# MATH 6101 - HOMEWORK ASSIGNMENT 3

DUE THURSDAY, FEBRUARY 9TH BY 6:00PM

**Exercise 0.1.** Let $X$ be a $\sigma$-compact, locally compact Hausdorff space. Consider $C(X)$ endowed with the topology of uniform convergence on compact sets. Then $C(X)$ is a Fréchet space.

**Exercise 0.2.** Let $X$ be a vector space over $\mathbb{K}$ and suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two complete norms on $X$ such that there exists $C > 0$ with $\|x\|_1 \leq C\|x\|_2$ for all $x \in X$. Show that there exists $C' > 0$ so that $\|x\|_2 \leq C'\|x\|_1$ for all $x \in X$.

**Exercise 0.3.** Let $X$ be a topological vector space and suppose $\varphi \in X^*$ is not the zero functional, then $\varphi$ is an open map, i.e., if $G \subset X$ is open then $\varphi(G)$ is also open.

**Exercise 0.4.** Let $X$ be a locally convex topological vector space whose topology is defined by a family of seminorms $\mathcal{F}$. If $\varphi \in X^*$ then there exist $\rho_1, \ldots, \rho_n \in \mathcal{F}$ and $K > 0$ so that $|\varphi(x)| \leq K \sum_{i=1}^n \rho_i(x)$, for all $x \in X$.

**Exercise 0.5.** Consider $L^\infty([0,1]) = L^1([0,1])^*$ where $[0,1]$ is endowed with Lebesgue measure.

(a) $\mathrm{span}\{e^{i2\pi nt}\}_{n\in\mathbb{Z}}$ is weak$^*$-dense in $L^\infty([0,1])$. Hint: Use Lusin's theorem and the Stone-Weierstrass theorem.

(b) The sequence $\{e^{i2\pi nt}\}_{n\in\mathbb{N}} \subset L^\infty([0,1]) = L^1([0,1])^*$ converges to 0 in the weak$*$-topology.

**Exercise 0.6.** If $X$ is a Fréchet space then a subspace $Y \subset X$ is a Fréchet space (with respect to the subspace topology) if and only if $Y$ is closed.

*Date*: February 2, 2017.