Notes on operator algebras

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Chapter 1

Spectral theory

If A is a complex unital algebra then we denote by $G(A)$ the set of elements which have a two sided inverse. If $x \in A$, the **spectrum** of x is

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \notin G(A)\}.$$

The complement of the spectrum is called the **resolvent** and denoted $\rho_A(x)$.

Proposition 1.0.1. *Let A be a unital algebra over $\mathbb{C}$, and consider $x, y \in A$. Then $\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}$.*

Proof. If $1 - xy \in G(A)$ then we have

$$\begin{aligned}(1 - yx)(1 + y(1 - xy)^{-1}x) &= 1 - yx + y(1 - xy)^{-1}x - yxy(1 - xy)^{-1}x \\ &= 1 - yx + y(1 - xy)(1 - xy)^{-1}x = 1.\end{aligned}$$

Similarly, we have

$$(1 + y(1 - xy)^{-1}x)(1 - yx) = 1,$$

and hence $1 - yx \in G(A)$. ∎

Knowing the formula for the inverse beforehand of course made the proof of the previous proposition quite a bit easier. But this formula is quite natural to consider. Indeed, if we just consider formal power series then we have

$$(1 - yx)^{-1} = \sum_{k=0}^{\infty}(yx)^k = 1 + y(\sum_{k=0}^{\infty}(xy)^k)x = 1 + y(1 - xy)^{-1}x.$$

1.1 Banach and C^*-algebras

A **Banach algebra** is a Banach space A, which is also an algebra such that

$$\|xy\| \leq \|x\|\|y\|.$$

A Banach algebra A is **involutive** if it possesses an anti-linear involution $*$, such that $\|x^*\| = \|x\|$, for all $x \in A$.

If an involutive Banach algebra A additionally satisfies

$$\|x^*x\| = \|x\|^2,$$

for all $x \in A$, then we say that A is a **C^*-algebra**. If a Banach or C^*-algebra is unital, then we further require $\|1\| = 1$.

Note that if A is a unital involutive Banach algebra, and $x \in G(A)$ then $(x^{-1})^* = (x^*)^{-1}$, and hence $\sigma_A(x^*) = \overline{\sigma_A(x)}$.

Example 1.1.1. Let K be a locally compact Hausdorff space. Then the space $C_0(K)$ of complex valued continuous functions which vanish at infinity is a C^*-algebra when given the supremum norm $\|f\|_\infty = \sup_{x\in K} |f(x)|$. This is unital if and only if K is compact.

Example 1.1.2. Let $\mathcal{H}$ be a complex Hilbert space. Then the space of all bounded operators $\mathcal{B}(\mathcal{H})$ is a C^*-algebra when endowed with the operator norm $\|x\| = \sup_{\xi\in\mathcal{H}, \|\xi\|\le 1} \|x\xi\|$.

Lemma 1.1.3. *Let A be a unital Banach algebra and suppose $x \in A$ such that $\|1 - x\| < 1$, then $x \in G(A)$.*

Proof. Since $\|1 - x\| < 1$, the element $y = \sum_{k=0}^{\infty}(1 - x)^k$ is well defined, and it is easy to see that $xy = yx = 1$. ■

Proposition 1.1.4. *Let A be a unital Banach algebra, then $G(A)$ is open, and the map $x \mapsto x^{-1}$ is a continuous map on $G(A)$.*

Proof. If $y \in G(A)$ and $\|x - y\| < \|y^{-1}\|$ then $\|1 - xy^{-1}\| < 1$ and hence by the previous lemma $xy^{-1} \in G(A)$ (hence also $x = xy^{-1}y \in G(A)$) and

$$\begin{aligned}\|xy^{-1}\| &\le \sum_{n=0}^{\infty} \|(1 - xy^{-1})\|^n \\ &\le \sum_{n=0}^{\infty} \|y^{-1}\|^n \|y - x\|^n = \frac{1}{1 - \|y\|^{-1}\|y - x\|}.\end{aligned}$$

Hence,

$$\begin{aligned}\|x^{-1} - y^{-1}\| &= \|x^{-1}(y - x)y^{-1}\| \\ &\le \|y^{-1}(xy^{-1})^{-1}\|\|y^{-1}\|\|y - x\| \le \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\|\|y - x\|}\|y - x\|.\end{aligned}$$

Thus continuity follows from continuity of the map $t \mapsto \frac{\|y^{-1}\|^2}{1-\|y^{-1}\|t}t$, at $t = 0$. ■

Proposition 1.1.5. *Let A be a unital Banach algebra, and suppose $x \in A$, then $\sigma_A(x)$ is a non-empty compact set.*

Proof. If $\|x\| < |\lambda|$ then $\frac{x}{\lambda} - 1 \in G(A)$ by Lemma 1.1.3, also $\sigma_A(x)$ is closed by Proposition 1.1.4, thus $\sigma_A(x)$ is compact.

To see that $\sigma_A(x)$ is non-empty note that for any linear functional $\varphi \in A^*$, we have that $f(z) = \varphi((x-z)^{-1})$ is analytic on $\rho_A(x)$. Indeed, if $z, z_0 \in \rho_A(x)$ then we have

$$(x-z)^{-1} - (x-z_0)^{-1} = (x-z)^{-1}(z-z_0)(x-z_0)^{-1}.$$

Since inversion is continuous it then follows that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \varphi((x-z_0)^{-2}).$$

We also have $\lim_{z\to\infty} f(z) = 0$, and hence if $\sigma_A(x)$ were empty then f would be a bounded entire function and we would then have $f = 0$. Since $\varphi \in A^*$ were arbitrary this would then contradict the Hahn-Banach theorem. ■

Theorem 1.1.6 (Gelfand-Mazur)**.** *Suppose A is a unital Banach algebra such that every non-zero element is invertible, then $A \cong \mathbb{C}$.*

Proof. Fix $x \in A$, and take $\lambda \in \sigma(x)$. Since $x - \lambda$ is not invertible we have that $x - \lambda = 0$, and the result then follows. ■

If $f(z) = \sum_{k=0}^n a_k z^k$ is a polynomial, and $x \in A$, a unital Banach algebra, then we define $f(x) = \sum_{k=0}^n a_k x^k \in A$.

Proposition 1.1.7. *Let A be a unital Banach algebra, $x \in A$ and f a polynomial. then $\sigma_A(f(x)) = f(\sigma_A(x))$.*

Proof. If $\lambda \in \sigma_A(x)$, and $f(z) = \sum_{k=0}^n a_k z^k$ then

$$\begin{aligned} f(x) - f(\lambda) &= \sum_{k=1}^n a_k (x^k - \lambda^k) \\ &= (x-\lambda) \sum_{k=1} a_k \sum_{j=0}^{k-1} x^j \lambda^{k-j-1}, \end{aligned}$$

hence $f(\lambda) \in \sigma_A(x)$. conversely if $\mu \notin f(\sigma_A(x))$ and we factor $f - \mu$ as

$$f - \mu = \alpha_n (x - \lambda_1) \cdots (x - \lambda_n),$$

then since $f(\lambda) - \mu \neq 0$, for all $\lambda \in \sigma_A(x)$ it follows that $\lambda_i \notin \sigma_A(x)$, for $1 \leq i \leq n$, hence $f(x) - \mu \in G(A)$. ■

If A is a unital Banach algebra and $x \in A$, the **spectral radius** of x is

$$r(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|.$$

Note that by Proposition 1.1.5 the spectral radius is finite, and the supremum is attained. Also note that by Proposition 1.0.1 we have the very useful equality $r(xy) = r(yx)$ for all x and y in a unital Banach algebra A. A priori the spectral radius depends on the Banach algebra in which x lives, but we will show now that this is not the case.

