ANALYSIS PRACTICE PROBLEMS

Problem 0.1. Let X be a countably infinite set. There exists an uncountable family $\mathcal{F} \subset 2^X$ so that for any distinct pair $A, B \in \mathcal{F}$ we have that $A \cap B$ is finite. (Hint: It may help to consider the case $X = 2^{<\mathbb{N}}$.)

Problem 0.2. A compact metric space is separable.

Problem 0.3. Is the Banach space $\ell^\infty(\mathbb{N})$ separable? Prove your answer is correct.

Problem 0.4. Let $\{a_n\}_n$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$ but such that $\sum_{n=1}^\infty a_n$ is divergent. Show that for any real number r there exists a sequence $\sigma_n \in \{-1, 1\}$ such that $\sum_{n=1}^\infty \sigma_n a_n = r$.

Problem 0.5. Suppose K is a compact Hausdorff space and $\{f_n\}_n$ is a sequence of continuous complex-valued functions on K such that f_n converges pointwise to a continuous function f. Does it follow that f_n converges to f uniformly. Prove or give a counterexample.

Problem 0.6. Let X be a locally compact Hausdorff space and suppose $f : X \to \mathbb{C}$ is continuous except at countably many points, show that f is Borel measurable.

Problem 0.7. Let K be a compact Hausdorff space and suppose $Y \subset K$ is a dense subset, show that Y is locally compact in the relative topology if and only if Y is an open subset of K.

Problem 0.8. Show that a metrizable space X is compact if and only if every continuous real valued function on X is bounded.

Exercise 0.9. Let X be a compact Hausdorff space. Show that X is a second countable if and only if $C(X)$ is separable.

Problem 0.10. Let $(X, \mathcal{T})$ be a compact Hausdorff space. Show that if $\mathcal{T}'$ is any weaker topology then $(X, \mathcal{T}')$ is not Hausdorff. Show that if $\mathcal{T}'$ is any stronger topology then $(X, \mathcal{T}')$ is not compact.

Exercise 0.11. Suppose X and Y are compact Hausdorff spaces and $f \in C(X \times Y)$. Show that for all $\varepsilon > 0$ there exist $g_1, \ldots, g_n \in C(X)$ and $h_1, \ldots, h_n \in C(Y)$ so that $|f(x, y) - \sum_{i=1}^n g_i(x)h_i(y)| < \varepsilon$ for all $(x, y) \in X \times Y$.

Problem 0.12. Let V be a finite dimensional $\mathbb{K}$-vector space, and suppose $\| \cdot \|_1$, and $\| \cdot \|_2$ are norms on V. Then the identity map from $(V, \| \cdot \|_1)$ to $(V, \| \cdot \|_2)$ is a homeomorphism.

Problem 0.13. Suppose $(X, \mathcal{M})$ is a measurable space and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(X; \mathbb{R})$. Set

$$\mathcal{C} = \{x \in X \mid \{f_n(x)\}_{n\in\mathbb{N}} \text{ converges}\}.$$

Then $\mathcal{C}$ is measurable.

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Problem 0.14. Find an example of Borel sets $A, B \subset \mathbb{R}$ so that $\lambda(A) = \lambda(B) = 0$, but $\mathbb{R} = A + B := \{a + b \mid a \in A, b \in B\}$.

Problem 0.15. Find an example of a meager set in $[0,1]$ which has Lebesgue measure 1.

Problem 0.16. If $(X, \mathcal{M}, \mu)$ is a measure space and $\{A_j\}_{j=1}^\infty \subset \mathcal{M}$, we set $\liminf_{j\to\infty} A_j = \cup_{N=1}^\infty \cap_{k=N}^\infty A_k$. Then $\mu(\liminf A_j) \leq \liminf \mu(A_j)$.

Problem 0.17. Show that there exists a Borel set $A \subset [0,1]$ such that $0 < \lambda(A \cap I) < \lambda(I)$ for every subinterval I of $[0,1]$.

