MATH 6101 - HOMEWORK ASSIGNMENT 2

DUE THURSDAY, FEBRUARY 2ND BY 6:00PM

Exercise 0.1. Show that a function $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz with Lipschitz constant M if and only if f is absolutely continuous and satisfies $\|f'\|_\infty \leq M$.

Exercise 0.2. Let f and g be integrable functions on $[0, 1]$. Set

$$F(x) = \int_0^x f(y)\, d\lambda(y), \qquad G(x) = \int_0^x g(y)\, d\lambda(y)$$

Prove

$$\int_0^1 F(x)g(x)\, d\lambda(x) = F(1)G(1) - \int_0^1 f(x)G(x)\, d\lambda(x).$$

Exercise 0.3. Let (X, μ) be a measure space. If $0 < p < q < r \leq \infty$ then $L^p(X, \mu) \cap L^r(X, \mu) \subset L^q(X, \mu) \subset L^p(X, \mu) + L^r(X, \mu)$. Hint: For the first inclusion apply Holder's inequality with $|f|^{\lambda q} \in L^{p/\lambda q}(X, \mu)$ and $|f|^{(1-\lambda)q} \in L^{r/(1-\lambda)q}(X, \mu)$.

Exercise 0.4. Suppose (X, μ) is a measure space and $f \in L^p(X, \mu) \cap L^\infty(X, \mu)$ for some $p < \infty$ (hence $f \in L^q(X, \mu)$ for $q \geq p$), then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

Exercise 0.5. Suppose (X, μ) and (Y, ν) are σ-finite measure spaces and $K \in L^2(X \times Y, \mu \times \nu)$. For $f \in L^2(X, \mu)$ set $T_K f(y) = \int K(x, y) f(x)\, d\mu(x)$. Show that T_K gives a well defined operator from $L^2(X, \mu)$ into $L^2(Y, \nu)$, and show that $\|T_K\| \leq \|K\|_2$.

Date: January 25, 2017.