Proposition 1.1.8. *Let A be a unital Banach algebra, and suppose $x \in A$. Then $\lim_{n\to\infty} \|x^n\|^{1/n}$ exists and we have*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

Proof. By Proposition 1.1.7 we have $r(x^n) = r(x)^n$, and hence

$$r(x)^n \leq \|x^n\|,$$

showing that $r(x) \leq \liminf_{n\to\infty} \|x^n\|^{1/n}$.

To show that $r(x) \geq \limsup_{n\to\infty} \|x^n\|^{1/n}$, consider the domain $\Omega = \{z \in \mathbb{C} \mid |z| > r(x)\}$, and fix a linear functional $\varphi \in A^*$. We showed in Proposition 1.1.5 that $z \mapsto \varphi((x-z)^{-1})$ is analytic in Ω and as such we have a Laurent expansion

$$\varphi((z-x)^{-1}) = \sum_{n=0}^{\infty} \frac{a_n}{z^n},$$

for $|z| > r(x)$. However, we also know that for $|z| > \|x\|$ we have

$$\varphi((z-x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n}.$$

By uniqueness of the Laurent expansion we then have that

$$\varphi((z-x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n},$$

for $|z| > r(x)$.

Hence for $|z| > r(x)$ we have that $\lim_{n\to\infty} \frac{\varphi(x^{n-1})}{|z|^n} = 0$, for all linear functionals $\varphi \in A^*$. By the uniform boundedness principle we then have $\lim_{n\to\infty} \frac{\|x^{n-1}\|}{|z|^n} = 0$, hence $|z| > \limsup_{n\to\infty} \|x^n\|^{1/n}$, and thus

$$r(x) \geq \limsup_{n\to\infty} \|x^n\|^{1/n}.$$ ∎

Exercise 1.1.9. Suppose A is a unital Banach algebra, and $I \subset A$ is a closed two sided ideal, then A/I is again a unital Banach algebra, when given the norm $\|a + I\| = \inf_{y\in I} \|a + y\|$, and $(a + I)(b + I) = (ab + I)$.

Exercise 1.1.10. Let A be a unital Banach algebra and suppose $x, y \in A$ such that $xy = yx$. Show that $r(xy) \leq r(x)r(y)$.

1.2 The Gelfand transform

Let A be a abelian Banach algebra, the **spectrum** of A, denoted by $\sigma(A)$, is the set of continuous $*$-homomorphsims $\varphi : A \to \mathbb{C}$ such that $\|\varphi\| = 1$, which we endow with the weak*-topology as a subset of A^*.

Note that if A is unital, and $\varphi : A \to \mathbb{C}$ is a $*$-homomorphism, then it follows easily that $\ker(\varphi) \cap G(A) = \emptyset$. In particular, this shows that $\varphi(x) \in \sigma(x)$, since $x - \varphi(x) \in \ker(\varphi)$. Hence, for all $x \in A$ we have $|\varphi(x)| \leq r(x) \leq \|x\|$. Since, $\varphi(1) = 1$ this shows that the condition $\|\varphi\| = 1$ is automatic in the unital case.

It is also easy to see that when A is unital $\sigma(A)$ is closed and bounded, by the Banach-Alaoglu theorem it is then a compact Hausdorff space.

Proposition 1.2.1. *Let A be a unital Banach algebra. Then the association $\varphi \mapsto \ker(\varphi)$ gives a bijection between the spectrum of A and the space of maximal ideals.*

Proof. If $\varphi \in \sigma(A)$ then $\ker(\varphi)$ is clearly an ideal, and if we have a larger ideal I, then there exists $x \in I$ such that $\varphi(x) \neq 0$, hence $1 - x/\varphi(x) \in \ker(\varphi) \subset I$ and so $1 = (1 - x/\varphi(x)) + x/\varphi(x) \in I$ which implies $I = A$.

Conversely, if $I \subset A$ is a maximal ideal, then $I \cap G(A) = \emptyset$ and hence $\|1 - y\| \geq 1$ for all $y \in I$. Thus, $\overline{I}$ is also an ideal and $1 \notin I$ which shows that $I = \overline{I}$ by maximality. We then have that A/I is a unital Banach algebra, and since I is maximal we have that all non-zero elements of A/I are invertible. Thus, by the Gelfand-Mazur theorem we have $A/I \cong \mathbb{C}$ and hence the projection map $\pi : A \to A/I \cong \mathbb{C}$ gives a continuous homomorphism with kernel I. ∎

Suppose A is a unital C^*-algebra which is generated (as a unital C^*-algebra) by a single element x, if $\lambda \in \sigma_A(x)$ then we can consider the closed ideal generated by $x - \lambda$ which is maximal since x generates A. This therefore induces a map from $\sigma_A(x)$ to $\sigma(A)$. We leave it to the reader to check that this map is actually a homeomorphism.

Let A be a unital abelian Banach algebra, the **Gelfand transform** is the map $\Gamma : A \to C(\sigma(A))$ defined by

$$\Gamma(x)(\varphi) = \varphi(x).$$

Theorem 1.2.2. *Let A be a unital abelian Banach algebra, then the Gelfand transform is a contractive homomorphism, and $\Gamma(x)$ is invertible in $C(\sigma(A))$ if and only if x is invertible in A.*

Proof. It is easy to see that the Gelfand transform is a contractive homomorphism. Also, if $x \in G(A)$, then $\Gamma(a)\Gamma(a^{-1}) = \Gamma(aa^{-1}) = \Gamma(1) = 1$, hence $\Gamma(x)$ is invertible. Conversely, if $x \notin G(A)$ then since A is abelian we have that the ideal generated by x is non-trivial, hence by Zorn's lemma we see that x is contained in a maximal ideal $I \subset A$, and from Proposition 1.2.1 there exists $\varphi \in \sigma(A)$ such that $\Gamma(x)(\varphi) = \varphi(x) = 0$. Hence, in this case $\Gamma(x)$ is not invertible. ∎

Corollary 1.2.3. *Let A be a unital abelian Banach algebra, then $\sigma(\Gamma(x)) = \sigma(x)$, and in particular $\|\Gamma(x)\| = r(\Gamma(x)) = r(x)$, for all $x \in A$.*

1.3 Continuous functional calculus

Let A be a C^*-algebra. An element $x \in A$ is:

- **normal** if $xx^* = x^*x$.
- **self-adjoint** if $x = x^*$, and **skew-adjoint** if $x = -x^*$.
- **positive** if $x = y^*y$ for some $y \in A$.
- a **projection** if $x^* = x^2 = x$.
- **unitary** if A is unital, and $x^*x = xx^* = 1$.
- **isometric** if A is unital, and $x^*x = 1$.
- **partially isometric** if x^*x is a projection.