Problem 0.18. Show that there does not exist a Borel set $A \subset [0,1]$ such that $\lambda(A \cap I) = \frac{1}{2}\lambda(I)$ for every subinterval I of $[0,1]$. Hint: If A existed show that the function $f(x) = \frac{1}{2} - 1_A$ would have to be zero.

Problem 0.19. Let B denote the open unit ball in $\mathbb{R}^d$ and suppose μ is a finite Borel measure on $\mathbb{R}^d$, show that $x \mapsto \mu(B + x)$ is lower semi-continuous.

Problem 0.20. Let $E, F \subset \mathbb{R}$ be Borel, with $\lambda(E) < \infty$, show that the function $t \mapsto \lambda((E + t) \cap F)$ is continuous.

Problem 0.21. Let A be a Borel subset of the unit square $[0,1]^2$. For $x \in [0,1]$ denote

$$A_x = \{y \in [0,1] \mid (x,y) \in A\}.$$

Suppose $\lambda^2(A) = 1/4$. Prove that $\lambda(\{x \in [0,1] \mid \lambda(A_x) > 1/8\}) \geq 1/8$.

Problem 0.22. Let μ be a finite, positive, Borel measure on $\mathbb{R}^2$, and let $\mathcal{G}$ be the family of finite unions of squares of the form

$$S = \{(x,y) \mid j2^{-n} \leq x \leq (j+1)2^{-n}; k2^{-n} \leq y \leq (k+1)2^{-n}\},$$

where j, k, and n are integers. Prove that the set of linear combinations of characteristic functions of elements from $\mathcal{G}$ is dense in $L^1(\mathbb{R}^2, \mu)$.

Problem 0.23. Let (X, μ) be a σ-finite measure space and suppose $\{f_\alpha\}_{\alpha \in I}$ is a family of non-negative valued measurable functions such that there exists $K < \infty$ with $\|f_\alpha\|_\infty \leq K$ for each $\alpha \in I$. Note that I is an arbitrary, possibly uncountable, set.

Show that there exists a measurable function $f : X \to [0, K]$ such that for each $\alpha \in I$ we have $f - f_\alpha \geq 0$ almost everywhere, and such that if $g : X \to [0, K]$ is any other measurable function such that for each $\alpha \in I$ we have $g - g_\alpha \geq 0$ almost everywhere, then we have $f - g \geq 0$ almost everywhere.

Problem 0.24. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and suppose $f \in L^\infty(X, \mu)$. Set $a_n = \int |f|^n \, d\mu$. Show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \|f\|_\infty$.

Problem 0.25. Let $K \subset \mathbb{R}$ be an uncountable compact set. Show that there exists a Borel measure μ such that $\mu(K) = 1$, $\mu(\mathbb{R} \setminus K) = 0$, and $\mu(\{x\}) = 0$ for any $x \in \mathbb{R}$.

Problem 0.26. Show that there exists no topology on $L^\infty([0,1], \lambda)$ so that a sequence of functions $\{f_n\}_{n=1}^\infty \subset L^\infty([0,1], \lambda)$ converge in this topology to f if and only if the sequence $\{f_n\}_{n=1}^\infty$ converges almost everywhere to f.

Problem 0.27. Suppose (X, μ) is a measure space, $f \in L^1(X, \mu)$ is non-negative, and $g_n \in L^1(X, \mu)$ are non-negative such that $g_n \to f$ a.e., then $\|g_n - f\|_1 \to 0$ if and only if $\int g_n \, d\mu \to \int f \, d\mu$.

Problem 0.28. Suppose (X, μ) is a measure space, and f_n, g_n are integrable functions with $0 \leq f_n \leq g_n$, suppose f is measurable and g is integrable such that $f_n \to f$ a.e., and $\|g_n - g\|_1 \to 0$. Prove that f is integrable and $\|f - f_n\|_1 \to 0$.

Problem 0.29. Let $1 \leq p < \infty$ and suppose $f \in L^p(\mathbb{R})$. Set $f_t(x) = f(x - t)$. Show that $\lim_{t \to 0} \|f - f_t\|_p = 0$.

