suppose $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Γ-invariant. Thus, $T^*T \in \mathcal{B}(\mathcal{H})$ is Γ-invariant and hence any spectral projection of T^*T gives a Γ-invariant subspace. Since π is irreducible it then follows that $T^*T \in \mathbb{C}$. If $T^*T \neq 0$ then by multiplying T by a scalar we may assume that T is an isometry. Hence, $TT^* \in \mathcal{B}(\mathcal{K})$ is a non-zero Γ-invariant projection, and since ρ is irreducible it follows that $TT^* = TT^* = 1$. ■

Corollary 7.1.10. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a representation of a discrete group Γ. Then π is irreducible if and only if $\pi(\Gamma)'' = \mathcal{B}(\mathcal{H})$.*

A function $\varphi : \Gamma \to \mathbb{C}$ is a **character**[3] if it is of positive type, is constant on conjugacy classes, and is normalized so that $\varphi(e) = 1$. Characters arise from representations into finite von Neumann algebras. Indeed, if M is a finite von Neumann algebra with a normal faithful trace τ, and if $\pi : \Gamma \to \mathcal{U}(M) \subset \mathcal{U}(L^2(M, \tau))$ is a representation then $\varphi(g) = \tau(\pi(g)) = \langle \pi(g)1_\tau, 1_\tau\rangle$ defines a character on Γ. Conversely, if $\varphi : \Gamma \to \mathbb{C}$ is a character then the cyclic vector ξ in the corresponding GNS-representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ satisfies $\langle \pi(gh)\xi, \xi\rangle = \varphi(gh) = \varphi(hg) = \langle \pi(hg)\xi, \xi\rangle$ for all $g, h \in \Gamma$. Since the linear functional $T \mapsto \langle T\xi, \xi\rangle$ is normal we may then extend this to a normal faithful trace $\tau : \pi(\Gamma)'' \to \mathbb{C}$ by the formula $\tau(x) = \langle x\xi, \xi\rangle$. In particular this shows that $\pi(\Gamma)''$ is finite since it has a normal faithful trace.

If (M_i, τ_i) are finite von Neumann algebras with normal faithful traces τ_i, $i \in \{1, 2\}$, and $\pi_i : \Gamma \to \mathcal{U}(M_i)$ then we will consider π_1 and π_2 to be equivalent if there is a trace preserving automorphism $\alpha : M_1 \to M_2$, such that $\alpha(\pi_1(g)) = \pi_2(g)$ for all $g \in \Gamma$. Clearly, this is equivalent to requiring that there exist a unitary $U : L^2(M_1, \tau_1) \to L^2(M_2, \tau_2)$ such that $U1_{\tau_1} = 1_{\tau_2}$, and $U\pi_1(g) = \pi_2(g)U$ for all $g \in G$.

[3]Some authors use the term character to refer to a homomorphism into the circle $\mathbb{T}$, we will allow a more general definition and refer to homomorphisms into $\mathbb{T}$ as unitary characters.

Note that the space of characters is a convex set, which is closed in the topology of pointwise convergence.

Theorem 7.1.11 (Thoma)**.** *Let Γ be a discrete group. There is a one to one correspondence between:*

1. *Equivalence classes of embeddings $\pi : \Gamma \to \mathcal{U}(M)$ where M is a finite von Neumann algebra with a given normal faithful trace τ, and such that $\pi(\Gamma)'' = M$, and*

2. *Characters $\varphi : \Gamma \to \mathbb{C}$,*

which is given by $\varphi(g) = \tau(\pi(g))$. Moreover, M is a factor if and only if φ is an extreme point in the space of characters.

Proof. The one to one correspondence follows from the discussion preceding the theorem, thus we only need to show the correspondence between factors and extreme points. If $p \in \mathcal{P}(\mathcal{Z}(M))$, is a non-trivial projection then we obtain characters φ_1, and φ_2 by the formulas $\varphi_1(g) = \frac{1}{\tau(p)}\tau(\pi(g)p)$, and $\varphi_2(g) = \frac{1}{\tau(1-p)}\tau(\pi(g)(1-p))$, and we have $\varphi = \tau(p)\varphi_1 + \tau(1-p)\varphi_2$. Since $p \in M = \pi(\Gamma)''$, there exists a sequence $x_n \in \mathbb{C}\Gamma$ such that $\frac{1}{\tau(p)}\tau(\pi(x_n)p) \to 1$, and $\frac{1}{\tau(1-p)}\tau(\pi(x_n)(1-p)) \to 0$, it then follows that $\varphi_1 \neq \varphi_2$ and hence φ is not an extreme point.

Conversely, if $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2)$ with $\varphi_1 \neq \varphi_2$ then if we consider the corresponding representations $\pi_i : \Gamma \to \mathcal{U}(N_i)$, we obtain a trace preserving embedding $\alpha : N \to N_1 \oplus N_2$, which satisfies $\alpha(\pi(g)) = \pi_1(g) \oplus \pi_2(g)$. If we denote by p the projection $1 \oplus 0$ then it need not be the case that $p \in \alpha(N)$, however by considering p we may then define a new trace τ' on N by $\tau'(x) = \frac{1}{4}\tau_1(\alpha(x)p) + \frac{3}{4}\tau_2(\alpha(x)(1-p))$. Since $\varphi_1 \neq \varphi_2$ we must have that $\tau'(\pi(g)) \neq \tau(\pi(g))$ for some $g \in \Gamma$. Thus, N does not have unique trace and so is not a factor by Corollary 6.5.3. ∎

7.2 The group-measure space construction

Let Γ be a discrete group and (X, μ) a σ-finite measure space. An action $\Gamma \curvearrowright (X, \mu)$ is **quasi-invariant** (or non-singular) if for each $g \in \Gamma$, and each measurable set $E \subset X$ we have that gE is also measurable, and $\mu(gE) = 0$ if and only if $\mu(E) = 0$. If in addition we have $\mu(gE) = \mu(E)$ for each $g \in \Gamma$ and each measurable set $E \subset X$, then we say that the action is **measure-preserving**.

If $\Gamma \curvearrowright (X, \mu)$ is quasi-invariant then we have an induced action $\Gamma \curvearrowright^{\sigma} \mathcal{M}(X, \mu)$ on the space of measurable functions $\mathcal{M}(X, \mu)$, given by $\sigma_g(a) = a \circ g^{-1}$ for all $g \in \Gamma$, $a \in \mathcal{M}(X, \mu)$. Note that if $a \in L^\infty(X, \mu)$, then $\|\sigma_g(a)\|_\infty = \|a\|_\infty$, and thus this action restricts to an action also on $L^\infty(X, \mu)$.

For $g \in \Gamma$ the **push-forward measure** $g\mu$ is given by $g\mu(E) = \mu(g^{-1}E)$ for $E \subset X$ measurable. Since the action is quasi-invariant we have $g\mu \prec \mu$ and hence we may consider the Radon-Nikodym derivative $\frac{dg\mu}{d\mu} \in L^1(X,\mu)_+$, which satisfies

$$\int \sigma_{g^{-1}}(a)\, d\mu = \int a\, dg\mu = \int a \frac{dg\mu}{d\mu}\, d\mu,$$

for all $a \in L^\infty(X,\mu)$. Note for $g, h \in \Gamma$, we have a cocycle relation

$$\frac{dgh\mu}{d\mu} = \frac{dg\mu}{d\mu}\frac{dgh}{dg\mu} = \frac{dg\mu}{d\mu}\sigma_g\left(\frac{dh\mu}{d\mu}\right).$$

The **Koopman representation** is the representation $\pi : \Gamma \to \mathcal{U}(L^2(X,\mu))$ given by $\pi_g\xi = (\frac{dg\mu}{d\mu})^{1/2}\sigma_g(\xi)$. Note that this is indeed a unitary representation since

$$\begin{aligned}\langle \pi_g\xi, \pi_g\eta\rangle &= \int \sigma_g(\xi)\overline{\sigma_g(\eta)}\left(\frac{dg\mu}{d\mu}\right) d\mu \\ &= \int \sigma_g(\xi\overline{\eta})\, dg\mu = \int \xi\overline{\eta}\, d\mu = \langle \xi, \eta\rangle,\end{aligned}$$

and,

$$\begin{aligned}\pi_{gh}\xi &= \left(\frac{dgh\mu}{d\mu}\right)^{1/2} \sigma_{gh}(\xi) \\ &= \left(\frac{dg\mu}{d\mu}\right)^{1/2}\left(\sigma_g\left(\frac{dh\mu}{d\mu}\right)\right)^{1/2} \sigma_g(\sigma_h(\xi)) = \pi_g\pi_h\xi.\end{aligned}$$

If $a \in L^\infty(X,\mu)$, $\xi \in L^2(X,\mu)$, and $g \in \Gamma$ then we have

$$\begin{aligned}\pi_g M_a \pi_{g^{-1}}\xi &= \pi_g(a\left(\frac{dg^{-1}\mu}{d\mu}\right)^{1/2} \sigma_{g^{-1}}(\xi)) \\ &= \sigma_g(a)\left(\frac{dg\mu}{d\mu}\sigma_g\left(\frac{dg^{-1}\mu}{d\mu}\right)\right)^{1/2} \xi = M_{\sigma_g(a)}\xi.\end{aligned}$$

Hence $\pi_g M_a \pi_{g^{-1}} = M_{\sigma_g(a)}$, and in particular the action of Γ on the abelian von Neumann algebra $L^\infty(X,\mu)$ is normal.

If we consider the Hilbert space $\mathcal{H} = L^2(X,\mu)\overline{\otimes}\ell^2\Gamma$ then we have a normal representation of $L^\infty(X,\mu)$ on $\mathcal{H}$ given by $a \mapsto M_a \otimes 1 \in \mathcal{B}(\mathcal{H})$. We also may consider the diagonal action of Γ on $\mathcal{H}$ given by $u_g = \pi_g \otimes \lambda_g \in \mathcal{U}(\mathcal{H})$.

The **group-measure space construction** associated to the action $\Gamma \curvearrowright (X,\mu)$ is the von Neumann algebra $L^\infty(X,\mu) \rtimes \Gamma$, generated by all the operators $M_a \otimes 1$, and u_g. We will consider $L^\infty(X,\mu)$ as a von Neumann subalgebra of $L^\infty(X,\mu) \rtimes \Gamma$, and note that we have $u_g a u_{g^{-1}} = \sigma_g(a)$ under this identification. Note also that by Fell's absorption principle we have $\pi \otimes \lambda \sim 1 \otimes \lambda$, and hence it follows that the map $\lambda_g \mapsto u_g$ extends to $L\Gamma$, giving an inclusion $L\Gamma \subset L^\infty(X,\mu) \rtimes \Gamma$.

We will also consider $L^2(X,\mu)$ as a subspace of $L^2(X,\mu)\overline{\otimes}\ell^2\Gamma$ given by the isometry $U\xi = \xi \otimes \delta_e$. We then let $e : L^2(X,\mu)\overline{\otimes}\ell^2\Gamma \to L^2(X,\mu)$ be the orthogonal projection, and we denote by $E : L^\infty(X,\mu) \rtimes \Gamma \to \mathcal{B}(L^2(X,\mu))$ the map $E(x) = exe$.

Lemma 7.2.1. *Suppose $\Gamma \curvearrowright (X,\mu)$ is a quasi-invariant action, and $L^\infty(X,\mu) \rtimes \Gamma$ is the associated group-measure space construction. If E is defined as above then the range of E is contained in $L^\infty(X,\mu)$, and for $x \in L^\infty(X,\mu) \rtimes \Gamma$, $E(x^*x) = 0$ if and only if $x = 0$. Moreover, for $g \in \Gamma$, and $x \in L^\infty(X,\mu) \rtimes \Gamma$, we have $\sigma_g(E(x)) = E(u_g x u_g^*)$.*

Proof. If $x = \sum_{g\in\Gamma} a_g u_g$, where $a_g \in L^\infty(X,\mu)$ with only finitely many non-zero terms then we may compute directly $E(x) = a_e \in L^\infty(X,\mu)$, and in particular $E(u_h x u_h^*) = \sigma_h(a_e) = \sigma_h(E(x))$ for all $h \in \Gamma$. Since E is normal and the algebra generated by $L^\infty(X,\mu)$ and Γ is weak operator topology dense it then follows that the range of E is contained in $L^\infty(X,\mu)$, and $E(u_h x u_h^*) = \sigma_h(E(x))$, for all $x \in L^\infty(X,\mu) \rtimes \Gamma$, and $h \in \Gamma$.