We denote by A_+ the set of positive elements, and $a, b \in A$ are two self-adjoint elements then we write $a \leq b$ if $b - a \in A_+$. Note that if $x \in A$ then $x^*A_+x \subset A_+$, in particular, if $a, b \in A$ are self-adjoint such that $a \leq b$, then $x^*ax \leq x^*bx$.

Proposition 1.3.1. *Let A be a C^*-algebra and $x \in A$ normal, then $\|x\| = r(x)$.*

Proof. We first show this if x is self-adjoint, in which case we have $\|x^2\| = \|x\|^2$, and by induction we have $\|x^{2^n}\| = \|x\|^{2^n}$ for all $n \in \mathbb{N}$. Therefore, $\|x\| = \lim_{n\to\infty} \|x^{2^n}\|^{2^n} = r(x)$.

If x is normal then by Exercise 1.1.10 we have

$$\|x\|^2 = \|x^*x\| = r(x^*x) \leq r(x^*)r(x) = r(x)^2 \leq \|x\|^2. \qquad \blacksquare$$

Corollary 1.3.2. *Let A and B be two unital C^*-algebras and $\Phi : A \to B$ a unital $*$-homomorphism, then Φ is contractive. If Φ is a $*$-isomorphism, then Φ is isometric.*

Proof. Since Φ is a unital $*$-homomorphism we clearly have $\Phi(G(A)) \subset G(B)$, from which it follows that $\sigma_B(\Phi(x)) \subset \sigma_A(x)$, and hence $r(\Phi(x)) \leq r(x)$, for all $x \in A$. By Proposition 1.3.1 we then have

$$\|\Phi(x)\|^2 = \|\Phi(x^*x)\| = r(\Phi(x^*x)) \leq r(x^*x) = \|x^*x\| = \|x\|^2.$$

If Φ is a $*$-isomorphism then so is Φ^{-1} which then shows that Φ is isometric. $\blacksquare$

Corollary 1.3.3. *Let A be a unital complex involutive algebra, then there is at most one norm on A which makes A into a C^*-algebra.*

Proof. If there were two norms which gave a C^*-algebra structure to A then by the previous corollary the identity map would be an isometry. $\blacksquare$

Lemma 1.3.4. *Let A be a unital C^*-algebra, if $x \in A$ is self-adjoint then $\sigma_A(x) \subset \mathbb{R}$.*

Proof. Suppose $\lambda = \alpha + i\beta \in \sigma_A(x)$ where $\alpha, \beta \in \mathbb{R}$. If we consider $y = x - \alpha + it$ where $t \in \mathbb{R}$, then we have $i(\beta + t) \in \sigma_A(y)$ and y is normal. Hence,

$$\begin{aligned}(\beta + t)^2 \leq r(y)^2 &= \|y\|^2 = \|y^* y\| \\ &= \|(x-\alpha)^2 + t^2\| \leq \|x - \alpha\|^2 + t^2,\end{aligned}$$

and since $t \in \mathbb{R}$ was arbitrary it then follows that $\beta = 0$. ∎

Lemma 1.3.5. *Let A be a unital Banach algebra and suppose $x \notin G(A)$. If $x_n \in G(A)$ such that $\|x_n - x\| \to 0$, then $\|x_n^{-1}\| \to \infty$.*

Proof. If $\|x_n^{-1}\|$ were bounded then we would have that $\|1 - x x_n^{-1}\| < 1$ for some n. Thus, we would have that $x x_n^{-1} \in G(A)$ and hence also $x \in G(A)$. ∎

Proposition 1.3.6. *Let B be a unital C^*-algebra and $A \subset B$ a unital C^*-subalgebra. If $x \in A$ then $\sigma_A(x) = \sigma_B(x)$.*

Proof. Note that we always have $G(A) \subset G(B)$. If $x \in A$ is self-adjoint such that $x \notin G(A)$, then by Lemma 1.3.4 we have $it \in \rho_A(x)$ for $t > 0$. By the previous lemma we then have

$$\lim_{t \to 0} \|(x - it)^{-1}\| = \infty,$$

and thus $x \notin G(B)$ since inversion is continuous in $G(B)$.

For general $x \in A$ we then have

$$x \in G(A) \Leftrightarrow x^* x \in G(A) \Leftrightarrow x^* x \in G(B) \Leftrightarrow x \in G(B).$$

In particular, we have $\sigma_A(x) = \sigma_B(x)$ for all $x \in A$. ∎

Because of the previous result we will henceforth write simply $\sigma(x)$ for the spectrum of an element in a C^*-algebra.

Theorem 1.3.7. *Let A be a unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C(\sigma(A))$ gives an isometric isomorphism between A and $C(\sigma(A))$.*

Proof. If x is self-adjoint then from Lemma 1.3.4 we have $\sigma(\Gamma(x)) = \sigma(x) \subset \mathbb{R}$, and hence $\overline{\Gamma(x)} = \Gamma(x^*)$. In general, if $x \in A$ we can write x as $x = a + ib$ where $a = \frac{x + x^*}{2}$ and $b = \frac{i(x^* - x)}{2}$ are self-adjoint. Hence, $\Gamma(x^*) = \Gamma(a - ib) = \Gamma(a) - i\Gamma(b) = \overline{\Gamma(a) + i\Gamma(b)} = \overline{\Gamma(x)}$ and so Γ is a $*$-homomorphism.

By Proposition 1.3.1, if $x \in A$ we then have

$$\begin{aligned}\|x\|^2 &= \|x^* x\| = r(x^* x) \\ &= r(\Gamma(x^* x)) = \|\Gamma(x^*)\Gamma(x)\| = \|\Gamma(x)\|^2,\end{aligned}$$

and so Γ is isometric, and in particular injective.

To show that Γ is surjective note that $\Gamma(A)$ is self-adjoint, and closed since Γ is isometric. Moreover, $\Gamma(A)$ contains the constants and clearly separates points, hence $\Gamma(A) = C(\sigma(A))$ by the Stone-Weierstrauss theorem. ∎

Since we have seen above that if A is generated as a unital C^*-algebra by a single normal element $x \in A$, then we have a natural homeomorphism $\sigma(x) \cong \sigma(A)$. Thus by considering the inverse Gelfand transform we obtain an isomorphism between $C(\sigma(x))$ and A which we denote by $f \mapsto f(x)$.

Theorem 1.3.8 (Continuous functional calculus)**.** *Let A and B be a unital C^*-algebras, with $x \in A$ normal, then this functional calculus satisfies the following properties:*

(i) *The map $f \mapsto f(x)$ is a homomorphism from $C(\sigma(x))$ to A, and if $f(z) = \sum_{k=0}^n a_k z^k$ is a polynomial, then $f(x) = \sum_{k=0}^n a_k x^k$.*

(ii) *For $f \in C(\sigma(x))$ we have $\sigma(f(x)) = f(\sigma(x))$.*

(iii) *If $\Phi : A \to B$ is a C^*-homomorphism then $\Phi(f(x)) = f(\Phi(x))$.*

(iv) *If $x_n \in A$ is a sequence of normal elements such that $\|x_n - x\| \to 0$, Ω is a compact neighborhood of $\sigma(x)$, and $f \in C(\Omega)$, then for large enough n we have $\sigma(x_n) \subset \Omega$ and $\|f(x_n) - f(x)\| \to 0$.*

Proof. Parts (i), and (ii) follow easily from Theorem 1.3.7. Part (iii) is obvious for polynomials and then follows for all continuous functions by approximation.