Problem 0.30. For each interval $I \subset [1, \infty]$ give an example of a function $f : \mathbb{R}^2 \to [0, \infty)$ so that $f \in L^p(\mathbb{R}^2)$ if and only if $p \in I$.

Problem 0.31. Let (X, μ) be a measure space and suppose $1 \leq p < \infty$. If $f_n, f \in L^1(X, \mu)$ such that $\{\|f_n - f\|_p\}_n \in \ell^p\mathbb{N}$ then $f_n \to f$ a.e.

Problem 0.32. Let $f \in L^1(\mathbb{R}^d)$, and $g \in L^\infty(\mathbb{R}^d)$ with compact support, set $g_t(x) = g(x - t)$. Show that $\lim_{|t| \to \infty} \int g_t f \, d\lambda = 0$. Find an example with $f, g \in L^1(\mathbb{R}^d)$ and g having compact support so that $g_t f \in L^1(\mathbb{R}^d)$ for all $t \in \mathbb{R}^d$ but $\lim_{|t| \to \infty} \int g_t f \, d\lambda \neq 0$.

Problem 0.33. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a uniformly continuous function and suppose $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Then $\lim_{|x| \to \infty} f(x) = 0$.

Problem 0.34. Show that a function $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz with Lipschitz constant M if and only if f is absolutely continuous and satisfies $\|f'\|_\infty \leq M$.

Problem 0.35. Let $f : [a, b] \to \mathbb{R}$ and suppose $A \subset (a, b)$ is a set such that f' exists for every $x \in A$ and such that $|f'(x)| < C$ for all $x \in A$. Prove that $\lambda^*(f(A)) \leq C\lambda^*(A)$ where λ^* is Lebesgue outer measure.

Problem 0.36. Let $f : [a, b] \to \mathbb{R}$ and suppose $A \subset (a, b)$ is Borel such that f' exists and is measurable on A then $\lambda^*(f(A)) \leq \int_A |f'| \, d\lambda$. Hint: Use the previous problem.

Problem 0.37 (The Banach-Zaretsky theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function of bounded variation, then f is absolutely continuous if and only if for any set $E \subset [a, b]$ with $\lambda^*(E) = 0$ we have $\lambda^*(f(E)) = 0$. Hint for showing absolute continuity: For $[a_k, b_k] \subset [a, b]$, if we set $A_k = \{x \in (a_k, b_k) \mid f' \text{ exists.}\}$ then we have $\lambda^*([a_k, b_k] \setminus A_k) = 0$, hence $\lambda^*(f([a_k, b_k])) = \lambda^*(f(A_k))$. Now use the previous problem.

Problem 0.38. Let X be an infinite dimensional Banach space. Then X does not have a countable basis as an abstract vector space.

Problem 0.39. Let (X, μ) be a σ-finite measure space. Then $L^1(X, \mu)$ is reflexive if and only if $L^1(X, \mu)$ is finite dimensional.

Problem 0.40. Show that there exists a real valued absolutely continuous function on $[0, 1]$ which is not monotone on any interval of positive length.

Problem 0.41. Show that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is not of bounded variation on any interval of positive length.

Problem 0.42. Let f and g be integrable functions on $[0, 1]$. Set

$$F(x) = \int_0^x f(y) \, d\lambda(y), \qquad G(x) = \int_0^x g(y) \, d\lambda(y)$$

Prove

$$\int_0^1 F(x)g(x) \, d\lambda(x) = F(1)G(1) - \int_0^1 f(x)G(x) \, d\lambda(x).$$

Problem 0.43. Let (X,μ) be a measure space. If $0 < p < q < r \leq \infty$ then $L^p(X,\mu) \cap L^r(X,\mu) \subset L^q(X,\mu) \subset L^p(X,\mu) + L^r(X,\mu)$. Hint: For the first inclusion apply Holder's inequality with $|f|^{\lambda q} \in L^{p/\lambda q}(X,\mu)$ and $|f|^{(1-\lambda)q} \in L^{r/(1-\lambda)q}(X,\mu)$.