If we consider $h \in \Gamma$ then $(1 \otimes \rho_h^*)e(1 \otimes \rho_h)$ is the orthogonal projection from $L^2(X,\mu)\overline{\otimes}\ell^2\Gamma$ onto $L^2(X,\mu) \otimes \delta_h$, and hence $1 = \sum_{h\in\Gamma}(1 \otimes \rho_h^*)e(1 \otimes \rho_h)$. If $E(x^*x) = 0$, then $xe = 0$, and hence

$$x(1 \otimes \rho_h^*)e(1 \otimes \rho_h) = (1 \otimes \rho_h^*)xe(1 \otimes \rho_h) = 0$$

for every $h \in \Gamma$ (note that $1 \otimes \rho_h \in (L^\infty(X,\mu) \rtimes \Gamma)'$), thus

$$x = x(\sum_{h\in\Gamma}(1 \otimes \rho_h^*)e(1 \otimes \rho_h)) = 0. \qquad \blacksquare$$

If $x \in L^\infty(X,\mu) \rtimes \Gamma$, then as we did for the group von Neumann algebra, we may define the Fourier coefficients $a_g \in L^\infty(X,\mu)$ by $a_g = E(xu_g^*)$. From the previous lemma we have that the Fourier coefficients completely determine the operator x and so we will write $x = \sum_{g\in\Gamma} a_g u_g$. Note that just as in the case for the group von Neumann algebra this summation does not in general converge in an operator space topology. However, it gives us a useful way to view operators in $L^\infty(X,\mu) \rtimes \Gamma$, and this behaves well with respect to multiplication so that we may calculate the Fourier coefficients of a product as

$$(\sum_{g\in\Gamma} a_g u_g)(\sum_{h\in\Gamma} b_h u_h) = \sum_{g\in\Gamma}(\sum_{h\in\Gamma} a_{gh}\sigma_{(gh)^{-1}}(b_h))u_g.$$

Where for each $g \in \Gamma$, $\sum_{h\in\Gamma} a_{gh}\sigma_{(gh)^{-1}}(b_h)$ converges in $L^2(X,\mu)$ to a function in $L^\infty(X,\mu)$.

A quasi-invariant action of a discrete group Γ on (X,μ) is (essentially) **free** if for all $E \subset X$ with $\mu(E) > 0$, and $g \in \Gamma \setminus \{e\}$ there exists $a \in L^\infty(X,\mu)$ such that $(a - \sigma_g(a))1_E \neq 0$. If X is a compact Hausdorff space and μ is a σ-finite Randon measure then it is not hard to see that an action is free if and only if for any $g \in \Gamma \setminus \{e\}$ we have $\mu(\{x \in X \mid gx = x\}) = 0$, or equivalently, the stabilizer subgroup Γ_x is trivial for almost every $x \in X$.

An action is **ergodic** if whenever $E \subset X$ is a measurable subset such that $gE = E$ for all $g \in \Gamma$, then we have $\mu(E) = 0$, or $\mu(X \setminus E) = 0$.

Theorem 7.2.2. *Let $\Gamma \curvearrowright (X,\mu)$ be a quasi-invariant action of a discrete group Γ on a σ-finite measure space (X,μ).*

(*i*) *The action* $\Gamma \curvearrowright (X, \mu)$ *is free if and only if* $L^\infty(X, \mu) \subset L^\infty(X, \mu) \rtimes \Gamma$ *is a maximal abelian subalgebra.*

(*ii*) *If* $L^\infty(X, \mu) \rtimes \Gamma$ *is a factor then* $\Gamma \curvearrowright (X, \mu)$ *is ergodic.*

(*iii*) *If the action* $\Gamma \curvearrowright (X, \mu)$ *is free and ergodic then* $L^\infty(X, \mu) \rtimes \Gamma$ *is a factor.*

Proof. For $g \in \Gamma \setminus \{e\}$ let p_g be the supremum of all projections p in $L^\infty(X, \mu)$ such that $(a - \sigma_g(a))p = 0$ for all $a \in L^\infty(X, \mu)$. If $p_g \neq 0$ for some $g \in \Gamma \setminus \{e\}$ then for all $a \in L^\infty(X, \mu)$ we have $p_g u_g a = \sigma_g(a) p_g u_g = a p_g u_g$, and hence $p_g u_g$ gives a non-trivial element in $L^\infty(X, \mu)'$, showing that $L^\infty(X, \mu)$ is not maximal abelian in $L^\infty(X, \mu) \rtimes \Gamma$. Conversely, if $x \in L^\infty(X, \mu)' \cap (L^\infty(X, \mu) \rtimes \Gamma)$, but $x \notin L^\infty(X, \mu)$, then considering the Fourier decomposition $x = \sum_{g \in \Gamma} a_g u_g$ we must have that $a_g \neq 0$ for some $g \in \Gamma \setminus \{e\}$. Since $ax = xa$ for each $a \in L^\infty(X, \mu)$ we may use the uniqueness for the Fourier decomposition to conclude that $(a - \sigma_g(a))a_g = 0$ for each $a \in L^\infty(X, \mu)$. Hence, the action is not free.

Next, suppose the action is not ergodic, then there exists $E \subset X$ such that $gE = E$ for all $g \in \Gamma$, and $1_E \notin \mathbb{C}$. Then $u_g 1_E u_g^* = \sigma_g(1_E) = 1_E$ for all $g \in \Gamma$ and hence 1_E commutes with $L^\infty(X, \mu)$ and $L\Gamma$. Since these two subalgebras generate $L^\infty(X, \mu) \rtimes \Gamma$ it follows that 1_E is a non-trivial element in the center.

Finally, suppose that the action is free and ergodic and fix p a projection in the center of $L^\infty(X, \mu) \rtimes \Gamma$. Since the action is free we have from the first part that $z \in L^\infty(X, \mu)$. Thus $z = 1_E$ for some measurable subset $E \subset X$. Since 1_E commutes with u_g for each $g \in \Gamma$ we see that $gE = E$ a.e. for each $g \in \Gamma$. By ergodicity we then have either $\mu(E) = 0$ in which case $z = 0$, or $\mu(X \setminus E) = 0$ in which case $z = 1$. ∎

We next turn to the question of the type of $L^\infty(X, \mu) \rtimes \Gamma$. For this we need a lemma which is reminiscent of Dixmier's property, the difference being that we consider only conjugating by unitaries in a subalgebra, and we consider the weak operator topology rather than the norm topology.

Lemma 7.2.3. *Let* M *be a von Neumann algebra and* $A \subset M$ *an abelian von Neumann subalgebra. For each* $x \in M$ *let* $\mathcal{K}_x$ *be the weak operator topology convex closure of* $\{uxu^* \mid u \in \mathcal{U}(A)\}$, *then* $\mathcal{K}_x \cap (A' \cap M) \neq \emptyset$.

Proof. Consider the space $\mathcal{F}$ of all finite dimensional subalgebras of A, directed by inclusion. Note that since A is abelian, if $A_1, A_2 \in \mathcal{F}$, then $(A_1 \cup A_2)'' \in \mathcal{F}$. Also, note that $\cup_{A_0 \in \mathcal{F}} A_0$ is weak operator topology dense in A by the spectral theorem.

Since each $B \in \mathcal{F}$ is finite dimensional $\mathcal{U}(B)$ is a compact group, and if we consider the Haar measure λ_B on $\mathcal{U}(B)$ then we have that $\int uxu^* \, d\lambda_B \in \mathcal{K}_x \cap B'$. Thus, if we denote by $\mathcal{K}_B = \mathcal{K}_x \cap B'$ then $\{\mathcal{K}_B\}_{B \in \mathcal{F}}$ has the finite intersection property, and by weak operator topology compactness we then have $\mathcal{K}_x \cap (A' \cap M) = \cap_{B \in \mathcal{F}} \mathcal{K}_B \neq \emptyset$. ∎

A von Neumann algebra M is **completely atomic** if 1 is an orthogonal sum of minimal projections in M, if M has no minimal projections then M is

diffuse. If (X,μ) is a σ-finite measure space then (X,μ) is completely atomic (resp. diffuse) if $L^\infty(X,\mu)$ is.

Theorem 7.2.4. *Let $\Gamma \curvearrowright (X,\mu)$ be a quasi-invariant free ergodic action of a discrete group Γ on a σ-finite measure space. Then $L^\infty(X,\mu) \rtimes \Gamma$ is*

(i) *type I if and only if (X,μ) is completely atomic;*

(ii) *type II_1 if and only if (X,μ) is diffuse and there exists a Γ-invariant probability measure $\nu \sim \mu$;*

(iii) *type II_∞ if and only if (X,μ) is diffuse and there exists an infinite Γ-invariant σ-finite measure $\nu \sim \mu$;*

(iv) *type III if and only if there is no Γ-invariant σ-finite measure $\nu \sim \mu$.*

Proof. We prove part (iv) first. Suppose first that $\nu \sim \mu$ is a σ-finite Γ-invariant measure. Then we obtain a normal weight on $L^\infty(X,\mu) \rtimes \Gamma$ by the formula $\mathrm{Tr}(x) = \int E(x)\, d\nu$. Note that if $F \subset X$ such that $\nu(F) < \infty$ then $\mathrm{Tr}(1_F x) < \infty$ for all $x \geq 0$, thus Tr is semi-finite. Also, if $\mathrm{Tr}(x^*x) = 0$ then $E(x^*x) = 0$ and hence $x = 0$ by Lemma 7.2.1, thus Tr is faithful. If $x = \sum_{g\in\Gamma} a_g u_g \in L^\infty(X,\mu) \rtimes \Gamma$, then we can compute directly $E(x^*x) = \sum_{h\in\Gamma} \sigma_{h^{-1}}(a_h^* a_h)$,and $E(xx^*) = \sum_{h\in\Gamma} a_h^* a_h$. Since ν is measure preserving we then see that $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$ and hence Tr is a semi-finite normal faithful trace which shows that $L^\infty(X,\mu) \rtimes \Gamma$ is semi-finite by Theorem 6.6.4.

Conversely, if $\mathrm{Tr} : (L^\infty(X,\mu) \rtimes \Gamma)_+ \to [0,\infty]$ is a semi-finite normal faithful trace then by restriction we define a normal faithful weight ω on $L^\infty(X,\mu)$. We claim that ω is again semi-finite. Indeed, if $x_n \in L^\infty(X,\mu) \rtimes \Gamma$ is a net of positive operators increasing to 1 such that $\mathrm{Tr}(x_n) < \infty$, then by Lemma 7.2.3 there exists an increasing net of operators $y_n \in L^\infty(X,\mu)' \cap (L^\infty(X,\mu) \rtimes \Gamma) = L^\infty(X,\mu)$ such that each y_n is in the weak operator topology convex closure of $\{u x_n u^* \mid u \in \mathcal{U}(L^\infty(X,\mu))\}$. Since Tr is normal we then have $\mathrm{Tr}(y_n) = \mathrm{Tr}(x_n) < \infty$, and y_n is increasing to 1, and hence ω is semi-finite. By the Riesz representation theorem there then exists a σ-finite measure $\nu \sim \mu$ such that $\omega(a) = \int a\, d\nu$ for all $a \in L^\infty(X,\mu)_+$. Since Tr is a trace we have that ν is Γ-invariant.

Having established part (iv) we now consider the other parts. Note that in the correspondence describe above we have that ν is finite if and only if $\mathrm{Tr}(1) < \infty$, thus the only thing left to show is that $L^\infty(X,\mu) \rtimes \Gamma$ is completely atomic if and only if $L^\infty(X,\mu)$ is completely atomic. Since we are in the semi-finite case we let Tr be as above.

Suppose that $L^\infty(X,\mu) \rtimes \Gamma$ is completely atomic then from above we have that Tr restricted to $L^\infty(X,\mu)$ is semi-finite and hence $L^\infty(X,\mu)$ has finite projections. Since every finite projection is a finite sum of minimal projections it follows that $L^\infty(X,\mu)$ has a minimal projection p_0. Since the action is free we have that $\sigma_g(p_0)$ is orthogonal to p_0 for all $g \in \Gamma$, and since the action is ergodic we then have $1 = \sum_{g\in\Gamma} \sigma_g(p_0)$, showing that (X,μ) is atomic.

Conversely, if $p_0 \in L^\infty(X, \mu)$ is a minimal projection then we claim that p_0 is also a minimal projection in $L^\infty(X, \mu) \rtimes \Gamma$. Indeed, if $0 \neq q \leq p_0$, then for all $a \in L^\infty(X, \mu)$ we have $aq = (ap_0)q + (a(1 - p_0))q = q(ap_0) = qa$, since $ap_0 \in \mathbb{C}p_0$. By freeness $L^\infty(X, \mu)$ is maximal abelian and hence $q \in L^\infty(X, \mu)$. Thus, $q = p_0$ showing that p_0 is a minimal projection in $L^\infty(X, \mu) \rtimes \Gamma$, which must then by type I. ■

Example 7.2.5. Consider the action $\mathbb{Z} \curvearrowright (\mathbb{T}, \lambda)$ by an irrational rotation. Then this action is clearly measure preserving and free. If $E \subset \mathbb{T}$ were invariant with $\lambda(E) > 0$, then we could consider the measure ν on $\mathbb{T}$ given by $\nu(F) = \frac{1}{\lambda(E)}\lambda(E \cap F)$. since E is invariant we have that ν is an invariant Randon measure on $\mathbb{T}$. Moreover, since the rotation is irrational we see that ν is invariant under a dense subgroup of $\mathbb{T}$ and hence it is invariant for all of $\mathbb{T}$. Uniqueness of the Haar measure implies $\nu = \lambda$ and hence $\lambda(E) = 1$. Thus, $\mathbb{Z} \curvearrowright (\mathbb{T}, \lambda)$ is ergodic and $L^\infty(\mathbb{T}, \lambda) \rtimes \mathbb{Z}$ is a II_1 factor.

Example 7.2.6. Consider the action $\mathbb{Q} \curvearrowright (\mathbb{R}, \lambda)$ by addition. Then this is measure preserving and free, and again uniqueness of the Haar measure up to scaling implies that this action is ergodic. Thus, $L^\infty(\mathbb{R}, \lambda) \rtimes \mathbb{Q}$ is a II_∞ factor.