For part (iv), the fact that $\sigma(x_n) \subset \Omega$ for large n follows from continuity of inversion. If we write $C = \sup_n \|x_n\|$ and we have $\varepsilon > 0$ arbitrary, then we may take a polynomial $g : \Omega \to \mathbb{C}$ such that $\|f - g\|_\infty < \varepsilon$ and we have

$$\limsup_{n\to\infty} \|f(x_n) - f(x)\| \leq 2\|f - g\|_\infty C + \limsup_{n\to\infty} \|g(x_n) + g(x)\| < 2C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have $\lim_{n\to\infty} \|f(x_n) - f(x)\| = 0$. ■

1.3.1 The non-unital case

If A is not a unital C^*-algebra then we may consider the space $\tilde{A} = A \oplus \mathbb{C}$ which is a $*$-algebra with multiplication

$$(x \oplus \alpha) \cdot (y \oplus \beta) = (xy + \alpha y + \beta x) \oplus \alpha\beta,$$

and involution $(x \oplus \alpha)^* = x^* \oplus \overline{\alpha}$. We may also place a norm on $\tilde{A}$ given by

$$\|x \oplus \alpha\| = \sup_{y \in A, \|y\| \leq 1} \|xy + \alpha y\|.$$

We call $\tilde{A}$ the **unitization** of A.

Proposition 1.3.9. *Let A be a non-unital C^*-algebra, then the unitization $\tilde{A}$ is again a C^*-algebra, and the map $x \mapsto x \oplus 0$ is an isometric $*$-isomorphism of A onto a maximal ideal in $\tilde{A}$.*

Proof. The map $x \mapsto x \oplus 0$ is indeed isometric since on one hand we have $\|x \oplus 0\| = \sup_{y \in A, \|y\| \leq 1} \|xy\| \leq \|x\|$, while on the other hand if $x \neq 0$, and we set $y = x^*/\|x^*\|$ then we have $\|x\| = \|xx^*\|/\|x^*\| = \|xy\| \leq \|x \oplus 0\|$.

The norm on $\tilde{A}$ is nothing but the operator norm when we view $\tilde{A}$ as acting on A by left multiplication and hence we have that this is at least a semi-norm such that $\|xy\| \leq \|x\|\|y\|$, for all $x, y \in \tilde{A}$. To see that this is actually a norm note that if $\alpha \neq 0$, but $\|x \oplus \alpha\| = 0$ then for all $y \in A$ we have $\|xy + \alpha y\| \leq \|x \oplus \alpha\|\|y\| = 0$, and hence $e = -x/\alpha$ is a left identity for A. Taking adjoints we see that e^* is a right identity for A, and then $e = ee^* = e^*$ is an identity for A which contradicts that A is non-unital. Thus, $\|\cdot\|$ is indeed a norm.

It is easy to see then that $\tilde{A}$ is then complete, and hence all that remains to be seen is the C^*-identity. Since, each for each $y \in A$, $\|y\| \leq 1$ we have $(y \oplus 0)^*(x \oplus \alpha) \in A \oplus 0 \cong A$ it follows that the C^*-identity holds here, and so

$$\begin{aligned}\|(x \oplus \alpha)^*(x \oplus \alpha)\| &\geq \|(y \oplus 0)^*(x \oplus \alpha)^*(x \oplus \alpha)(y \oplus 0)\| \\ &= \|(x \oplus \alpha)(y \oplus 0)\|^2.\end{aligned}$$

Taking the supremum over all $y \in A$, $\|y\| \leq 1$ we then have

$$\|(x \oplus \alpha)^*(x \oplus \alpha)\| \geq \|x \oplus \alpha\|^2 \geq \|(x \oplus \alpha)^*(x \oplus \alpha)\|. \qquad \blacksquare$$

Lemma 1.3.10. *If A is a non-unital abelian C^*-algebra, then any norm 1 multiplicative linear functional $\varphi \in \sigma(A)$ has a unique extension $\tilde{\varphi} \in \tilde{A}$.*

Proof. If we consider $\tilde{\varphi}(x \oplus \alpha) = \varphi(x) + \alpha$ then the result follows easily. $\blacksquare$

In particular, this shows that $\sigma(A)$ is homeomorphic to $\sigma(\tilde{A}) \setminus \{\varphi_0\}$ where φ_0 is defined by $\varphi(x, \alpha) = \alpha$. Thus, $\sigma(A)$ is locally compact.

If $x \in A$ then the **spectrum** $\sigma(x)$ of x is defined to be the spectrum of $x \oplus 0 \in \tilde{A}$. Note that for a non-unital C^*-algebra A, since $A \subset \tilde{A}$ is an ideal it follows that $0 \in \sigma(x)$ whenever $x \in A$.

By considering the embedding $A \subset \tilde{A}$ we are able to extend the spectral theorem and continuous functional calculus to the non-unital setting. We leave the details to the reader.

Theorem 1.3.11. *Let A be a non-unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C_0(\sigma(A))$ gives an isometric isomorphism between A and $C_0(\sigma(A))$.*

Theorem 1.3.12. *Let A be a C^*-algebra, and $x \in A$ a normal element, then if $f \in C(\sigma(x))$ such that $f(0) = 0$, then $f(x) \in A \subset \tilde{A}$.*

Exercise 1.3.13. Suppose K is a non-compact, locally compact Hausdorff space, and $K \cup \{\infty\}$ is the one point compactification. Show that we have a natural isomorphism $C(K \cup \{\infty\}) \cong \widetilde{C_0(K)}$.

1.4 Applications of functional calculus

Given any element x in a C^*-algebra A, we can decompose x uniquely as a sum of a self-adoint and skew-adjoint elements $\frac{x+x^*}{2}$ and $\frac{x-x^*}{2}$. We refer to the self-adjoint elements $\frac{x+x^*}{2}$ and $i\frac{x^*-x}{2}$ the **real** and **imaginary** parts of x, note that the real and imaginary parts of x have norms no grater than that of x.

Also, if $x \in A$ is self-adjoint then from above we know that $\sigma(x) \subset \mathbb{R}$, hence by considering $x_+ = (0 \vee t)(x)$ and $x_- = -(0 \wedge t)(x)$ it follows easily from functional calculus that $\sigma(x_+), \sigma(x_-) \subset [0, \infty)$, $x_+x_- = x_-x_+ = 0$, and $x = x_+ - x_-$. We call x_+ and x_- the **positive** and **negative** parts of x.