Problem 0.44. Suppose (X,μ) is a measure space and $f \in L^p(X,\mu) \cap L^\infty(X,\mu)$ for some $p < \infty$ (hence $f \in L^q(X,\mu)$ for $q \geq p$), then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

Problem 0.45. Suppose (X,μ) and (Y,ν) are σ-finite measure spaces and $K \in L^2(X \times Y, \mu \times \nu)$. For $f \in L^2(X,\mu)$ set $T_K f(y) = \int K(x,y) f(x)\, d\mu(x)$. Show that T_K gives a well defined operator from $L^2(X,\mu)$ into $L^2(Y,\nu)$, and show that $\|T_K\| \leq \|K\|_2$.

Problem 0.46. Let (X,μ) be a σ-finite measure space and suppose $T : L^2(X,\mu) \to L^2(X,\mu)$ is a bounded operator such that for all $f \in L^2(X,\mu)$ and $g \in L^\infty(X,\mu)$ we have $T(gf) = gT(f)$. Show that there exists $h \in L^\infty(X,\mu)$ so that $T(f) = hf$ for all $f \in L^2(X,\mu)$.

Problem 0.47. Let X be a σ-compact, locally compact Hausdorff space. Consider the space of all complex valued continuous functions $C(X)$ endowed with the topology of uniform convergence on compact sets. Show that $C(X)$ is a Fréchet space.

Problem 0.48. If X is a Fréchet space and $Y \subset X$ a subspace which is a G_δ-set, then Y is closed.

Problem 0.49. Show that $\{e^{i2\pi nt}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0,1],\lambda)$.

Problem 0.50. Suppose $x \in [0,1]$ has unique decimal expansion

$$x = 0.a_1 a_2 a_3 \dots.$$

Define

$$A_n(x) = \begin{cases} 1, & \text{if } a_n \text{ is even} \\ -1, & \text{if } a_n \text{ is odd.} \end{cases}$$

Note that A_n is defined almost everywhere. Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f A_n \, d\lambda = 0.$$

Problem 0.51. Suppose $E \subset \mathbb{R}$ is Borel such that $\lambda(E) > 0$. Show that $E - E = \{x - y \mid x, y \in E\}$ contains an interval of positive length containing 0.

Problem 0.52. Let X and Y be Banach spaces and $T : X \to Y$ a linear map. Suppose that $\varphi \circ T \in X^*$ for any $\varphi \in Y^*$, show that T is bounded.

Problem 0.53. Let X and Y be normed spaces with $X \neq \{0\}$, and suppose that $\mathcal{B}(X,Y)$ is complete. Show that Y is complete.

Problem 0.54. Let X be a vector space over $\mathbb{K}$ and suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two complete norms on X such that there exists $C > 0$ with $\|x\|_1 \leq C\|x\|_2$ for all $x \in X$. Show that there exists $C' > 0$ so that $\|x\|_2 \leq C'\|x\|_1$ for all $x \in X$.

Problem 0.55. Let X and Y be Banach space, and suppose $\{T_n\} \subset \mathcal{B}(X,Y)$ is a sequence of bounded linear maps such that $T_n x$ has a limit for each $x \in X$. If we set $Tx = \lim_{n\to\infty} T_n x$ prove that $T \in \mathcal{B}(X,Y)$.

Problem 0.56. Define $\varphi_n \in \ell^\infty(\mathbb{N})^*$ by $\varphi_n(f) = \frac{1}{n}\sum_{k=1}^n f(n)$, show that if φ is a weak$*$ cluster point of $\{\varphi_n\}_n$ then $\varphi \notin \ell^1(\mathbb{N})$.

Problem 0.57. Show that $L^1([0,1],\lambda)$ is not isometric to the dual of any Banach space. Here λ is Lebesgue measure. Hint: What are the extreme points in the unit ball?

Problem 0.58. Let $\mathcal{H}$ be a separable Hilbert space and fix an orthonormal basis $\{e_n\}_n$. Show that 0 belongs to the weak closure of $\{\sqrt{n}e_n\}_n$, but $\sqrt{n}e_n \not\to 0$ weakly.

