MATH 6101 - HOMEWORK ASSIGNMENT 5

DUE THURSDAY, FEBRUARY 23RD BY 6:00PM

Exercise 0.1. Let X be a real normed space with norm $\|\cdot\|$. Then $\|\cdot\|$ comes from an inner product if and only if the parallelogram identity $\|\xi+\eta\|^2+\|\xi-\eta\|^2 = 2(\|\xi\|^2+\|\eta\|^2)$ holds.

Exercise 0.2. Define $U : L^2(\mathbb{R},\lambda) \to L^2(\mathbb{R},\lambda)$ by $Uf(x) = f(x-1)$. Show that U has no non-zero eigenvectors.

Exercise 0.3 (The Banach-Saks theorem)**.** Let $\mathcal{H}$ be a Hilbert space and $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ a uniformly bounded sequence, then there exists a subsequence $\{\xi_{n_k}\}_k$ so that the Cesàro means $\frac{1}{K}\sum_{k=1}^{K}\xi_{n_k}$ converges in $\mathcal{H}$. Hint: Using the Banach-Alaoglu theorem you may assume that ξ_n has a weak limit.

Exercise 0.4. If $\mathcal{H}$ is a Hilbert space and $A \subset \mathcal{H}$ is a subspace then $(A^\perp)^\perp = \overline{A}$. Show that this does not hold for general inner product spaces.

Exercise 0.5. Let $\mathcal{H}$ be a Hilbert space, and set

$$G = \{T \in \mathcal{B}(\mathcal{H}) \mid T \text{ has a bounded inverse }\}.$$

Show that G is an open subset of $\mathcal{B}(\mathcal{H})$ with respect to the operator norm.

Exercise 0.6. Let M be a closed subspace of $L^2([0,1],\lambda)$ such that M is contained in $C([0,1])$.

(1) There exists $C > 0$ such that $\|f\|_\infty \leq C\|f\|_{L^2}$ for all $f \in M$.
(2) For each $x \in [0,1]$ there exists $g_x \in M$ so that $f(x) = \langle f, g_x\rangle$ for all $f \in M$. Moreover, $\|g_x\|_{L^2} \leq C$.
(3) $\dim M \leq C^2$. Hint: If $\{f_i\}_i$ is an orthonormal sequence in M then $\sum_i |f_i(x)|^2 \leq C^2$ for all $x \in [0,1]$.

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