1.4.1 The positive cone

Lemma 1.4.1. *Suppose we have self-adjoint elements $x, y \in A$ such that $\sigma(x), \sigma(y) \subset [0, \infty)$ then $\sigma(x + y) \subset [0, \infty)$.*

Proof. Let $a = \|x\|$, and $b = \|y\|$. Since x is self-adjoint and $\sigma(x) \subset [0, \infty)$ we may use the spectral radius formula to see that $\|a - x\| = r(a - x) = a$. Similarly we have $\|b - y\| = b$ and since $\|x + y\| \leq a + b$ we have

$$\sup_{\lambda \in \sigma(x+y)} \{a + b - \lambda\} = r((a + b) - x) = \|(a + b) - (x + y)\|$$
$$\leq \|x - a\| + \|y - b\| = a + b.$$

Therefore, $\sigma(x + y) \subset [0, \infty)$. ■

Proposition 1.4.2. *Let A be a C^*-algebra. A normal element $x \in A$ is*

(i) *self-adjoint if and only if $\sigma(x) \subset \mathbb{R}$.*

(ii) *positive if and only if $\sigma(x) \subset [0, \infty)$.*

(iii) *unitary if and only if $\sigma(x) \subset \mathbb{T}$.*

(iv) *a projection if and only if $\sigma(x) \subset \{0, 1\}$.*

Proof. Parts (i), (iii), and (iv) all follow easily by applying continuous functional calculus. For part (ii) if x is normal and $\sigma(x) \subset [0, \infty)$ then $x = (\sqrt{x})^2 = (\sqrt{x})^* \sqrt{x}$ is positive. It also follows easily that if $x = y^*y$ where y is normal then $\sigma(x) \subset [0, \infty)$. Thus, the difficulty arises only when $x = y^*y$ where y is perhaps not normal.

Suppose $x = y^*y$ for some $y \in A$, to show that $\sigma(x) \subset [0, \infty)$, decompose x into its positive and negative parts $x = x_+ - x_-$ as described above. Set $z = yx_-$ and note that $z^*z = x_-(y^*y)x_- = -x_-^3$, and hence $\sigma(zz^*) \subset \sigma(z^*z) \subset (-\infty, 0]$.

If $z = a + ib$ where a and b are self-adjoint, then we have $zz^* + z^*z = 2a^2 + 2b^2$, hence we also have $\sigma(zz^* + z^*z) \subset [0, \infty)$ and so by Lemma 1.4.1 we have $\sigma(z^*z) = \sigma((2a^2 + 2b^2) - zz^*) \subset [0, \infty)$. Therefore $\sigma(-x_-^3) = \sigma(z^*z) \subset \{0\}$ and since x_- is normal this shows that $x_-^3 = 0$, and consequently $x_- = 0$. ■

Corollary 1.4.3. *Let A be a C^*-algebra. An element $x \in A$ is a partial isometry if and only if x^* is a partial isometry.*

Proof. Since x^*x is normal, it follows from the previous proposition that x is a partial isometry if and only if $\sigma(x^*x) \subset \{0, 1\}$. Since $\sigma(x^*x) \cup \{0\} = \sigma(xx^*) \cup \{0\}$ this gives the result. ∎

Corollary 1.4.4. *Let A be a C^*-algebra, then the set of positive elements forms a closed cone. Moreover, if $a \in A$ is self-adjoint, and A is unital, then we have $a \leq \|a\|$.*

Note that if $x \in A$ is an arbitrary element of a C^*-algebra A, then from above we have that x^*x is positive and hence we may define the **absolute value** of x as the unique element $|x| \in A$ such that $|x|^2 = x^*x$.

Proposition 1.4.5. *Let A be a unital C^*-algebra, then every element is a linear combination of four unitaries.*

Proof. If $x \in A$ is self-adjoint and $\|x\| \leq 1$, then $u = x + i(1 - x^2)^{1/2}$ is a unitary and we have $x = u + u^*$. In general, we can decompose x into its real and imaginary parts and then write each as a linear combination of two unitaries. ∎

Proposition 1.4.6. *Let A be a C^*-algebra, and suppose $x, y \in A_+$ such that $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$. Moreover, if A is unital and $x, y \in A$ are invertible, then $y^{-1} \leq x^{-1}$.*

Proof. First consider the case that A is unital and x and y are invertible, then we have
$$y^{-1/2}xy^{-1/2} \leq 1,$$
hence
$$\begin{aligned} x^{1/2}y^{-1}x^{1/2} &\leq \|x^{1/2}y^{-1}x^{1/2}\| = r(x^{1/2}y^{-1}x^{1/2}) \\ &= r(y^{-1/2}xy^{-1/2}) \leq 1. \end{aligned}$$
Conjugating by $x^{-1/2}$ gives $y^{-1} \leq x^{-1}$.

We also have
$$\|y^{-1/2}x^{1/2}\|^2 = \|y^{-1/2}xy^{-1/2}\| \leq 1,$$
therefore
$$\begin{aligned} y^{-1/4}x^{1/2}y^{-1/4} &\leq \|y^{-1/4}x^{1/2}y^{-1/4}\| = r(y^{-1/4}x^{1/2}y^{-1/4}) \\ &= r(y^{-1/2}x^{1/2}) \leq \|y^{-1/2}x^{1/2}\| \leq 1. \end{aligned}$$
Conjugating by $y^{1/4}$ gives $x^{1/2} \leq y^{1/2}$.

In the general case we may consider the unitization of A, and note that if $\varepsilon > 0$, then we have $0 \leq x + \varepsilon \leq y + \varepsilon$, where $x + \varepsilon$, and $y + \varepsilon$ are invertible, hence from above we have
$$(x + \varepsilon)^{1/2} \leq (y + \varepsilon)^{1/2}.$$
Taking the limit as $\varepsilon \to 0$ we obtain the result. ∎

In general, a continuous real valued function f defined on an interval I is said to be **operator monotone** if $f(a) \leq f(b)$ whenever $\sigma(a), \sigma(b) \subset I$, and $a \leq b$. The previous proposition shows that the functions $f(t) = \sqrt{t}$, and $f(t) = -1/t$, $t > 0$ are operator monotone.

Corollary 1.4.7. *Let A be a C^*-algebra, then for $x, y \in A$ we have $|xy| \leq \|x\||y|$.*

Proof. Since $|xy|^2 = y^* x^* xy \leq \|x\|^2 y^* y$, this follows from the previous proposition. ∎

1.4.2 Extreme points

Given a involutive normed algebra A, we denote by $(A)_1$ the unit ball of A, and $A_{\text{s.a.}}$ the subspace of self-adjoint elements.

Proposition 1.4.8. *Let A be a C^*-algebra.*

(i) *The extreme points of $(A_+)_1$ are the projections of A.*

(ii) *The extreme points of $(A_{\text{s.a.}})_1$ are the self-adjoint unitaries in A.*

(iii) *Every extreme point of $(A)_1$ is a partial isometry in A.*

Proof. (i) If $x \in (A_+)_1$, then we have $x^2 \leq 2x$, and $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$. Hence if x is an extreme point then we have $x = x^2$ and so x is a projection. For the converse we first consider the case when A is abelian, and so we may assume $A = C_0(K)$ for some locally compact Hausdorff space K. If x is a projection then $x = 1_E$ is the characteristic function on some open and closed set $E \subset K$, hence the result follows easily from the fact that 0 and 1 are extreme points of $[0, 1]$.

For the general case, suppose $p \in A$ is a projection, if $p = \frac{1}{2}(a + b)$ then $\frac{1}{2}a = p - b \leq p$, and hence $0 \leq (1 - p)a(1 - p) \leq 0$, thus $a = ap = pa$. We therefore have that a, b, and p commute and hence the result follows from the abelian case.

