MATH 6101 - HOMEWORK ASSIGNMENT 6

DUE THURSDAY, MARCH 2ND BY 6:00PM

Exercise 0.1. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and suppose $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $\ker(T) = \mathrm{Range}(T^*)^\perp$.

Exercise 0.2. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces.

(1) A bounded linear operator $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection operator if and only if $P = P^*$ and $P^2 = P$.
(2) A bouned operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is isometric if and only if $U^*U = \mathrm{id}$.

A linear operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a **partial isometry** if V^*V is an orthogonal projection.

Exercise 0.3. Suppose $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a partial isometry.

(1) V^*V is the orthogonal projection onto $\ker(V)^\perp$.
(2) $\mathrm{Range}(V)$ is closed and VV^* is the orthogonal projection onto $\mathrm{Range}(V)$. In particular, V^* is also a partial isometry.

Exercise 0.4 (Von Neumann's mean ergodic theorem)**.** Let U be a unitary operator on a Hilbert space $\mathcal{H}$, set $\mathcal{K} = \{\xi \in \mathcal{H} \mid U\xi = \xi\}$, and let P denote the orthogonal projection onto $\mathcal{K}$. If $S_n = \frac{1}{n}\sum_{k=0}^{n-1} U^k$ then for all $\xi \in \mathcal{U}$ we have $S_n\xi \to P\xi$. Hint: Use Exercise 0.1 applied to the operator $1 - U$.

Exercise 0.5. Let $\mathcal{H}$ be a Hilbert space. A **sesquilinear form** on $\mathcal{H}$ is a map $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ so that $B(\alpha\xi + \eta, \zeta) = \alpha B(\xi, \zeta) + B(\eta, \zeta)$, and $B(\xi, \alpha\eta + \zeta) = \overline{\alpha}B(\xi, \eta) + B(\xi, \zeta)$ for all $\alpha \in \mathbb{C}$ and $\xi, \eta, \zeta \in \mathcal{H}$. The sesquilinear form is **bounded** if there exists $C \geq 0$ so that $|B(\xi, \eta)| \leq C\|\xi\|\|\eta\|$, for all $\xi, \eta \in \mathcal{H}$, and the norm of B, denoted $\|B\|$ is defined to be the infemum over all such C.

Show that if B is a bounded sesquilinear form then there exists a unique bounded linear map A such that $B(\xi, \eta) = \langle A\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$. Moreover, show that $\|B\| = \|A\|$.

Exercise 0.6 (Birkhoff-Rota)**.** Let $\mathcal{H}$ be a Hilbert space and suppose $\{e_n\}_n$ is an orthonormal basis and $\{f_n\}_n$ is an orthonormal set such that

$$\sum_n \|e_n - f_n\|^2 < \infty,$$

then $\{f_n\}_n$ is also an orthonormal basis. Hint: Take N so that $\sum_{n>N} \|e_n - f_n\|^2 < 1$ and use Parseval's inequality to show that the span of $\{e_1, \ldots, e_N, f_{N+1}, F_{N+1}, \ldots\}$ is dense. Use this to show that if $\mathcal{K} = \{f_{N+1}, \ldots\}^\perp$ then $\dim \mathcal{K} \leq N$. Since $\{f_1, \ldots, f_N\} \subset \mathcal{K}$ are orthogonal it then follows that $\mathcal{K} = \mathrm{sp}\{f_1, \ldots, f_N\}$.

Date: February 16, 2017.