Example 7.2.7. Consider the action $\mathbb{Q} \rtimes \mathbb{Q}^* \curvearrowright (\mathbb{R}, \lambda)$ where $\mathbb{Q}$ acts by addition, and $\mathbb{Q}^*$ acts by multiplication. This is (essentially) free, and is ergodic since it is ergodic when restricted to $\mathbb{Q}$. Moreover, if $\nu \sim \lambda$ were a σ-finite invariant measure then ν would be invariant under the action of $\mathbb{Q}$ and hence be a multiple of Haar measure. But then it would not be preserved by $\mathbb{Q}^*$. Thus $\mathbb{Q} \rtimes \mathbb{Q}^* \curvearrowright (\mathbb{R}, \lambda)$ has no σ-finite invariant measure and so $L^\infty(\mathbb{R}, \lambda) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}^*)$ is a type III factor.

Chapter 8

Completely positive maps

An **operator system** E is a closed self adjoint subspace of a unital C^*-algebra A such that $1 \in E$. We denote by $\mathbb{M}_n(E)$ the space of $n \times n$ matrices over E. If A is a C^*-algebra, then $\mathbb{M}_n(A) \cong A \otimes \mathbb{M}_n(\mathbb{C})$ has a unique norm for which it is again a C^*-algebra, where the adjoint given by $[a_{i,j}]^* = [a_{j,i}^*]$. This can be seen easily for C^*-subalgebras of $\mathcal{B}(\mathcal{H})$, and by Corollary 2.3.7 every C^*-algebra is isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. In particular, if E is an operator system then $\mathbb{M}_n(E)$ is again an operator system when viewed as a subspace of the C^*-algebra $\mathbb{M}_n(A)$.

As was the case for C^*-algebras, a bounded linear functional $\varphi \in E^*$ is a **state** if it is positive and satisfies $\varphi(1) = 1$. We denote by $S(E)$ the state space of E which is a compact subset of E^* when given the weak$*$-topology. We remark that the proof in Lemma 2.3.2 shows that a linear functional $\varphi \in E^*$ is positive if and only if $\varphi(1) = \|\varphi\|$. In particular, it follows form the Hahn-Banach theorem that any positive linear functional on E extends to a positive linear functional on A which has the same norm.

8.1 Function systems

If X is a locally convex topological vector space and $A \subset X$ is a closed convex subset, we let $\mathrm{Aff}(A)$ denote the space of continuous complex valued affine functions on A, i.e., a continuous function $f : A \to \mathbb{C}$ is in $\mathrm{Aff}(A)$ if and only if $f(\sum_{i=1}^n t_i x_i) = \sum_{i=1}^n t_i f(x_i)$ whenever $t_1, \ldots, t_n \in [0,1]$ satisfy $\sum_{i=1}^n t_i = 1$.

Proposition 8.1.1. *Let X be a locally convex topological vector space and suppose that $K \subset X$ is a non-empty compact convex subset. Consider the operator system $\mathrm{Aff}(K) \subset C(K)$. Then the map $\mu : K \to S(\mathrm{Aff}(K))$ given by $\mu(k)(f) = f(k)$ is an affine homeomorphism.*

Proof. Clearly μ is continuous, affine, and injective. If $\varphi \in S(\mathrm{Aff}(K))$ then we may extend φ to a state on $C(K)$ and hence by the Riesz representation theorem there is a probability measure $\nu \in \mathrm{Prob}(K)$ so that $\varphi(f) = \int f \, d\nu$ for all $f \in E$.

Since f is continuous and affine we then have $\varphi(f) = \int f\, d\nu = f(\mathrm{bar}(\nu))$ where $\mathrm{bar}(\nu)$ denotes the barycenter of ν. Hence, μ is surjective. ■

Lemma 8.1.2. *If X is a locally convex topological vector space, and $f \in \mathrm{Aff}(X;\mathbb{R})$ then there exist $\varphi \in X^*$ and $t_0 \in \mathbb{R}$ such that $f(x) = \varphi(x) + t_0$ for all $x \in X$.*

Proof. Considering the affine function $X \ni x \mapsto f(x) - f(0)$ we may assume $f(0) = 0$. If $t \in [0,1]$ and $x \in X$ we have

$$f(tx) = f(tx + (1-t)0) = tf(x) + (1-t)f(0) = tf(x).$$

It then follows that $f(tx) = tf(x)$ for all $t \in \mathbb{R}$. If $x, y \in X$, then

$$f(x+y) = 2f\left(\frac{1}{2}x + \frac{1}{2}y\right) = 2\left(\frac{1}{2}f(x) + \frac{1}{2}f(y)\right) = f(x) + f(y).$$

Hence, $f \in X^*$. ■

Lemma 8.1.3. *Let X be a real locally convex topological vector space, and suppose $K \subset X$ is a non-empty compact convex subset. Let $f : K \to (0,\infty)$ be a continuous affine function. Consider the set*

$$G = \{g \in \mathrm{Aff}(X;\mathbb{R}) \mid g(k) < f(k),\ \ k \in K\}.$$

1. *For all $\varepsilon > 0$ and $k \in K$ there exists $g \in G$ such that $f(k) - g(k) < \varepsilon$.*
2. *If $g_1, g_2 \in G$ then there exists $g \in G$ such that $\max\{g_1(k), g_2(k)\} < g(k)$, for all $k \in K$.*

Proof. Set $M = \{(k,t) \mid t \leq f(k)\}$, so that M is a closed convex subset of $X \times \mathbb{R}$.

Suppose first that $k \in K$ and $\varepsilon > 0$. Then $(k, f(k)-\varepsilon) \notin M$ and hence by the Hahn-Banach separation theorem there exists a closed hyperspace $H \subset X \times \mathbb{R}$ which strictly separates $(k, f(k) - \varepsilon)$ from M. Clearly H is not of the form $X_0 \times \mathbb{R}$ for some hyperspace $X_0 \subset X$, and hence H is the graph of a continuous affine map $g \in G$, such that $g(k) > f(k) - \varepsilon$.

For the second assertion we fix $\beta = \inf\{g_1(k), g_2(k) \mid k \in K\}$ and set $M_i = \{(k,t) \mid \beta \leq t \leq g_i(k)\}$, which is a compact convex set for $i = 1, 2$. By hypothesis we have $M_i \cap M = \emptyset$ and since f is affine we then have $M \cap (\mathrm{co}(M_1, M_2)) = \emptyset$. Just as above we may then use the Hahn-Banach separation theorem to strictly separate M from $(\mathrm{co}(M_1, M_2))$ and this then gives a continuous affine map $g \in G$ such that $\max\{g_1(k), g_2(k)\} < g(k)$ for all $k \in K$. ■

Theorem 8.1.4 (Kadison)**.** *Suppose K is a compact Hausdorff space and $E \subset C(K)$ is an operator system, then the map $\Gamma : E \to \mathrm{Aff}(S(E))$ given by $\Gamma(a)(f) = f(a)$ is an isometric surjection that preserves positivity.*

Proof. We clearly have that Γ preserves positivity. If $a \in E$ then

$$\|\Gamma(a)\| = \sup_{\varphi \in S(E)} |\varphi(a)| = \sup_{\mu \in \mathrm{Prob}(K)} \left| \int a \, d\mu \right| = \|a\|,$$

hence Γ is isometric. In particular it follows that $\Gamma(E) \subset \mathrm{Aff}(S(E))$ is closed.

If $g \in \mathrm{Aff}(E^*; \mathbb{R})$, then by Lemma 8.1.2 there exists $a \in E$ and $t_0 \in \mathbb{R}$ so that $g(\varphi) = \varphi(a) + t_0$ for all $\varphi \in E^*$, and hence $g(\varphi) = \varphi(a + t_0) = \Gamma(a + t_0)(\varphi)$ for all $\varphi \in S(E)$. As $\Gamma(E)$ is closed, to show that Γ is surjective it then suffices to show that any affine map $f \in \mathrm{Aff}(S(E); \mathbb{R})$ can be uniformly approximated on $S(E)$ by affine maps in $\mathrm{Aff}(E^*; \mathbb{R})$. It also suffices to consider the case when $f : S(E) \to (0, \infty)$ and this then follows from Lemma 8.1.3, together with Dini's theorem. ∎

If $\phi : E \to F$ is a linear map between operator systems, then we denote by $\phi^{(n)} : \mathbb{M}_n(E) \to \mathbb{M}_n(F)$ the map defined by $\phi^{(n)}([a_{i,j}]) = [\phi(a_{i,j})]$. We say that ϕ is **positive** if $\phi(a) \geq 0$, whenever $a \geq 0$. If $\phi^{(n)}$ is positive then we say that ϕ is **n-positive** and if ϕ is n-positive for every $n \in \mathbb{N}$ then we say that ϕ is **completely positive**. If A and B are unital and $\phi : A \to B$ such that $\phi(1) = 1$ then we say that ϕ is unital.

Lemma 8.1.5. *A matrix $a = [a_{i,j}] \in \mathbb{M}_n(A)$ is positive if and only if*

$$\sum_{i,j=1}^{n} x_i^* a_{i,j} x_j \geq 0,$$

for all $x_1, \ldots, x_n \in A$.

Proof. For all $x_1, \ldots, x_n \in A$ we have that $\sum_{i,j=1}^n x_i^* a_{i,j} x_j$ is the conjugation of a by the $1 \times n$ column matrix with entries $x_1, \ldots, x_n$, hence if a is positive then so is $\sum_{i,j=1}^n x_i^* a_{i,j} x_j$.

Conversely, if $\sum_{i,j=1}^n x_i^* a_{i,j} x_j \geq 0$, for all $x_1, \ldots, x_n$ then for any representation $\pi : A \to \mathcal{B}(\mathcal{H})$, and $\xi \in \mathcal{H}$ we have

$$\begin{aligned} \left\langle (\mathrm{id} \otimes \pi)(a) \begin{pmatrix} \pi(x_1)\xi \\ \vdots \\ \pi(x_n)\xi \end{pmatrix}, \begin{pmatrix} \pi(x_1)\xi \\ \vdots \\ \pi(x_n)\xi \end{pmatrix} \right\rangle &= \sum_{i,j=1}^{n} \langle \pi(a_{i,j})\pi(x_j)\xi, \pi(x_i)\xi \rangle \\ &= \left\langle \pi\left(\sum_{i,j=1}^{n} x_i^* a_{i,j} x_j \right) \xi, \xi \right\rangle \geq 0. \end{aligned}$$

Thus, if $\mathcal{H}$ has a cyclic vector, then $(\mathrm{id} \otimes \pi)(a) \geq 0$. But since every representation is decomposed into a direct sum of cyclic representations it then follows that $(\mathrm{id} \otimes \pi)(a) \geq 0$ for any representation, and hence $a \geq 0$ by considering a faithful representation. ∎

Proposition 8.1.6. *Let E be an operator system, and let B be an abelian C^*-algebras. If $\phi : E \to B$ is positive, then ϕ is completely positive.*

Proof. Since B is commutative we may assume $B = C_0(X)$ for some locally compact Hausdorff space X. If $a = [a_{i,j}] \in \mathbb{M}_n(E)$ such that $a \geq 0$, then for all $x_1, \ldots, x_n \in B$, and $\omega \in X$ we have

$$\left(\sum_{i,j} x_i^* \phi(a_{i,j}) x_j\right)(\omega) = \left(\sum_{i,j} \phi(\overline{x_i(\omega)} x_j(\omega) a_{i,j})\right)(\omega)$$
$$= \phi\left(\begin{pmatrix} x_1(\omega) \\ \vdots \\ x_n(\omega) \end{pmatrix}^* a \begin{pmatrix} x_1(\omega) \\ \vdots \\ x_n(\omega) \end{pmatrix}\right)(\omega) \geq 0.$$

By Lemma 8.1.5, and since n was arbitrary, we then have that ϕ is completely positive. ■

Example 8.1.7. Let E be an operator system and K a compact Hausdorff space. If $\pi : K \to S(E)$ is a continuous map, then we obtain a unital (completely) positive map $\phi : E \to C(K)$ given by

$$\phi(a)(x) = \pi(x)(a), \qquad a \in E, x \in K.$$

Conversely, this formula also defines a continuous map $\pi : K \to S(E)$ whenever $\phi : E \to C(K)$ is unital (completely) positive.

Similarly, if (X, μ) is σ-finite measure space and $\pi : X \to S(E)$ is a measurable map, then we obtain a unital (completely) positive map $\phi : E \to L^\infty(X, \mu)$ given by the same formula. Less obvious is that if E is separable then every unital (completely) positive map from E into $L^\infty(X, \mu)$ arrises in this way. A proof of this may be adapted from the proof of Theorem 3.9.14.

Proposition 8.1.8. *Let A and B be unital C^*-algebras such that A is abelian. If $\phi : A \to B$ is positive, then ϕ is completely positive.*

Proof. We may identify A with $C(K)$ for some compact Hausdorff space K, hence for $n \in \mathbb{N}$ we may identify $\mathbb{M}_n(A)$ with $C(K, \mathbb{M}_n(\mathbb{C}))$ where the norm is given by $\|f\| = \sup_{k \in K} \|f(k)\|$.

Suppose $f \in C(K, \mathbb{M}_n(\mathbb{C}))$ is positive, with $\|f\| \leq 1$, and let $\varepsilon > 0$ be given. Since K is compact f is uniformly continuous and hence there exists a finite open cover $\{U_1, U_2, \ldots, U_m\}$ of K, and $a_1, a_2, \ldots, a_m \in \mathbb{M}_n(\mathbb{C})_+$ such that $\|f(k) - a_j\| \leq \varepsilon$ for all $k \in U_j$.