(ii) First note that if A is unital then 1 is an extreme point in the unit ball. Indeed, if $1 = \frac{1}{2}(a+b)$ where $a, b \in (A)_1$, then we have the same equation when replacing a and b by their real parts. Thus, assuming a and b are self-adjoint we have $\frac{1}{2}a = 1 - \frac{1}{2}b$ and hence a and b commute. By considering the unital C^*-subalgebra generated by a and b we may assume $A = C(K)$ for some compact Hausdorff space K, and then it is an easy exercise to conclude that $a = b = 1$.

If u is a unitary in A, then the map $x \mapsto ux$ is a linear isometry of A, thus since 1 is an extreme point of $(A)_1$ it follows that u is also an extreme point. In particular, if u is self-adjoint then it is an extreme point of $(A_{\text{s.a.}})_1$.

Conversely, if $x \in (A_{\text{s.a.}})_1$ is an extreme point then if $x_+ = \frac{1}{2}(a + b)$ for $a, b \in (A_+)_1$, then $0 = x_- x_+ x_- = \frac{1}{2}(x_- a x_- + x_- b x_-) \geq 0$, hence we have $(a^{1/2}x_-)^*(a^{1/2}x_-) = x_- a x_- = 0$. We conclude that $ax_- = x_- a = 0$, and similarly $bx_- = x_- b = 0$. Thus, $a - x_-$ and $b - x_-$ are in $(A_{\text{s.a.}})_1$ and $x =$

$\frac{1}{2}((a - x_-) + (b - x_-))$. Since x is an extreme point we conclude that $x = a - x_- = b - x_-$ and hence $a = b = x_+$.

We have shown now that x_+ is an extreme point in $(A_+)_1$ and thus by part (i) we conclude that x_+ is a projection. The same argument shows that x_- is also a projection, and thus x is a self-adjoint unitary.

(iii) If $x \in (A)_1$ such that x^*x is not a projection then by applying functional calculus to x^*x we can find an element $y \in A_+$ such that $x^*xy = yx^*x \neq 0$, and $\|x(1 \pm y)\|^2 = \|x^*x(1 \pm y)^2\| \leq 1$. Since $xy \neq 0$ we conclude that $x = \frac{1}{2}((x + xy) + (x - xy))$ is not an extreme point of $(A)_1$. ■

1.4.3 Ideals and quotients

Theorem 1.4.9. *Let A be a C^*-algebra, and let $I \subset A$ be a left ideal, then there exists an increasing net $\{a_\lambda\}_\lambda \subset I$ of positive elements such that for all $x \in I$ we have*

$$\|xa_\lambda - x\| \to 0.$$

Moreover, if A is separable then the net can be taken to be a sequence.

Proof. Consider Λ to be the set of all finite subsets of $I \subset A \subset \tilde{A}$, ordered by inclusion. If $\lambda \in \Lambda$ we consider

$$h_\lambda = \sum_{x \in \lambda} x^*x, \qquad a_\lambda = |\lambda| h_\lambda (1 + |\lambda| h_\lambda)^{-1}.$$

Then we have $a_\lambda \in I$ and $0 \leq a_\lambda \leq 1$. If $\lambda \leq \lambda'$ then we clearly have $h_\lambda \leq h_{\lambda'}$ and hence by Proposition 1.4.6 we have that

$$\frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_{\lambda}\right)^{-1}.$$

Therefore

$$a_\lambda = 1 - \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_\lambda\right)^{-1} \leq 1 - \frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} = a_{\lambda'}.$$

If $y \in \lambda$ then we have

$$(y(1 - a_\lambda))^*(y(1 - a_\lambda)) \leq \sum_{x \in \lambda} (x(1 - a_\lambda))^*(x(1 - a_\lambda)) = (1 - a_\lambda)h_\lambda(1 - a_\lambda).$$

But $\|(1 - a_\lambda)h_\lambda(1 - a_\lambda)\| = \|h_\lambda(1 + |\lambda|h_\lambda)^{-2}\| \leq \frac{1}{4|\lambda|}$, from which it follows easily that $\|y - ya_\lambda\| \to 0$, for all $y \in I$.

If A is separable then so is $\overline{I}$, hence there exists a countable subset $\{x_n\}_{n \in \mathbb{N}} \subset I$ which is dense in I. If we take $\lambda_n = \{x_1, \ldots, x_n\}$, then clearly $a_n = a_{\lambda_n}$ also satisfies

$$\|y - ya_n\| \to 0.$$

■

We call such a net $\{a_\lambda\}$ a **right approximate identity** for I. If I is self-adjoint then we also have $\|a_\lambda x - x\| = \|x^* a_\lambda - x^*\| \to 0$ and in this case we call $\{a_\lambda\}$ an **approximate identity**. Using the fact that the adjoint is an isometry we also obtain the following corollary.

Corollary 1.4.10. *Let A be a C^*-algebra, and $I \subset A$ a closed two sided ideal. Then I is self-adjoint. In particular, I is a C^*-algebra.*

Exercise 1.4.11. Show that if A is a C^*-algebra such that $x \leq y \implies x^2 \leq y^2$, for all $x, y \in A_+$, then A is abelian.

Exercise 1.4.12. Let A be a C^*-algebra and $I \subset A$ a non-trivial closed two sided ideal. Show that A/I is again a C^*-algebra.

Chapter 2

Bounded linear operators

Recall that if $\mathcal{H}$ is a Hilbert space then $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators is a C^*-algebra with norm

$$\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|,$$

and involution given by the adjoint, i.e., x^* is the unique bounded linear operator such that

$$\langle \xi, x^* \eta \rangle = \langle x\xi, \eta \rangle,$$

for all $\xi, \eta \in \mathcal{H}$.

Lemma 2.0.13. *Let $\mathcal{H}$ be a Hilbert space and consider $x \in \mathcal{B}(\mathcal{H})$, then* $\ker(x) = R(x^*)^\perp$.

Proof. If $\xi \in \ker(x)$, and $\eta \in \mathcal{H}$, then $\langle \xi, x^*\eta \rangle = \langle x\xi, \eta \rangle = 0$, hence $\ker(x) \subset R(x^*)^\perp$. If $\xi \in R(x^*)^\perp$ then for any $\eta \in \mathcal{H}$ we have $\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle = 0$, hence $\xi \in \ker(x)$. ■

Lemma 2.0.14. *Let $\mathcal{H}$ be a Hilbert space, then an operator $x \in \mathcal{B}(\mathcal{H})$ is*

(*i*) *normal if and only if $\|x\xi\| = \|x^*\xi\|$, for all $\xi \in \mathcal{H}$.*

(*ii*) *self-adjoint if and only if $\langle x\xi, \xi \rangle \in \mathbb{R}$, for all $\xi \in \mathcal{H}$.*

(*iii*) *positive if and only if $\langle x\xi, \xi \rangle \geq 0$, for all $\xi \in \mathcal{H}$.*

(*iv*) *an isometry if and only if $\|x\xi\| = \|\xi\|$, for all $\xi \in \mathcal{H}$.*

(*v*) *a projection if and only if x is the orthogonal projection onto some closed subspace of $\mathcal{H}$.*

(*vi*) *a partial isometry if and only if there is a closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $x_{|\mathcal{K}}$ is an isometry while $x_{|\mathcal{K}^\perp} = 0$.*

Proof.