Problem 0.59. Let X be a Fréchet space, then X^* is a Fréchet space in the weak*-topology if and only if X is isomorphic to a separable Banach space. Hint: If $\{\rho_n\}_{n\in\mathbb{N}}$ is a family of seminorms which give the topology on X, first show that $A_n = \{\varphi \in X^* \mid |\varphi(x)| \leq n\sum_{k=1}^n \rho_k(x) \text{ for all } x \in X\}$ is weak*-compact by the Banach-Alaoglu theorem, then show that $X = \cup_n A_n$ and use the Baire property.

Problem 0.60. Let $\mathcal{H}$ be a Hilbert space and suppose $\mathcal{C} \subset \mathcal{H}$ is a non-empty closed convex set. Show that $\mathcal{C}$ has a unique element of minimal norm.

Problem 0.61. Let $\mathcal{H}$ be a Hilbert space. A **sesquilinear form** on $\mathcal{H}$ is a map $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ so that $B(\alpha\xi + \eta, \zeta) = \alpha B(\xi, \zeta) + B(\eta, \zeta)$, and $B(\xi, \alpha\eta + \zeta) = \overline{\alpha}B(\xi, \eta) + B(\xi, \zeta)$ for all $\alpha \in \mathbb{C}$ and $\xi, \eta, \zeta \in \mathcal{H}$. The sesquilinear form is **bounded** if there exists $C \geq 0$ so that $|B(\xi, \eta)| \leq C\|\xi\|\|\eta\|$, for all $\xi, \eta \in \mathcal{H}$, and the norm of B, denoted $\|B\|$ is defined to be the infemum over all such C.

Show that if B is a bounded sesquilinear form then there exists a unique bounded linear map A such that $B(\xi, \eta) = \langle A\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$. Moreover, show that $\|B\| = \|A\|$.

Problem 0.62. Let $\mathcal{H}$ be a Hilbert space. Suppose $T \in \mathcal{B}(\mathcal{H})$ such that $\|\mathrm{id}_{\mathcal{H}} - T\| < 1$. Show that T has a bounded inverse.

Problem 0.63. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and suppose μ is a Borel measure on $\mathcal{H}$ such that $\mu(E + \xi) = \mu(E)$ for each borel set E and $\xi \in \mathcal{H}$, and suppose μ gives positive measure to the unit ball. Then $\mu(E) = \infty$ for any open set E. (Bonus: Show this is also the case for any infinite dimensional Banach space. Hint: Use Riesz' lemma.)

Problem 0.64 (Birkhoff-Rota)**.** Let $\mathcal{H}$ be a Hilbert space and suppose $\{e_n\}_n$ is an orthonormal basis and $\{f_n\}_n$ is an orthonormal set such that

$$\sum_n \|e_n - f_n\|^2 < \infty,$$

then $\{f_n\}_n$ is also an orthonormal basis. Hint: Take N so that $\sum_{n>N} \|e_n - f_n\|^2 < 1$ and use Parseval's inequality to show that the span of $\{e_1, \ldots, e_N, f_{N+1}, F_{N+1}, \ldots\}$ is dense. Use this to show that if $\mathcal{K} = \{f_{N+1}, \ldots\}^\perp$ then $\dim \mathcal{K} \leq N$. Since $\{f_1, \ldots, f_N\} \subset \mathcal{K}$ are orthogonal it then follows that $\mathcal{K} = \mathrm{sp}\{f_1, \ldots, f_N\}$.

Problem 0.65 (Banach-Saks)**.** Let $\mathcal{H}$ be a Hilbert space and $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ a uniformly bounded sequence, then there exists a subsequence $\{\xi_{n_k}\}_k$ so that the Cesàro means $\frac{1}{K}\sum_{k=1}^K \xi_{n_k}$ converges in $\mathcal{H}$.

Problem 0.66. Define $U : L^2(\mathbb{R}, \lambda) \to L^2(\mathbb{R}, \lambda)$ by $Uf(x) = f(x - 1)$. Show that U has no non-zero eigenvectors.