For each $j \leq m$ chose $g_j \in C(K)$ such that $0 \leq g_j \leq 1$, $\sum_{j=1}^m g_j = 1$, and ${g_j}_{|U_j^c} = 0$. If we consider $f_0 = \sum_{j=1}^m g_j a_j$ then we have $\|f - f_0\| \leq \varepsilon$. Therefore we have $\|\phi^{(n)}(f) - \phi^{(n)}(f_0)\| \leq \|\phi^{(n)}\|\varepsilon$.

Since $\phi^{(n)}(g_j a_j) = \phi(g_j) a_j \geq 0$, for all $1 \leq j \leq m$, we have that $\phi^{(n)}(f_0) \geq 0$, and hence since $\varepsilon > 0$ was arbitrary it follows that $\phi^{(n)}(f) \geq 0$, and it follows that ϕ is completely positive. ■

Example 8.1.9. Let K be a locally compact Hausdorff space and let $\mathcal{H}$ be a Hilbert space. An **operator valued measure** E on K relative to $\mathcal{H}$ is a mapping from the Borel subsets of K to $\mathcal{B}(\mathcal{H})$ such that

(i) $E(\emptyset) = 0, E(K) = 1$.

(ii) $E(B_1 \cup B_2) = E(B_1) + E(B_2)$ for all disjoint Borel sets B_1 and B_2.

(iii) For all $\xi, \eta \in \mathcal{H}$ the function

$$B \mapsto E_{\xi,\eta}(B) = \langle E(B)\xi, \eta \rangle$$

is a finite Radon measure on K.

We say that E is **positive** if $E(B) \geq 0$, for all $B \subset K$ Borel.

If $\tilde{E}$ is a spectral measure on K relative to $\mathcal{K}$ and $V : \mathcal{H} \to \mathcal{K}$ is an isometry, then we may define a positive operator valued measure E on K relative to $\mathcal{H}$ by setting $E(B) = V^*\tilde{E}(B)V$ for all $B \subset K$ Borel.

As was the case for spectral measures, if we are given an operator valued measure E on K relative to $\mathcal{H}$ we may define a linear map $f \mapsto \int f \, \mathrm{d}E$ from $B_\infty(K)$ into $\mathcal{B}(\mathcal{H})$ so that for all $f \in B_\infty(K)$ and $\xi, \eta \in \mathcal{H}$ we have $\left\langle \left(\int f \, \mathrm{d}E\right) \xi, \eta \right\rangle = \int f(x) \, \mathrm{d}E_{\xi,\eta}(x)$. Note that if E is positive, then $E_{\xi,\xi}$ is a probability measure for each $\xi \in \mathcal{H}$. It then follows that $\int f \, \mathrm{d}E \geq 0$, whenever $f \geq 0$ and hence by the previous proposition we have that $f \mapsto \int f \, \mathrm{d}E$ is a completely positive map.

Note that just as in the case of states, if $\phi : E \to F$ is positive then $\phi(x^*) = \phi(x)^*$, for all $x \in E$. Also note that positive maps are continuous. Indeed, if $\phi : E \to F$ is positive and $\{x_n\}_n$ is a sequence which converges to 0 in E, such that $\lim_{n\to\infty} \phi(x_n) = y$, then since $\omega \circ \phi$ is positive (and hence continuous) for any state $\omega \in S(B)$ we have $\omega(y) = 0$, Proposition 2.3.5 then gives that $y = 0$. The closed graph theorem then shows that ϕ is bounded.

Lemma 8.1.10. *If E is an operator system, B is a unital C^*-algebras, and $\phi : E \to B$ is a unital contraction then ϕ is positive.*

Proof. Suppose $a \in E_+$, and $\omega \in B^*$ is a state. Then $\omega \circ \phi$ is a linear functional which satisfies $\|\omega \circ \phi\| \leq 1$ and $\omega \circ \phi(1) = 1$, hence, $\omega \circ \phi$ is a state on E, so that $\omega \circ \phi(a) \geq 0$. Since ω was an arbitrary state Proposition 2.3.4 then shows that $\phi(a) \geq 0$, hence ϕ is positive. ■

The following proposition follows easily from Proposition **??**.

Proposition 8.1.11. *Let M and N be von Neumann algebras, and $\phi : M \to N$ a positive map, then the following conditions are equivalent.*

(i) *ϕ is normal.*

(ii) *For any bounded increasing net $\{x_i\}_i$ we have $\phi(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} \phi(x_i)$ where the limits are taken in the strong operator topologies.*

(iii) *For any family $\{p_i\}_i$ of pairwise orthogonal projections we have $\phi(\sum_i p_i) = \sum_i \phi(p_i)$.*

8.2 Dilation theorems

8.2.1 Stinespring's Dilation Theorem

If $\pi : A \to \mathcal{B}(\mathcal{K})$ is a representation of a C^*-algebra A and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the operator $\phi : A \to \mathcal{B}(\mathcal{H})$ given by $\phi(x) = V^* \pi(x) V$ is completely positive. Indeed, if we consider the operator $V^{(n)} \in \mathcal{B}(\mathcal{H}^{\oplus n}, \mathcal{K}^{\oplus n})$ given by $V^{(n)}((\xi_i)_i) = (V\xi_i)_i$ then for all $x \in \mathbb{M}_n(A)$ we have

$$\begin{aligned}\phi^{(n)}(x^*x) &= {V^{(n)}}^* \pi^{(n)}(x^*x) V^{(n)} \\ &= (\pi^{(n)}(x)V^{(n)})^*(\pi^{(n)}(x)V^{(n)}) \geq 0.\end{aligned}$$

Generalizing the GNS construction Stinespring showed that every completely positive map from A to $\mathcal{B}(\mathcal{H})$ arises in this way.

Theorem 8.2.1 (Stinespring's dilation theorem)**.** *Let A be a unital C^*-algebra, and suppose $\phi : A \to \mathcal{B}(\mathcal{H})$, then ϕ is completely positive if and only if there exists a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ and a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\phi(x) = V^*\pi(x)V$. We also have $\|\phi\| = \|V\|^2$, and if ϕ is unital then V is an isometry. Moreover, if A is a von Neumann algebra and ϕ is a normal completely positive map, then π is a normal representation.*

Proof. Consider the sesquilinear form on $A \otimes \mathcal{H}$ given by $\langle a \otimes \xi, b \otimes \eta \rangle_\phi = \langle \phi(b^*a)\xi, \eta \rangle$, for $a, b \in A$, $\xi, \eta \in \mathcal{H}$. If $(a_i)_i \in A^{\oplus n}$, and $(\xi_i)_i \in \mathcal{H}^{\oplus n}$, then we have

$$\begin{aligned}\langle \sum_i a_i \otimes \xi_i, \sum_j a_j \otimes \xi_j \rangle_\phi &= \sum_{i,j} \langle \phi(a_j^* a_i)\xi_i, \xi_j \rangle \\ &= \langle \phi((a_i)_i^*(a_i)_i)(\xi_i)_i, (\xi_i)_i \rangle \geq 0.\end{aligned}$$

Thus, this form is non-negative definite and we can consider N_ϕ the kernel of this form so that $\langle \cdot, \cdot \rangle_\phi$ is positive definite on $\mathcal{K}_0 = (A \otimes \mathcal{H})/N_\phi$. Hence, we can take the Hilbert space completion $\mathcal{K} = \overline{\mathcal{K}_0}$.

As in the case of the GNS construction, we define a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ by first setting $\pi_0(x)(a \otimes \xi) = (xa) \otimes \xi$ for $a \otimes \xi \in A \otimes \mathcal{H}$. Note that since ϕ is positive we have $\phi(a^*x^*xa) \leq \|x\|^2 \phi(a^*a)$, applying this to $\phi^{(n)}$ we see that $\|\pi_0(x) \sum_i a_i \otimes \xi_i\|_\phi^2 \leq \|x\|^2 \| \sum_i a_i \otimes \xi_i\|_\phi^2$. Thus, $\pi_0(x)$ descends to a well defined bounded map on $\mathcal{K}_0$ and then extends to a bounded operator $\pi(x) \in \mathcal{B}(\mathcal{K})$.

If we define $V_0 : \mathcal{H} \to \mathcal{K}_0$ by $V_0(\xi) = 1 \otimes \xi$, then we see that V_0 is bounded by $\|\phi(1)\|$ and hence extends to a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. For any $x \in A$, $\xi, \eta \in \mathcal{H}$ we then check that

$$\begin{aligned}\langle V^*\pi(x)V\xi, \eta \rangle &= \langle \pi(x)(1 \otimes \xi), 1 \otimes \eta \rangle_\phi \\ &= \langle x \otimes \xi, 1 \otimes \eta \rangle_\phi = \langle \phi(x)\xi, \eta \rangle.\end{aligned}$$

Thus, $\phi(x) = V^*\pi(x)V$ as claimed. ∎

Corollary 8.2.2 (Naimark's dilation theorem)**.** *Let K be a locally compact Hausdorff space and E a positive operator valued measure on K relative to $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K}$, a spectral measure $\tilde{E}$ on K relative to $\mathcal{K}$, and an isometry $V : \mathcal{H} \to \mathcal{K}$ so that $E(B) = V^*\tilde{E}(B)V$ for all $B \subset K$ Borel.*

Proof. By considering the one point compactification of K it is easy to see that it is enough to consider the case when K is compact. If E is a positive operator valued measure on K relative to $\mathcal{H}$ then we saw from Example 8.1.9 that the map $f \mapsto \int f \, \mathrm{d}E$ is unital and completely positive from $C(K)$ into $\mathcal{B}(\mathcal{H})$. By Stinespring's dilation theorem it then follows that there exists a Hilbert space $\mathcal{K}$, a unital $*$-representation $\pi : C(K) \to \mathcal{B}(\mathcal{K})$, and an isometry $V : \mathcal{H} \to \mathcal{K}$ so that $\int f \, \mathrm{d}E = V^*\pi(f)V$ for all $f \in C(K)$. If we let $\tilde{E}$ be the spectral measure on K relative to $\mathcal{K}$ which is given from π by the spectral theorem, then it is easy to see that we have $E(B) = V^*\tilde{E}(B)V$ for all Borel sets $B \subset K$. ■

Corollary 8.2.3 (Kadison's inequality)**.** *If A and B are unital C^*-algebras, and $\phi : A \to B$ is unital compleley positive then for all $x \in A$ we have $\phi(x)^*\phi(x) \leq \phi(x^*x)$*

Proof. By Corollary 2.3.7 we may assume that $B \subset \mathcal{B}(\mathcal{H})$. If we consider the Stinespring dilation $\phi(x) = V^*\pi(x)V$, then since ϕ is unital we have that V is an isometry. Hence $1 - VV^* \geq 0$ and so we have

$$\begin{aligned} \phi(x^*x) - \phi(x)^*\phi(x) &= V^*\pi(x^*x)V - V^*\pi(x)^*VV^*\pi(x)V \\ &= V^*\pi(x^*)(1 - VV^*)\pi(x)V \geq 0. \end{aligned}$$ ■

Proposition 8.1.8 gives us the following strengthening of Kadison's inequality for normal operators:

Corollary 8.2.4 (Kadison's inequality for positive maps)**.** *Let A and B be unital C^*-algebras, and $\phi : A \to B$ a unital positive map. Then for all $x \in A$ normal we have $\phi(x)^*\phi(x) \leq \phi(x^*x)$.*

Proof. Restricting ϕ to the abelian unital C^*-algebra generated by x we may then assume, by Proposition 8.1.8, that ϕ is completely positive. Hence this follows from Kadison's inequality for completely positive maps. ■

Corollary 8.2.5 (Kadison's inequality for 2-positive maps)**.** *Let A and B be unital C^*-algebras, and $\phi : A \to B$ is a unital 2-positive map, then for all $x \in A$ we have $\phi(x)^*\phi(x) \leq \phi(x^*x)$.*

Proof. Fix $x \in A$. Since ϕ is 2-positive and since $\left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(A)$ is self-adjoint we may apply the previous corollary to deduce

$$\begin{pmatrix} \phi(x)\phi(x)^* & 0 \\ 0 & \phi(x)^*\phi(x) \end{pmatrix} = \begin{pmatrix} 0 & \phi(x) \\ \phi(x)^* & 0 \end{pmatrix} \begin{pmatrix} 0 & \phi(x) \\ \phi(x)^* & 0 \end{pmatrix} \leq \begin{pmatrix} \phi(xx^*) & 0 \\ 0 & \phi(x^*x) \end{pmatrix},$$

and the result then follows. ■

Lemma 8.2.6. *Let A be a unital C^*-algebra. An element of the form $\begin{pmatrix} a & x \\ y^* & b \end{pmatrix} \in \mathbb{M}_2(A)$ is positive if and only if a and b are positive, $x = y$, and for all $\varepsilon > 0$ we have $\|(a+\varepsilon)^{-1/2} x (b+\varepsilon)^{-1/2}\| \leq 1$.*

Proof. We may assume $A \subset \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Since positive elements are self-adjoint, if $\begin{pmatrix} a & x \\ y^* & b \end{pmatrix} \geq 0$, then we have $x = y$, $a = a^*$, and $\langle a\xi, \xi \rangle = \langle \begin{pmatrix} a & x \\ y^* & b \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} \xi \\ 0 \end{pmatrix} \rangle \geq 0$, for all $\xi \in \mathcal{H}$. Hence $a \geq 0$. A similar argument shows that we also have $b \geq 0$.