(i) If x is normal than for all $\xi \in \mathcal{H}$ we have $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \langle xx^*\xi, \xi\rangle = \|x^*\xi\|^2$. Conversely, is $\langle (x^*x - xx^*)\xi, \xi\rangle = 0$, for all $\xi \in \mathcal{H}$, then for all $\xi, \eta \in \mathcal{H}$, by polarization we have

$$\langle (x^*x - xx^*)\xi, \eta\rangle = \sum_{k=0}^{3} i^k \langle (x^*x - xx^*)(\xi + i^k\eta), (\xi + i^k\eta)\rangle = 0.$$

Hence $x^*x = xx^*$.

(ii) If $x = x^*$ then $\overline{\langle x\xi, \xi\rangle} = \langle \xi, x\xi\rangle = \langle x\xi, \xi\rangle$. The converse follows again by a polarization argument.

(iii) If $x = y^*y$, then $\langle x\xi, \xi\rangle = \|y\xi\|^2 \geq 0$. Conversely, if $\langle x\xi, \xi\rangle \geq 0$, for all $\xi \in \mathcal{H}$ then we know from part (b) that x is self-adjoint, and for all $a > 0$ we have $\langle (x + a)\xi, \xi\rangle \geq a\|\xi\|^2$. This shows that $x + a$ is an injective operator with dense image (since the orthogonal complement of the range is trivial). Moreover, by the Cauchy-Schwarz inequality we have

$$a\|\xi\|^2 \leq \langle (x + a)\xi, \xi\rangle \leq \|(x + a)\xi\|\|\xi\|,$$

and hence $a\|\xi\| \leq \|(x + a)\xi\|$, for all $\xi \in \mathcal{H}$. In particular this shows that the image of $x + a$ is closed since if $\{(x + a)\xi_n\}$ is Cauchy then $\{\xi_n\}$ is also Cauchy. Therefore $(x + a)$ is invertible and $a\|(x + a)^{-1}\xi\| \leq \|\xi\|$, for all $\xi \in \mathcal{H}$, showing that $(x + a)^{-1}$ is bounded. Since $a > 0$ was arbitrary this shows that $\sigma(x) \subset [0, \infty)$ and hence x is positive.

(iv) If x is an isometry then $x^*x = 1$ and hence $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \|\xi\|^2$ for all $\xi \in \mathcal{H}$. The converse again follows from the polarization identity.

(v) If x is a projection then let $\mathcal{K} = \overline{R(x)} = \ker(x)^\perp$, and note that for all $\xi \in \mathcal{K}, \eta \in \ker(x), x\zeta \in R(x)$ we have $\langle x\xi, \eta + x\zeta\rangle = \langle \xi, x\zeta\rangle$, hence $x\xi \in \mathcal{K}$, and $x\xi = \xi$. This shows that x is the orthogonal projection onto the subspace $\mathcal{K}$.

(vi) This follows directly from iv and v. ■

Proposition 2.0.15 (Polar decomposition)**.** *Let $\mathcal{H}$ be a Hilbert space, and $x \in \mathcal{B}(\mathcal{H})$, then there exists a partial isometry v such that $x = v|x|$, and $\ker(v) = \ker(|x|) = \ker(x)$. Moreover, this decomposition is unique, in that if $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$ then $y = |x|$, and $w = v$.*

Proof. We define a linear operator $v_0 : R(|x|) \to R(x)$ by $v_0(|x|\xi) = x\xi$, for $\xi \in \mathcal{H}$. Since $\||x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$ it follows that v_0 is well defined and extends to a partial isometry v from $\overline{R(|x|)}$ to $\overline{R(x)}$, and we have $v|x| = x$. We also have $\ker(v) = R(|x|)^\perp = \ker(|x|) = \ker(x)$.

To see the uniqueness of this decomposition suppose $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$. Then $|x|^2 = x^*x = yw^*wy = y^2$, and hence $|x| = (|x|^2)^{1/2} = (y^2)^{1/2} = y$. We then have $\ker(w) = \overline{R(|x|)}^\perp$, and $\|w|x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$, hence $w = v$. ■

2.1 Trace class operators

Given a Hilbert space $\mathcal{H}$, an operator $x \in \mathcal{B}(\mathcal{H})$ has finite rank if $\overline{R(x)} = \ker(x^*)^\perp$ is finite dimensional, the **rank** of x is $\dim(\overline{R(x)})$. We denote the space of finite rank operators by $\mathcal{FR}(\mathcal{H})$. If x is finite rank than $R(x^*) = R(x^*_{|\ker(x^*)^\perp})$ is also finite dimensional being the image of a finite dimensional space, hence we see that x^* also has finite rank. If $\xi, \eta \in \mathcal{H}$ are vectors we denote by $\xi \otimes \overline{\eta}$ the operator given by

$$(\xi \otimes \overline{\eta})(\zeta) = \langle \zeta, \eta \rangle \xi.$$

Note that $(\xi \otimes \overline{\eta})^* = \eta \otimes \overline{\xi}$, and if $\|\xi\| = \|\eta\| = 1$ then $\xi \otimes \overline{\eta}$ is a rank one partial isometry from $C\eta$ to $\mathbb{C}\xi$. Also note that if $x, y \in \mathcal{B}(\mathcal{H})$, then we have $x(\xi \otimes \overline{\eta})y = (x\xi) \otimes (\overline{y^*\eta})$.

From above we see that any finite rank operator is of the form pxq where $p, q \in \mathcal{B}(\mathcal{H})$ are projections onto finite dimensional subspaces. In particular this shows that $\mathcal{FR}(\mathcal{H}) = \mathrm{sp}\{\xi \otimes \overline{\eta} \mid \xi, \eta \in \mathcal{H}\}$

Lemma 2.1.1. *Suppose $x \in \mathcal{B}(\mathcal{H})$ has polar decomposition $x = v|x|$. Then for all $\xi \in \mathcal{H}$ we have*

$$2|\langle x\xi, \xi \rangle| \leq \langle |x|\xi, \xi \rangle + \langle |x|v^*\xi, v^*\xi \rangle.$$

Proof. If $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, then we have

$$\begin{aligned} 0 &\leq \|(|x|^{1/2} - \lambda |x|^{1/2} v^*)\xi\|^2 \\ &= \||x|^{1/2}\xi\|^2 - 2\mathrm{Re}(\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi \rangle) + \||x|^{1/2}v^*\xi\|^2. \end{aligned}$$

Taking λ such that $\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi \rangle \geq 0$, the inequality follows directly. ∎

If $\{\xi_i\}$ is an orthonormal basis for $\mathcal{H}$, and $x \in \mathcal{B}(\mathcal{H})$ is positive, then we define the trace of x to be

$$\mathrm{Tr}(x) = \sum_i \langle x\xi_i, \xi_i \rangle.$$

Lemma 2.1.2. *If $x \in \mathcal{B}(\mathcal{H})$ then $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$.*

Proof. By Parseval's identity and Fubini's theorem we have

$$\begin{aligned} \sum_i \langle x^*x\xi_i, \xi_i \rangle &= \sum_i \sum_j \langle x\xi_i, \xi_j \rangle \overline{\langle \xi_j, x\xi_i \rangle} \\ &= \sum_j \sum_i \langle \xi_i, x^*\xi_j \rangle \overline{\langle \xi_i, x^*\xi_j \rangle} = \sum_j \langle xx^*\xi_j, \xi_j \rangle. \end{aligned}$$ ∎