If $a, b \geq 0$ are invertible, then $\begin{pmatrix} a & x \\ x^* & b \end{pmatrix}$ is positive if and only if

$$\begin{pmatrix} 1 & a^{-1/2} x b^{-1/2} \\ b^{-1/2} x^* a^{-1/2} & 1 \end{pmatrix} = \begin{pmatrix} a^{-1/2} & 0 \\ 0 & b^{-1/2} \end{pmatrix} \begin{pmatrix} a & x \\ y & b \end{pmatrix} \begin{pmatrix} a^{-1/2} & 0 \\ 0 & b^{-1/2} \end{pmatrix}$$

is positive. Also, if $a \geq 0$ then $a + \varepsilon$ is invertible for all $\varepsilon > 0$. Thus, to finish the lemma it suffices to show that an operator of the form $\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}$ is positive if and only if $\|x\| \leq 1$. To see this we observe that

$$\langle \begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} \begin{pmatrix} -\xi \\ \eta \end{pmatrix}, \begin{pmatrix} -\xi \\ \eta \end{pmatrix} \rangle = \|\xi\|^2 + \|\eta\|^2 - 2\mathrm{Re}(\langle x\eta, \xi \rangle),$$

so that $\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}$ is positive if and only if for all $\xi, \eta \in \mathcal{H}$ we have

$$2\mathrm{Re}(\langle x\eta, \xi \rangle) \leq \|\xi\|^2 + \|\eta\|^2.$$

If x is a contraction then this holds due to Cauchy-Schwarz, and conversely, if this inequality holds, then taking $\xi = x\eta$ we see that x is a contraction. ∎

Corollary 8.2.7. *Let A be a C^*-algebra, if $\begin{pmatrix} 0 & x^* \\ x & b \end{pmatrix} \in \mathbb{M}_2(A)$ is positive, then $x = 0$, and $b \geq 0$.*

Proof. Applying the previous lemma, $\begin{pmatrix} 0 & x^* \\ x & b \end{pmatrix}$ is positive if and only if $b \geq 0$ and for each $\varepsilon > 0$ we have $\|x(b+\varepsilon)^{-1/2}\| \leq \varepsilon^{1/2}$. This then implies that for all $\varepsilon > 0$ we have

$$\|x\| \leq \|x(b+\varepsilon)^{-1/2}\| \|(b+\varepsilon)^{1/2}\| \leq \varepsilon^{1/2}(\|b\| + \varepsilon)^{1/2}.$$

Hence, $x = 0$. ∎

Theorem 8.2.8 (Choi)**.** *If $\phi : A \to B$ is a unital 2-positive map between C^*-algebras, then for $a \in A$ we have $\phi(a^*a) = \phi(a^*)\phi(a)$ if and only if $\phi(xa) = \phi(x)\phi(a)$, and $\phi(a^*x) = \phi(a^*)\phi(x)$, for all $x \in A$.*

Proof. Applying Kadison's inequality to $\phi^{(2)}$ it follows that for all $x \in A$ we have

$$\begin{pmatrix} \phi(a^*a) & \phi(a^*x) \\ \phi(x^*a) & \phi(aa^* + x^*x) \end{pmatrix} = \phi^{(2)} \left(\left| \begin{pmatrix} 0 & a^* \\ a & x \end{pmatrix} \right|^2 \right) \geq \left| \phi^{(2)} \left(\begin{pmatrix} 0 & a^* \\ a & x \end{pmatrix} \right) \right|^2$$
$$= \begin{pmatrix} \phi(a^*)\phi(a) & \phi(a^*)\phi(x) \\ \phi(x^*)\phi(a) & \phi(a)\phi(a^*) + \phi(x^*)\phi(x) \end{pmatrix}.$$

Since $\phi(a^*a) = \phi(a)^*\phi(a)$ it follows from the previous corollary that $\phi(x^*a) = \phi(x^*)\phi(a)$, and $\phi(a^*x) = \phi(a)^*\phi(x)$. ∎

If $\phi : A \to B$ is completely positive, then the **multiplicative domain** of ϕ is

$$\{a \in A \mid \phi(a^*a) = \phi(a^*)\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a^*)\}.$$

Note that by Theorem 8.2.8 the multiplicative domain is a C^*-subalgebra of A, and ϕ restricted to the multiplicative domain is a homomorphism.

Corollary 8.2.9. *If A is a unital C^*-algebra, $\phi : A \to A$ is unital and 2-positive, and $B \subset A$ is a C^*-subalgebra such that $\phi(b) = b$ for all $b \in B$ then ϕ is B-bimodular, i.e., for all $x \in A$, $b_1, b_2 \in B$ we have $\phi(b_1 x b_2) = b_1 \phi(x) b_2$.*

Theorem 8.2.10 (Choi)**.** *If A and B are unital C^*-algebras, and $\phi : A \to B$ is a unital 2-positive isometry onto B, then ϕ is a $*$-isomorphism.*

Proof. Since a self-adjoint element x of norm at most 1 in a unital C^*-algebra is positive if and only if $\|1 - x\| \leq 1$ it follows that ϕ^{-1} is positive.

Fix $a \in A$ self adjoint, and assume $\|a\| \leq 1$. Since ϕ is onto there exists $b \in A$ such that $\phi(b) = \phi(a)^2 \leq \phi(a^2)$.

Thus $b \leq a^2$, and since ϕ^{-1} is also positive we may apply the previous corollary to the map ϕ^{-1} to conclude that

$$a^2 = \phi^{-1}(\phi(a))\phi^{-1}(\phi(a)) \leq \phi^{-1}(\phi(a)^2) = b.$$

Hence, $\phi(a)^2 = \phi(b) = \phi(a^2)$.

Since a was an arbitrary self adjoint element, and since A is generated by its self adjoint elements, Theorem 8.2.8 then shows that ϕ is an isomorphism. ∎

Exercise 8.2.11. Show that a C^*-algebra A is abelian if and only if for any C^*-algebra B, every positive map from B to A is completely positive.

8.2.2 Bhat's Dilation Theorem

Lemma 8.2.12. *If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and $V : \mathcal{H} \to \mathcal{K}$ is a partial isometry, then for $A \subset \mathcal{B}(\mathcal{H})$, $B \subset \mathcal{B}(\mathcal{K})$, we have that $V^* \, *\text{-alg}(VBV^*, A)V = *\text{-alg}(B, V^*AV)$.*

Proof. Using the fact that $V^*V = 1$, this follows easily by induction on the length of alternating products for monomials in VBV^*, and A. ∎

If $A_0 \subset \mathcal{B}(\mathcal{H}_0)$ is a C^*-algebra, and $\phi : A_0 \to A_0$ is a unital completely positive map, then one can iterate Stinespring's dilation as follows:

Lemma 8.2.13. *Suppose $A_0 \subset \mathcal{B}(\mathcal{H}_0)$ is a unital C^*-algebra, and $\phi_0 : A_0 \to A_0$ is a unital completely positive map. Then there exists a sequence whose entries consist of:*

(1) *a Hilbert space $\mathcal{H}_n$;*

(2) *an isometry $V_n : \mathcal{H}_{n-1} \to \mathcal{H}_n$;*

(3) *a unital C^*-algebra $A_n \subset \mathcal{B}(\mathcal{H}_n)$;*

(4) *a unital representation* $\pi_n : A_{n-1} \to \mathcal{B}(\mathcal{H}_n)$*, such that* $\pi_n(A_{n-1})$*, and* $V_n A_{n-1} V_n^*$ *generate* A_n*;*

(5) *a unital completely positive map* $\phi_n : A_n \to A_n$*;*

such that the following relationships are satisfied for each $n \in \mathbb{N}$*,* $x \in A_{n-1}$*:*

$$V_n^* \pi_n(x) V_n = \phi_{n-1}(x); \tag{8.1}$$

$$V_n^* A_n V_n = A_{n-1}; \tag{8.2}$$

$$\phi_n(\pi_n(x)) = \pi_n(\phi_{n-1}(x)); \tag{8.3}$$

$$\pi_{n+1}(V_n x V_n^*) = V_{n+1} \pi_n(x) V_{n+1}^*. \tag{8.4}$$

Moreover, for each $n \in \mathbb{N}$ *we have that the central support of* $V_n V_n^*$ *in* A_n'' *is* 1*. Also, if* A_0 *is a von Neumann algebra and* ϕ_0 *is normal then* A_n *will also be a von Neumann algebra and* π_n *and* ϕ_n *will be normal for each* $n \in \mathbb{N}$*.*

Proof. We will first construct the objects and show the relationships (8.1), (8.2), and (8.3) by induction, with the base case being vacuous, and we will then show that (8.4) also holds for all $n \in \mathbb{N}$. So suppose $n \in \mathbb{N}$ and that (8.1), (8.2), and (8.3) hold for all $m < n$, (we leave V_0 undefined).

Recall from the proof of Stinespring's Dilation Theorem that we may construct a Hilbert space $\mathcal{H}_n$ by completing the vector space $A_{n-1} \otimes \mathcal{H}_{n-1}$ with respect to the non-negative definite sesquilinear form satisfying

$$\langle a \otimes \xi, b \otimes \eta \rangle = \langle \phi_{n-1}(b^* a)\xi, \eta \rangle,$$

for all $a, b \in A_{n-1}$, $\xi, \eta \in \mathcal{H}_{n-1}$.

We also obtain a partial isometry $V_n : \mathcal{H}_{n-1} \to \mathcal{H}_n$ from the formula

$$V_n(\xi) = 1 \otimes \xi,$$

for $\xi \in \mathcal{H}_{n-1}$.

We obtain a representation $\pi_n : A_{n-1} \to \mathcal{B}(\mathcal{H}_n)$ (which is normal when A_0 is a von Neumann algebra and ϕ_0 is normal) from the formula

$$\pi_n(x)(a \otimes \xi) = (xa) \otimes \xi,$$

for $x, a \in A_{n-1}$, $\xi \in \mathcal{H}_{n-1}$. And recall the fundamental relationship $V_n^* \pi_n(x) V_n = \phi_{n-1}(x)$ for all $x \in A_{n-1}$, which establishes (8.1).

If we let A_n be the C^*-algebra generated by $\pi_n(A_{n-1})$ and $V_n A_{n-1} V_n^*$, then $\pi_n : A_{n-1} \to A_n$, and from Lemma 8.2.12 we have that $V_n^* A_n V_n$ is generated by $V_n^* \pi_n(A_{n-1}) V_n$ and A_{n-1}. However, $V_n^* \pi_n(A_{n-1}) V_n = \phi_{n-1}(A_{n-1}) \subset A_{n-1}$, hence $V_n^* A_n V_n = A_{n-1}$, establishing (8.2). Also, when A_0 is a von Neumann algebra and π_n is normal it then follows easily that A_n is then also a von Neumann algebra.

Also note that $\pi_n(A_{n-1}) V_n V_n^* \mathcal{H}_n$ is dense in $\mathcal{H}_n$, and so since $\pi_n(A_{n-1}) \subset A_n$ we have that the central support of $V_n V_n^*$ in A_n'' is 1.

We then define $\phi_n : A_n \to A_n$ by $\phi_n(x) = \pi_n(V_n^* x V_n)$, for $x \in A_n$. This is well defined since $V_n^* A_n V_n = A_{n-1}$, unital, and completely positive. Note that for $x \in A_{n-1}$ we have $\phi_n(\pi_n(x)) = \pi_n(V_n^*\pi_n(x)V_n) = \pi_n(\phi_{n-1}(x))$, establishing (8.3).

Having established (8.1), (8.2), and (8.3) for all $n \in \mathbb{N}$, we now show that (8.4) holds as well. For this, notice first that for $a, b \in A_n$, $x \in A_{n-1}$, and $\xi, \eta \in \mathcal{H}_n$ we have

$$\begin{aligned}\langle \pi_{n+1}(V_n x V_n^*)(a \otimes \xi), b \otimes \eta\rangle &= \langle V_n x V_n^* a \otimes \xi, b \otimes \eta\rangle \\ &= \langle \phi_n(b^* V_n x V_n^* a)\xi, \eta\rangle \\ &= \langle \pi_n(V_n^* b^* V_n x V_n^* a V_n)\xi, \eta\rangle \\ &= \langle 1 \otimes \pi_n(x V_n^* a V_n)\xi, b \otimes \eta\rangle.\end{aligned}$$

Setting $x = 1$ and using that $V_{n+1}^*(1 \otimes \zeta) = \zeta$ for each $\zeta \in \mathcal{H}_n$, we see that

$$\begin{aligned}(V_{n+1}V_{n+1}^*)\pi_{n+1}(V_n V_n^*)(a \otimes \xi) &= (V_{n+1}V_{n+1}^*)(1 \otimes \pi_n(V_n^* a V_n)\xi) \\ &= 1 \otimes \pi_n(V_n^* a V_n)\xi \\ &= \pi_{n+1}(V_n V_n^*)(a \otimes \xi),\end{aligned}$$

and hence $\pi_{n+1}(V_n V_n^*) \leq V_{n+1}V_{n+1}^*$. If instead we set $a = 1$ then we have

$$V_{n+1}\pi_n(x)\xi = 1 \otimes \pi_n(x)\xi = \pi_{n+1}(V_n x V_n^*)V_{n+1}\xi,$$

and so $V_{n+1}\pi_n(x) = \pi_{n+1}(V_n x V_n^*)V_{n+1}$. Multiplying on the right by V_{n+1}^* and using that $\pi_n(V_n V_n^*) \leq V_{n+1}V_{n+1}^*$ then gives $V_{n+1}\pi_n(x)V_{n+1}^* = \pi_{n+1}(V_n x V_n^*)$. ■

Theorem 8.2.14 (Bhat)**.** *Let $A_0 \subset \mathcal{B}(\mathcal{H}_0)$ be a unital C^*-algebra, and $\phi_0 : A_0 \to A_0$ a unital completely positive map. Then there exists*

(1) a Hilbert space $\mathcal{K}$;

(2) an isometry $W : \mathcal{H}_0 \to \mathcal{K}$;

(3) a C^*-algebra $B \subset \mathcal{B}(\mathcal{K})$;

(4) a unital $*$-endomorphism $\alpha : B \to B$;

such that $W^*BW = A_0$, and for all $x \in A_0$ we have

$$\phi_0^k(x) = W^*\alpha^k(WxW^*)W.$$

Moreover, we have that the central support of P_0 in B'' is 1, and for $y \in \mathcal{B}(\mathcal{K})$ we have $y \in B$ if and only if $\alpha^k(WW^*)y\alpha^k(WW^*) \in \alpha^k(WA_0W^*)$ for all $k \geq 0$. Also, if A_0 is a von Neumann algebra and ϕ_0 is normal then B will also be a von Neumann algebra, and α will also be normal.

Proof. Using the notation from the previous lemma, we may define a Hilbert space $\mathcal{K}$ as the directed limit of the Hilbert spaces $\mathcal{H}_n$ with respect to the inclusions $V_{n+1} : \mathcal{H}_n \to \mathcal{H}_{n+1}$. We denote by $W_n : \mathcal{H}_n \to \mathcal{K}$ the associated sequence of isometries satisfying $W_{n+1}^*W_n = V_{n+1}$, for $n \in \mathbb{N}$, and we set $P_n = W_nW_n^*$, an increasing sequence of projections.

From (8.2) we have that $P_{n-1}W_nA_nW_n^*P_{n-1} = W_{n-1}A_{n-1}W_{n-1}^*$, and hence if we define the C^*-algebra $B = \{x \in \mathcal{B}(\mathcal{K}) \mid W_n^*xW_n \in A_n, n \geq 0\}$, then we have $W_n^*BW_n = A_n$, for all $n \geq 0$. Also, if A_0 is a von Neumann algebra, then so is A_n for each $n \in \mathbb{N}$ and from this it follows easily that B is also a von Neumann algebra.

We define the unital $*$-endomorphism $\alpha : B \to B$ (which is normal when A_0 is a von Neumann algebra and ϕ_0 is normal) by the formula

$$\alpha(x) = \lim_{n\to\infty} W_{n+1}\pi_{n+1}(W_n^*xW_n)W_{n+1},$$

where the limit is taken in the strong operator topology. Note that $\alpha(P_n) = P_{n+1} \geq P_n$. From (8.4) we see that in general, the strong operator topology limit exists in B, and that for $x \in A_n \cong P_nA_\infty P_n$ the limit stabilizes as $\alpha(W_nxW_n^*) = W_{n+1}\pi_{n+1}(x)W_{n+1}^*$.

From (8.1) we see that for $n \geq 0$, and $x \in A_n$ we have

$$\begin{aligned} P_n\alpha(W_nxW_n^*)P_n &= W_nW_n^*W_{n+1}\pi_{n+1}(x)W_{n+1}^*W_nW_n^* \\ &= W_nV_{n+1}^*\pi_{n+1}(x)V_{n+1}W_n^* \\ &= W_n\phi_n(x)W_n^*. \end{aligned}$$

By induction we then see that also for $k > 1$, and $x \in A_0$ we have

$$\begin{aligned} P_0\alpha^k(W_0xW_0^*)P_0 &= P_0\alpha^{k-1}(P_0\alpha(W_0xW_0^*)P_0)P_0 \\ &= P_0\alpha^{k-1}(W_0\phi_0(x)W_0^*)P_0 \\ &= W_0\phi_0^k(x)W_0^*. \end{aligned}$$

By the previous lemma we have that the central support of P_n in $W_nA_n''W_n^*$ is P_{n+1}. Hence it follows that the central support of P_0 in B is 1. ∎

Corollary 8.2.15 (Connes). *Let M be a countably decomposable properly infinite von Neumann algebra and suppose $\phi : M \to M$ is a normal unital completely positive map, then there exists a (possibly non-unital) $*$-endomorphism $\alpha : M \to M$, and an isometry $v \in M$ such that $\phi(x) = v^*\alpha(x)v$, for all $x \in M$. Also, if ϕ is normal then α is normal as well.*

Proof. Bhat's dilation provides a von Neumann algebra $\tilde{M}$, a projection $p \in \tilde{M}$ with central support 1 such that $p\tilde{M}p = M$, and a normal (unital) $*$-endomorphism $\alpha_0 : \tilde{M} \to \tilde{M}$, such that $\phi(x) = p\alpha_0(x)p$, for all $x \in M$. Since $M = p\tilde{M}p$ is property infinite and since p has central support 1 it follows from Corollary 5.2.10 that p and 1 are equivalent in $\tilde{M}$. Thus there exists a partial isometry $v_0 \in \tilde{M}$ such that $v_0v_0^* = 1$, and $v_0^*v_0 = p$.

If we then consider the possibly non-unital normal $*$-endomorphism $\alpha : M \to M$ given by $\alpha(x) = v_0^*\alpha_0(x)v_0$ then setting $v = v_0^*p$ we have $\phi(x) = p\alpha_0(x)p = v^*\alpha(x)v$, for all $x \in M = p\tilde{M}p$. ∎

8.2.3 Poisson boundaries

If $A \subset \mathcal{B}(\mathcal{H})$ is a unital C^*-algebra, and $\phi : A \to A$ a unital completely positive map, we consider the operator system $\mathrm{Har}(\phi) = \{x \in A \mid \phi(x) = x\}$, which we call the system of ϕ-**harmonic** operators. A projection $p \in A$ is said to be **coinvariant**, if $\{\phi^n(p)\}$ defines an increasing sequence of projections which strongly converge to 1 in $\mathcal{B}(\mathcal{H})$, and such for $y \in \mathcal{B}(\mathcal{H})$ we have $y \in A$ if and only if $\phi^n(p)y\phi^n(p) \in A$ for all $n \geq 0$. Note that for $n \geq 0$, $\phi^n(p)$ is in the multiplicative domain for ϕ, and is again coinvariant. We define $\phi_p : pAp \to pAp$ to be the map $\phi_p(x) = p\phi(x)p$, then ϕ_p is normal unital completely positive. Moreover, we have that $\phi_p^k(x) = p\phi^k(x)p$ for all $x \in pAp$, which can be seen by induction from

$$p\phi^k(x)p = p\phi^{k-1}(p)\phi^k(x)\phi^{k-1}(p)p = p\phi^{k-1}(\phi_p(x))p.$$

Theorem 8.2.16 (Prunaru)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a unital C^*-algebra, $\phi : A \to A$ a unital completely positive map, and $p \in A$ a coinvariant projection. Then the map $\theta : \mathrm{Har}(A, \phi) \to \mathrm{Har}(pAp, \phi_p)$ given by $\theta(x) = pxp$ defines a completely positive isometric surjection, between $\mathrm{Har}(A, \phi)$ and $\mathrm{Har}(pAp, \phi_p)$.*

Moreover, if A is a von Neumann algebra and ϕ is normal then θ is also normal.

Proof. First note that θ is well-defined since if $x \in \mathrm{Har}(A, \phi)$ we have $\phi_p(pxp) = p\phi(p)x\phi(p)p = pxp$. Clearly θ is completely positive (and normal in the case when A is a von Neumann algebra and ϕ is normal).

To see that it is surjective, if $x \in \mathrm{Har}(pAp, \phi_p)$ then consider the sequence $\phi^n(x)$. For each $m, n \geq 0$, we have

$$\phi^m(p)\phi^{m+n}(x)\phi^m(p) = \phi^m(p\phi^n(x)p) = \phi^m(\phi_p^n(x)) = \phi^m(x).$$

It follows that $\{\phi^n(x)\}$ converges in the strong operator topology to an element $y \in \mathcal{B}(\mathcal{H})$ such that $\phi^m(p)y\phi^m(p) = \phi^m(x)$ for each $m \geq 0$, consequently we have $y \in A$.

In particular, for $m = 0$ we have $pyp = x$. To see that $y \in \mathrm{Har}(A, \phi)$ we use that for all $z \in A$ we have the strong operator topology limit

$$\lim_{n\to\infty} \phi(\phi^n(p)z\phi^n(p)) = \phi^{n+1}(p)\phi(z)\phi^{n+1}(p) = \phi(z),$$

and hence

$$\phi(y) = \lim_{m\to\infty} \phi(\phi^m(p)y\phi^m(p)) = \lim_{m\to\infty} \phi^{m+1}(x) = y.$$

Thus θ is surjective, and since $\phi^n(p)$ converges strongly to 1, and each $\phi^n(p)$ is in the multiplicative domain of ϕ, it follows that if $x \in \mathrm{Har}(A, \phi)$ then $\phi^n(pxp)$ converges strongly to x and hence

$$\|x\| = \lim_{n\to\infty} \|\phi^n(pxp)\| \leq \|pxp\| \leq \|x\|.$$

Thus, θ is also isometric. ∎

Corollary 8.2.17. *Let A be a unital C^*-algebra, and $\phi : A \to A$ a unital completely positive map. Then there exists a C^*-algebra B and a completely positive isometric surjection $\theta : B \to \mathrm{Har}(A, \phi)$.*

Moreover B and θ are unique in the sense that if $\tilde{B}$ is another C^*-algebra, and $\tilde{\theta} : \tilde{B} \to \mathrm{Har}(A, \phi)$ is a completely positive isometric surjection, then $\theta^{-1} \circ \tilde{\theta}$ is an isomorphism.

Also, if A is a von Neumann algebra and ϕ is normal, then B is also a von Neumann algebra and θ is normal.

Proof. Note that we may assume $A \subset \mathcal{B}(\mathcal{H})$. Existence then follows by applying the previous theorem to Bhat's dilation. Uniqueness follows from Theorem 8.2.10. ■

The C^*-algebra B from the previous corollary is the **Poisson boundary of** ϕ, and the map θ as the **Poisson transform**. Poisson boundaries were introduced in the noncommutative setting by Izumi as a generalization of the commutative phenomenon introduced by Furstenberg (see Example 8.2.20 below).

Corollary 8.2.18 (Choi-Effros)**.** *Let A be a unital C^*-algebra and $F \subset A$ an operator system. If $E : A \to F$ is a completely positive map such that $E_{|F} = \mathrm{id}$, then F has a unique C^*-algebraic structure which is given by $x \cdot y = E(xy)$. Moreover, if A is a von Neumann algebra and F is weakly closed then this gives a von Neumann algebraic structure on F.*

Proof. When A is a C^*-algebra this follows from Corollary 8.2.17 since $\mathrm{Har}(A, E) = F$. Also note that since $E^n = E$ it follows from the proof of Theorem 8.2.16 that the product structure coming from the Poisson boundary is given by $x \cdot y = E(xy)$.

If A is a von Neumann algebra and F is weakly closed then F has a predual $F_\perp = \{\varphi \in A_* \mid \varphi(x) = 0, \text{ for all } x \in F\}$ and hence A is isomorphic to a von Neumann algebraic by Theorem 6.1.6. ■

Note that if A is a C^*-algebra, $F \subset A$ an operator system, and $E : A \to F$ completely positive with $E_{|F} = \mathrm{id}$, then we still have a form of bimodularity for E when we endow F with the Choi-Effros product from Corollary 8.2.18. In this case though the bimodularity is with respect to two different product structures, i.e., we have

$$E(xay) = x \cdot E(a) \cdot y,$$

for all $x, y \in F$, $a \in A$.

Proposition 8.2.19. *Let A be an abelian C^*-algebra and $\phi : A \to A$ a normal unital completely positive map. Then the Poisson boundary of ϕ is also abelian.*

Proof. Let B be the Poisson boundary of ϕ, and let $\theta : B \to \mathrm{Har}(A, \phi)$ be the Poisson transform. If C is a C^*-algebra and $\psi : C \to B$ is a positive map then $\theta \circ \psi : C \to \mathrm{Har}(A, \phi) \subset A$ is positive, and since A is abelian it is then completely positive by Proposition 8.1.8. Hence, ψ is also completely positive.

Since every positive map from a C^*-algebra to B is completely positive it then follows that B is abelian. ■

Example 8.2.20 (Furstenberg)**.** Let Γ be a discrete group and $\mu \in \mathrm{Prob}(\Gamma)$ a probability measure on Γ such that the support of μ generates Γ. Then on $\ell^\infty\Gamma$ we may consider the normal unital (completely) positive map ϕ_μ given by $\phi_\mu(f) = \mu * f$, where $\mu * f$ is the convolution $(\mu * f)(x) = \int f(g^{-1}x)\, d\mu(g)$. Then $\mathrm{Har}(\mu) = \mathrm{Har}(\ell^\infty\Gamma, \phi_\mu)$ has a unique von Neumann algebraic structure which is abelian by the previous proposition. Notice that Γ acts on $\mathrm{Har}(\mu)$ by right translation, and since this action preserves positivity it follows from Theorem 8.2.10 that Γ preserves the multiplication structure as well.

Since the support of μ generates Γ, for a non-negative function $f \in \mathrm{Har}(\mu)_+$, we have $f(e) = 0$ if and only if $f = 0$. Thus we obtain a natural normal faithful state φ on $\mathrm{Har}(\mu)$ which is given by $\varphi(f) = f(e)$.

Proposition 8.2.21. *Let A be a unital C^*-algebra and suppose that $\varphi \in A^*$ is a faithful state. If $\phi : A \to A$ is unital completely positive such that $\varphi \circ \phi = \varphi$, then* $\mathrm{Har}(\phi) \subset A$ *is a C^*-subalgebra of A.*

Proof. Suppose $x \in \mathrm{Har}(\phi)$. Then by Kadison's inequality we have $\phi(x^*x) - x^*x = \phi(x^*x) - \phi(x^*)\phi(x) \geq 0$, and since $\varphi\circ\phi = \varphi$ we have $\varphi(\phi(x^*x) - x^*x) = 0$. Since φ is faithful we then conclude that $x^*x = \phi(x^*x)$, and replacing x with x^* we see also that $xx^* = \phi(xx^*)$. By Theorem 8.2.8 we then have that $\mathrm{Har}(\phi)$ is a C^*-subalgebra of A. ■

8.3 Conditional expectations

If A is a unital C^*-algebra, and $B \subset A$ is a unital C^*-subalgebra, then a **conditional expectation** from A to B is a unital completely positive $E : A \to B$ such that $E_{|B} = \mathrm{id}$. Note that by Choi's theorem we have $E(axb) = aE(x)b$ for all $a, b \in B$, $x \in A$.

Theorem 8.3.1 (Tomiyama)**.** *Let A be a unital C^*-algebra, $B \subset A$ a unital C^*-subalgebra, and $E : A \to B$ a linear map such that $E_{|B} = \mathrm{id}$ and $\|E\| \leq 1$, then E is a conditional expectation.*

Proof. We first consider the case when A is a von Neumann algebra and $B \subset A$ is a von Neumann subalgebra. Then if $p \in \mathcal{P}(B)$ and $x \in A$ we have $(1-p)E(px) = E((1-p)E(px))$ and hence for all $t > 0$ we have

$$\begin{aligned}(1+t)^2\|pE((1-p)x)\|^2 &= \|pE((1-p)x + tpE((1-p)x))\|^2 \\ &\leq \|(1-p)x + tpE((1-p)x)\|^2 \\ &\leq \|(1-p)x\|^2 + t^2\|pE((1-p)x)\|^2\end{aligned}$$

Hence for all $t > 0$ we have $(1+2t)\|pE((1-p)x)\|^2 \leq \|(1-p)x\|^2$ which then shows that $pE((1-p)x) = 0$. Since this also holds when replacing p with $1-p$ we then have $pE(x) = pE((1-p)x + px) = pE(px) = E(px)$, and since the span of projections is norm dense in B it then follows that $E(yx) = yE(x)$ for all $y \in B$, $x \in A$. Taking adjoints shows that E is B-bimodular.

Since E is a unital contraction Lemma 8.1.10 shows that E is positive. To see that E is completely positive consider $[a_{i,j}] \in \mathbb{M}_n(A)_+$, and $x_1, \dots, x_n \in B$. Then by Lemma 8.1.5 we have

$$\sum_{i,j=1}^{n} x_i^* E(a_{i,j}) x_j = E(\sum_{i,j=1}^{n} x_i^* a_{i,j} x_j) \geq 0.$$

Hence $E^{(n)}([a_{i,j}]) \geq 0$ and so E is completely positive.

For the general case if we consider the double dual $A \subset A^{**}$ and $B \subset B^{**}$, then these are von Neumann algebras, and the result follows by considering the map $E^{**} : A^{**} \to B^{**}$. ■

Theorem 8.3.2 (Umegaki). *Let M be a finite von Neumann algebra with normal faithful trace τ, and let $N \subset M$ be a von Neumann subalgebra, then there exists a unique normal conditional expectation $E : M \to N$ such that $\tau \circ E = \tau$.*

Proof. Let $e_N \in \mathcal{B}(L^2(M, \tau))$ be the projection onto $L^2(N, \tau) \subset L^2(M, \tau)$, and let J be the conjugation operator on $L^2(M, \tau)$ which we also view as the conjugation operator on $L^2(N, \tau)$. Note that $N' \cap \mathcal{B}(L^2(N, \tau)) = JNJ$ by Proposition 6.4.12. Since $L^2(N, \tau)$ is invariant under JNJ we have $e_N(JyJ) = (JyJ)e_N$ for all $y \in N$.

If $x \in M$, $y \in N$, then

$$\begin{aligned} e_N x e_N JyJ = e_N x JyJ &= e_N JyJ x e_N \\ &= JyJ e_N x e_N. \end{aligned}$$

Thus, $e_N x e_N \in (JNJ)' = N$ and we denote this operator by $E(x)$. Clearly, $E : M \to N$ is normal unital completely positive, and $E_{|N} = \mathrm{id}$, thus E is a normal conditional expectation. Also, for $x \in M$ we have

$$\tau(E(x)) = \langle e_N x e_N 1_\tau, 1_\tau \rangle = \langle x 1_\tau, 1_\tau \rangle = \tau(x).$$

If $\tilde{E}$ were another trace preserving conditional expectation, then for $x \in M$, and $y \in N$ we would have

$$\begin{aligned} \tau(\tilde{E}(x)y) &= \tau(\tilde{E}(xy)) = \tau(xy) \\ &= \tau(E(xy)) = \tau(E(x)y), \end{aligned}$$

from which it follows that $\tilde{E} = E$. ■

Lemma 8.3.3 (Sakai). *Let M be a semi-finite factor, and $p \in M$ a finite projection. Then the adjoint operation is strongly continuous on bounded subsets of Mp.*

Proof. Let Tr be a semi-finite faithful normal trace on M, and let $\mathfrak{M}_{\mathrm{Tr}}$ be as in Lemma 6.6.2. Then the linear functionals of the form $x \mapsto \mathrm{Tr}(ax)$ for $a \in \mathfrak{M}_{\mathrm{Tr}}$ form a dense subset of M_*.

Suppose $\{x_i p\}$ is a net of bounded operators in M which converge strongly to 0, and consider $a \in \mathfrak{M}_{\mathrm{Tr}}$, then

$$|\,\mathrm{Tr}(a x_i p x_i^*)| = |\,\mathrm{Tr}(p x_i^* a x_i p)| \leq \mathrm{Tr}(p x_i^* x_i p)^{1/2}\,\mathrm{Tr}(p x_i^* a^* a x_i p)^{1/2} \to 0$$

Thus since $\{x_i p\}$ is bounded, $(x_i p)(x_i p)^*$ converges σ-weakly to 0 and hence $(x_i p)^*$ converges strongly to 0. ∎

Note that the previous lemma is not true if we considered bounded subsets of pM instead. Easy counter-example can be found by considering $M = \mathcal{B}(\mathcal{H})$ and p a rank one projection. Also note that if a von Neumann algebra is not finite then the adjoint operation is not continuous on bounded sets. We leave this as an exercise.

Theorem 8.3.4 (Tomiyama)**.** *Let M be a semi-finite factor and $N \subset M$ a purely infinite von Neumann subalgebra, then there exists no normal conditional expectation from M to N.*

Proof. Suppose $E : M \to N$ is a normal conditional expectation where M is semi-finite. Let $p \in M$ be a finite projection such that $E(p) \neq 0$, and take $q \in \mathcal{P}(N)$, $\lambda > 0$ such that $\lambda q \leq E(p)$. If $x_i \in qNq$ is a net which converges strongly to 0 then by the previous lemma $p x_i^*$ also converges strongly to 0, hence so does $E(p) x_i^* = E(p x_i^*)$, and consequently $x_i^* = (qE(p)q)^{-1} q E(p) x_i^*$ as well. Taking adjoints is therefore strongly continuous on bounded sets and so qNq is finite. A simple maximality argument then shows that N is semi-finite. ∎

Theorem 8.3.5 (Takesaki-Tomiyama)**.** *If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, then there exists a normal conditional expectation $E : \mathcal{B}(\mathcal{H}) \to M$ if and only if M is completely atomic.*

Proof. If M is completely atomic then $M = \oplus_{i \in I} M_i$ where each M_i is a type I factor. Restricting to each corner, to construct a normal conditional expectation from $\mathcal{B}(\mathcal{H})$ to M it is then enough to do so assuming M is a type I factor. In this case M' is also a type I factor by Theorem 5.3.9 and hence has a minimal projection $p \in M'$. Then for $T \in \mathcal{B}(\mathcal{H})$ we have $pTp \in (pM'p)' \cap \mathcal{B}(p\mathcal{H}) = Mp$, and since $z(p) = 1$ we have $xp \mapsto x$ is an isomorphism on M. Composing these maps then gives a normal conditional expectation.

For the converse, we first note that by Theorem 8.3.4 M must be semi-finite, and hence by restricting to corners of M it is enough to consider the case when M is finite. Similarly, by restricting to corners it is enough to consider the case when M is countably decomposable and hence we may assume by way of contradiction that M is diffuse and has a normal faithful trace τ.

If E is a normal conditional expectation then the the map $T \mapsto \tau(E(T))$ defines a positive normal linear functional on $\mathcal{B}(\mathcal{H})$ and so must be of the form

$\tau(E(T)) = \mathrm{Tr}(AT)$ for some positive trace class operator A. Moreover, for all $T \in \mathcal{B}(\mathcal{H})$ and $x \in M$ we have

$$\begin{aligned}\mathrm{Tr}(AxT) &= \tau(E(xT)) = \tau(xE(T)) \\ &= \tau(E(Tx)) = \mathrm{Tr}(ATx) = \mathrm{Tr}(xAT).\end{aligned}$$

Since this holds for all $T \in \mathcal{B}(\mathcal{H})$, and all $x \in M$ we conclude $A \in M'$. Taking a spectral projection of A we then produce a finite rank projection P such that $P \in M'$. But then if $z(P)$ is the central support of P in M' then we have $z(P)M \cong PM \subset P\mathcal{B}(\mathcal{H})P$, and the latter is finite dimensional contradicting the assumption that M was diffuse. ∎

8.4 Injective von Neumann algebras

If X and Y are Banach spaces and $x \in X$, and $y \in Y$ then we define the linear map $x \otimes y : \mathcal{B}(X, Y^*) \to \mathbb{C}$ by $(x \otimes y)(L) = L(x)(y)$. Note that $|(x \otimes y)(L)| \leq \|L\|\|x\|\|y\|$ and hence $x \otimes y$ is bounded and indeed $\|x \otimes y\| \leq \|x\|\|y\|$. Let Z be the norm closed linear span in $\mathcal{B}(X, Y^*)^*$ of all $x \otimes y$.

Lemma 8.4.1. *The pairing $\langle L, x \otimes y\rangle = (x \otimes y)(L)$ extends to an isometric identification between Z^* and $\mathcal{B}(X, Y^*)$.*

Proof. It is easy to see that this pairing gives an isometric embedding of $\mathcal{B}(X, Y^*)$ into Z^*. To see that this is onto consider $\varphi \in Z^*$, and for each $x \in X$ define $L_x : Y \to \mathbb{C}$ by $L_x(y) = \varphi(x \otimes y)$. Then we have $|L_x(y)| \leq \|\varphi\|\|x\|\|y\|$ and hence $L_x \in Y^*$. The mapping $x \mapsto L_x$ is easily seen to be linear and thus we have $L \in \mathcal{B}(X, Y^*)$, and clearly L is mapped to φ under this pairing. ∎

Lemma 8.4.2. *If X and Y are Banach spaces and L_i is a bounded net in $\mathcal{B}(X, Y^*)$ then $L_i \to L$ in the weak*-topology described above if and only if $L_i(x) \to L(x)$ in the weak*-topology for all $x \in X$.*

Proof. If L_i converges to L in the weak*-topology then for all $x \in X$ and $y \in Y$ we have $L_i(x)(y) = (x \otimes y)(L_i) \to (x \otimes y)(L) = L(x)(y)$ showing that $L_i(x) \to L(x)$ is the weak*-topology for all $x \in X$. Conversely, if $L_i(x) \to L(x)$ in the weak*-topology for all $x \in X$ then in particular we have $(x \otimes y)(L_i) \to (x \otimes y)(L)$ for each $x \in X$ and $y \in Y$, thus this also holds for the linear span of all $x \otimes y$ and since the net is bounded we then have convergence on the closed linear span. ∎

Corollary 8.4.3. *Let E be an operator system, then the set of contractive completely positive maps from E to $\mathcal{B}(\mathcal{H})$ is compact in the topology of point-wise weak operator topology convergence.*

Proof. First note that it is easy to see that the space of contractive completely positive maps from E to $\mathcal{B}(\mathcal{H})$ is closed in this topology. Since $\mathcal{B}(\mathcal{H})$ is a dual space, and on bounded sets the weak operator topology is the same as the weak*-topology, the result then follows from Alaoglu's theorem, together with the previous two lemmas. ∎

If A is a C^*-algebra and $\phi : A \to \mathbb{M}_n(\mathbb{C})$ is a linear map, then we define the linear functional $\hat{\phi} \in \mathbb{M}_n(A)^*$ by

$$\hat{\phi}([a_{i,j}]) = \frac{1}{n} \sum_{i,j=1}^{n} \phi(a_{i,j})_{i,j}.$$

Where $\phi(a_{i,j})_{i,j}$ denotes the i,jth entry of $[\phi(a_{i,j})]$. Note that the correspondence $\phi \mapsto \hat{\phi}$ is bijective and the inverse can be computed explicitly as $\phi(a)_{i,j} = n\hat{\phi}(a \otimes E_{i,j})$ where $E_{i,j}$ is the standard elementary matrix, and $a \otimes E_{i,j}$ is the matrix with a in the i,jth entry, and zeros elsewhere.

Lemma 8.4.4. *Let E be a unital C^*-algebra. A map $\phi : A \to \mathbb{M}_n(\mathbb{C})$ is unital completely positive if and only if $\hat{\phi}$ is a state.*

Proof. Let $\{e_i\}$, $1 \leq i \leq n$ denote that standard basis for $\mathbb{C}^n$, and let $\eta = [e_1, \dots, e_n]^T \in \mathbb{C}^{n^2}$, then for $[a_{i,j}] \in \mathbb{M}_n(A)$ a simple calculation shows

$$\hat{\phi}([a_{i,j}]) = \frac{1}{n} \langle \phi^{(n)}([a_{i,j}])\eta, \eta \rangle.$$

Thus, $\hat{\phi}$ is a state if ϕ is unital completely positive. Conversely, if $\hat{\phi}$ is a state then consider the GNS-construction $L^2(\mathbb{M}_n(A), \hat{\phi})$ with cyclic vector $1_{\hat{\phi}}$. If we define $V : \mathbb{C}^n \to L^2(\mathbb{M}_n(A), \hat{\phi})$ by $Ve_j = \pi(e_{1,j})1_{\hat{\phi}}$ then for $a \in A$ it follows easily that

$$\phi(a) = V^* \pi(a\mathrm{I})V,$$

Hence ϕ is completely positive. ■

Lemma 8.4.5. *Let A be a unital C^*-algebra, and $E \subset A$ an operator system. If $\phi : E \to \mathbb{M}_n(\mathbb{C})$ is a completely positive map then there exists a completely positive extension $\tilde{\phi} : A \to \mathbb{M}_n(\mathbb{C})$.*

Proof. If $\phi : E \to \mathbb{M}_n(\mathbb{C})$ is completely positive then the same argument as in Lemma 8.4.4 shows that $\hat{\phi}$ defines a state on $\mathbb{M}_n(E)$. By the Hahn-Banach theorem we can then extend this to norm 1 linear functional on $\mathbb{M}_n(A)$ which then must also be a state and so by Lemma 8.4.4 this corresponds to a unital completely positive extension $\tilde{\phi} : A \to \mathbb{M}_n(\mathbb{C})$. ■

Theorem 8.4.6 (Arveson's extension theorem)**.** *Let A be a unital C^*-algebra, and $E \subset A$ an operator system. If $\phi : E \to \mathcal{B}(\mathcal{H})$ is a completely positive map then there exists a completely positive extension $\tilde{\phi} : A \to \mathcal{B}(\mathcal{H})$.*

Proof. For each finite rank projection $P \in \mathcal{B}(\mathcal{H})$ we may consider the compression $E \ni x \mapsto P\phi(x)P$ and by Lemma 8.4.5, this has a completely positive extension $\tilde{\phi}_P : A \to P\mathcal{B}(\mathcal{H})P \subset \mathcal{B}(\mathcal{H})$. If we consider the net $\{\phi_P\}$ which is ordered by the usual order on projections, then by Corollary 8.4.3 this net must have a completely positive cluster point $\tilde{\phi}$. Since $\tilde{\phi}_P(x) = P\phi(x)P$ for all $x \in E$ it is then easy to see that $\tilde{\phi}$ is an extension of ϕ. ■

An operator system F is **injective** if for any C^*-algebra A and any operator system $E \subset A$, whenever $\phi : E \to F$ is completely positive then there exists a completely positive extension $\tilde{\phi} : A \to E$. Arveson's extension theorem states that $\mathcal{B}(\mathcal{H})$ is injective.

Corollary 8.4.7. *Let $F \subset \mathcal{B}(\mathcal{H})$ be an operator system, then F is injective if and only if there exists a completely positive map $E : \mathcal{B}(\mathcal{H}) \to F$ such that $E_{|F} = \mathrm{id}$. In particular, the existence of such a completely positive map does not depend on the representation $F \subset \mathcal{B}(\mathcal{H})$.*

Proof. If F is injective, then the identity map from F to F has an extension E to $\mathcal{B}(\mathcal{H})$. Conversely, if $E : \mathcal{B}(\mathcal{H}) \to F$ is completely positive such that $E_{|F} = \mathrm{id}$, and if A is a C^*-algebra and $F' \subset A$ is an operator space, such that $\phi : F' \to F \subset \mathcal{B}(\mathcal{H})$ is unital completely positive, then by Arveson's extension theorem there exists an extension $\tilde{\phi} : A \to \mathcal{B}(\mathcal{H})$, and $E \circ \tilde{\phi} : A \to F$ then gives an extension of ϕ showing that F is injective. ■

The previous corollary together with the Choi-Effros theorem give the following.

Corollary 8.4.8 (Choi-Effros)**.** *Let $F \subset \mathcal{B}(\mathcal{H})$ be an injective operator system, then there exists a unique C^*-algebraic structure on F. Moreover, if F is weakly closed then there exists a unique von Neumann algebraic structure on F.*

Example 8.4.9. Let Γ be a countable group which is locally finite, i.e., every finitely generated subgroup is finite. Then we can write $\Gamma = \cup_{n \in \mathbb{N}} \Gamma_n$ where Γ_n forms an increasing sequence of finite subgroups. For each $n \in \mathbb{N}$ we define the unital completely positive map $\phi_n : \mathcal{B}(\ell^2\Gamma) \to \mathcal{B}(\ell^2\Gamma)$ by $\phi_n(T) = \frac{1}{|\Gamma_n|} \sum_{g \in \Gamma_n} \rho_g T \rho_{g^{-1}}$. By Corollary 8.4.3 there exists a cluster point $E : \mathcal{B}(\ell^2\Gamma) \to \mathcal{B}(\ell^2\Gamma)$ for this sequence in the topology of point-wise weak convergence.

Note that $E_{|L\Gamma} = \mathrm{id}$ since this holds for each ϕ_n. Also, for each $n \in \mathbb{N}$, $h \in \Gamma_n$ and $T \in \mathcal{B}(\ell^2\Gamma)$ we have $\rho_h \phi_n(T) \rho_{h^{-1}} = \phi_n(T)$, and since the sequence Γ_n is increasing we then have that the range of ϕ_m is in $\rho(\Gamma_n)'$ whenever $m \geq n$. Since $\rho(\Gamma_n)'$ is closed in the weak operator topology we have that the range of E is in $\rho(\Gamma_n)'$ for every $n \in \mathbb{N}$. However, $\rho(\cup_{n \in \mathbb{N}} \Gamma_n)' = L\Gamma$ and hence E is a conditional expectation from $\mathcal{B}(\ell^2\Gamma)$ to $L\Gamma$ showing that $L\Gamma$ is injective.

8.5 An application

Theorem 8.5.1. *Let M be a separable finite von Neumann algebra with a normal faithful trace τ. Then for each operator $T \in \mathcal{B}(L^2(M, \tau))$ we have that weak operator toplology closed convex hull $\overline{\mathrm{co}}\{uJvJTJv^*Ju^* \mid u, v \in \mathcal{U}(M)\}$ has non-trivial intersection with $\mathcal{Z}(M)$.*

Proof. As M is separable it is generated by some countable collection of unitaries $\{u_n\}_{n \in \mathbb{N}}$, with $u_0 = 1$. Fix a probability measure $\mu \in \mathrm{Prob}(\mathbb{N})$ such that

$\mu(\{n\}) > 0$ for all $n \in \mathbb{N}$ and define a normal unital completely positive map $\phi : \mathcal{B}(L^2(M,\tau)) \to \mathcal{B}(L^2(M,\tau))$ by $\phi(T) = \int (Ju_nJ)T(Ju_n^*J)\, d\mu(n)$.

We let $E : \mathcal{B}(L^2(M,\tau)) \to \mathcal{B}(L^2(M,\tau))$ be a cluster point of the sequence $\left\{ \frac{1}{N}\sum_{n=1}^N \phi^n \right\}_{N=1}^{\infty}$ in the point-ultraweak topology. We then have that $E : \mathcal{B}(L^2(M,\tau)) \to \mathrm{Har}(\phi)$ and for all $T \in \mathcal{B}(L^2(M,\tau))$ we have $E(T) \in \overline{\mathrm{co}}\{(JuJ)T(Ju^*J) \mid u \in \mathcal{U}(M)\}$.

We let $\varphi : \mathrm{Har}(\phi) \to \mathbb{C}$ denote the state given by $\varphi(x) = \langle x\hat{1}, \hat{1}\rangle$. If $x \in \mathrm{Har}(\phi)_+$ such that $\varphi(x) = 0$, then we have

$$\sum_{(n_1,u_2,\ldots,n_k)\in\mathbb{N}^k} \mu(n_1)\mu(n_2)\cdots\mu(n_k)\langle xu_{n_1}u_{n_2}\cdots u_{n_k}\hat{1}, u_{n_1}u_{n_2}\cdots u_{n_k}\hat{1}\rangle$$
$$= \langle \phi^k(x)\hat{1}, \hat{1}\rangle = \varphi(x) = 0.$$

Hence for all $k \geq 1$ and for all $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$ we have

$$\langle xu_{n_1}u_{n_2}\cdots u_{n_k}\hat{1}, u_{n_1}u_{n_2}\cdots u_{n_k}\hat{1}\rangle = 0,$$

and since $\{u_n\}_{n\in\mathbb{N}}$ generate M as a von Neumann algebra we then have $x = 0$. Thus, $\varphi : \mathrm{Har}(\phi)$ is a faithful state.

We now consider the unital completely positive map $\psi : \mathrm{Har}(\phi) \to \mathrm{Har}(\phi)$ given by $\psi(x) = \int u_n^* x u_n \, d\mu(u)$. We remark that just as above if we let $\tilde{E}$ be a point-ultraweak limit point of $\left\{ \frac{1}{N}\sum_{n=1}^N \psi^n \right\}$ then we have $\tilde{E} : \mathrm{Har}(\phi) \to \mathrm{Har}(\psi)$ and $\tilde{E}(E(T)) \in \overline{\mathrm{co}}\{uJvJTJv^*Ju^* \mid u, v \in \mathcal{U}(M)\}$. Thus, to prove the theorem it suffices to show that $\mathrm{Har}(\psi) = \mathcal{Z}(M)$. As the inclusion $\mathcal{Z}(M) \subset \mathrm{Har}(\psi)$ is obvious we only prove the reverse inclusion.

For $x \in \mathrm{Har}(\phi)$ we have $\varphi \circ \psi(x) = \int \langle u_n x u_n \hat{1}, \hat{1}\rangle = \varphi(\phi(x)) = \varphi(x)$. Thus, by Corollary 8.2.17 we have that $\mathrm{Har}(\phi)$ has a C^*-algebraic structure, which we will write by $x \cdot y$ to distinguish from the multiplication in $\mathcal{B}(L^2(M,\tau))$. Note that since M is in the multiplicative domain of ϕ it follows that $\mathrm{Har}(\phi)$ is an M-M bimodule and $a \cdot x \cdot b = axb$ whenever $a, b \in M$ and $x \in \mathrm{Har}(\phi)$. By Proposition 8.2.21 we have $\mathrm{Har}(\psi)$ is a C^*-subalgebra of $\mathrm{Har}(\phi)$, and since ψ is a normal map it follows that $\mathrm{Har}(\psi)$ is a von Neumann sublagebra of $\mathrm{Har}(\phi)$.

We now suppose that we have a faithful normal representation of $\mathrm{Har}(\phi)$ as a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Suppose that $p \in \mathcal{P}(\mathrm{Har}(\psi))$ is a projection. Then for all $\xi \in \mathcal{K}$ we may compute

$$\int \|(p - u_n p u_n^*)\xi\| d\mu(n) = \|p\xi\|^2 - 2\mathrm{Re}(\langle p\xi, \psi(p)\xi\rangle) + \int \langle pu_n^*\xi, pu_n^*\xi\rangle = 0.$$

Hence it follows that $p = u_n \cdot p \cdot u_n^* = u_n p u_n^*$ for all $n \in \mathbb{N}$ and since $\{u_n\}_{n\in\mathbb{N}}$ generate M as a von Neumann algebra we then have that $p \in M' \cap \mathrm{Har}(\phi)$. Since p was an arbitrary projection we then have that $\mathrm{Har}(\psi) \subset M' \cap \mathrm{Har}(\phi)$. Viewing $\mathrm{Har}(\phi)$ as an operator subspace of $\mathcal{B}(L^2(M,\tau))$ we then have $M' \cap \mathrm{Har}(\phi) = \mathrm{Har}(\phi_{|M'})$. Thus, to complete the proof it suffices to show that $\mathrm{Har}(\phi_{|M'}) = \mathcal{Z}(M)$.

However, $\phi_{|M'}$ preserves the canonical trace on M' and thus again by Proposition 8.2.21 we have that $\text{Har}(\phi_{|M'})$ is a von Neumann subalgebra of M'. Proceeding just as we did above, if $q \in \mathcal{P}(M') \cap \text{Har}(\phi_{|M'})$ is a projection then for all $\xi \in L^2(M, \tau)$ we have $\int \|(q - Ju_n^* JqJu_n J)\xi\|^2 \, d\mu(n) = 0$ and from this it follows that $p \in \mathcal{Z}(M') = \mathcal{Z}(M)$. Since p was an arbitrary projection it then follows that $\text{Har}(\phi_{|M'}) = \mathcal{Z}(M)$. ∎