Corollary 2.1.3. *If $x \in \mathcal{B}(\mathcal{H})$ is positive and u is a unitary, then $\mathrm{Tr}(u^*xu) = \mathrm{Tr}(x)$. In particular, the trace is independent of the chosen orthonormal basis.*

Proof. If we write $x = y^*y$, then from the previous lemma we have

$$\mathrm{Tr}(y^*y) = \mathrm{Tr}(yy^*) = \mathrm{Tr}((yu)(u^*y^*)) = \mathrm{Tr}(u^*(y^*y)u).$$ ∎

An operator $x \in \mathcal{B}(\mathcal{H})$ is said to be of **trace class** if $\|x\|_1 := \operatorname{Tr}(|x|) < \infty$. We denote the set of trace class operators by $L^1(\mathcal{B}(\mathcal{H}))$ or $L^1(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

Given an orthonormal basis $\{\xi_i\}$, and $x \in L^1(\mathcal{B}(\mathcal{H}))$ we define the **trace** of x by

$$\operatorname{Tr}(x) = \Sigma_i \langle x\xi_i, \xi_i \rangle.$$

By Lemma 2.1.1 this is absolutely summable, and

$$2|\operatorname{Tr}(x)| \leq \operatorname{Tr}(|x|) + \operatorname{Tr}(v|x|v^*) \leq 2\|x\|_1.$$

Lemma 2.1.4. *$L^1(\mathcal{B}(\mathcal{H}))$ is a two sided self-adjoint ideal in $\mathcal{B}(\mathcal{H})$ which coincides with the span of the positive operators with finite trace. The trace is independent of the chosen basis, and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.*

Proof. If $x, y \in L^1(\mathcal{B}(\mathcal{H}))$ and we let $x+y = w|x+y|$ be the polar decomposition, then we have $w^*x, w^*y \in L^1(\mathcal{B}(\mathcal{H}))$, therefore $\sum_i \langle |x+y|\xi_i, \xi_i \rangle = \sum_i \langle w^*x\xi_i, \xi_i \rangle + \langle w^*y\xi_i, \xi_i \rangle$ is absolutely summable. Thus $x + y \in L^1(\mathcal{B}(\mathcal{H}))$ and

$$\|x + y\|_1 \leq \|w^*x\|_1 + \|w^*y\|_1 \leq \|x\|_1 + \|y\|_1.$$

Thus, it follows that $L^1(\mathcal{B}(\mathcal{H}))$ is a linear space which contains the span of the positive operators with finite trace, and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.

If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ then

$$4a|x| = \sum_{k=0}^{3} i^k (a + i^k)|x|(a + i^k)^*,$$

and for each k we have

$$\operatorname{Tr}((a + i^k)|x|(a + i^k)^*) = \operatorname{Tr}(|x|^{1/2}|a + i^k|^2|x|^{1/2}) \leq \|a + i^k\|^2 \operatorname{Tr}(|x|).$$

Thus if we take a to be the partial isometry in the polar decomposition of x we see that x is a linear combination of positive operators with finite trace, (in particular, the trace is independent of the basis). This also shows that $L^1(\mathcal{B}(\mathcal{H}))$ is a self-adjoint left ideal, and hence is also a right ideal. ■

Theorem 2.1.5. *If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a, b \in \mathcal{B}(\mathcal{H})$ then*

$$\|x\| \leq \|x\|_1$$

$$\|axb\|_1 \leq \|a\| \; \|b\| \; \|x\|_1,$$

and

$$\operatorname{Tr}(ax) = \operatorname{Tr}(xa).$$

Proof. Since the trace is independent of the basis, and $\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|$ it follows easily that $\|x\| \leq \|x\|_1$.

Since for $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ we have $|ax| \leq \|a\||x|$ it follows that $\|ax\|_1 \leq \|a\|\|x\|_1$. Since $\|x\|_1 = \|x^*\|_1$ we also have $\|xb\|_1 \leq \|b\|\|x\|_1$.

Since the definition of the trace is independent of the chosen basis, if $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $u \in \mathcal{U}(\mathcal{H})$ we have

$$\operatorname{Tr}(xu) = \sum_i \langle xu\xi_i, \xi_i\rangle = \sum_i \langle uxu\xi_i, u\xi_i\rangle = \operatorname{Tr}(ux).$$

Since every operator $a \in \mathcal{B}(\mathcal{H})$ is a linear combination of four unitaries this also gives

$$\operatorname{Tr}(xa) = \operatorname{Tr}(ax).$$ ■

We also remark that for all $\xi, \eta \in \mathcal{H}$, the operators $\xi \otimes \overline{\eta}$ satisfy $\operatorname{Tr}(\xi \otimes \overline{\eta}) = \langle \xi, \eta\rangle$. Also, it's easy to check that $\mathcal{FR}(\mathcal{H})$ is a dense subspace of $L^1(\mathcal{B}(\mathcal{H}))$, endowed with the norm $\| \cdot \|_1$.

Proposition 2.1.6. *The space of trace class operators $L^1(\mathcal{B}(\mathcal{H}))$, with the norm $\| \cdot \|_1$ is a Banach space.*

Proof. From Lemma 2.1.4 we know that $\| \cdot \|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$ and hence we need only show that $L^1(\mathcal{B}(\mathcal{H}))$ is complete. Suppose x_n is Cauchy in $L^1(\mathcal{B}(\mathcal{H}))$. Since $\|x_n - x_m\| \leq \|x_n - x_m\|_1$ it follows that x_n is also Cauchy in $\mathcal{B}(\mathcal{H})$, therefore we have $\|x - x_n\| \to 0$, for some $x \in \mathcal{B}(\mathcal{H})$, and by continuity of functional calculus we also have $\||x| - |x_n|\| \to 0$. Thus for any finite orthonormal set $\eta_1, \ldots, \eta_k$ we have

$$\begin{aligned}\sum_{i=1}^k \langle |x|\eta_i, \eta_i\rangle &= \lim_{n\to\infty} \sum_{i=1}^k \langle |x_n|\eta_i, \eta_i\rangle \\ &\leq \lim_{n\to\infty} \|x_n\|_1 < \infty.\end{aligned}$$

Hence $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $\|x\|_1 \leq \lim_{n\to\infty} \|x_n\|_1$.

If we let $\varepsilon > 0$ be given and consider $N \in \mathbb{N}$ such that for all $n > N$ we have $\|x_n - x_N\|_1 < \varepsilon/3$, and then take $\mathcal{H}_0 \subset \mathcal{H}$ a finite dimensional subspace such that $\|x_N P_{\mathcal{H}_0^\perp}\|_1, \|x P_{\mathcal{H}_0^\perp}\|_1 < \varepsilon/3$. Then for all $n > N$ we have

$$\begin{aligned}&\|x - x_n\|_1 \\ &\quad \leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|xP_{\mathcal{H}_0^\perp} - x_N P_{\mathcal{H}_0^\perp}\|_1 + \|(x_N - x_n)P_{\mathcal{H}_0^\perp}\|_1 \\ &\quad \leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \varepsilon.\end{aligned}$$

Since $\|x - x_n\| \to 0$ it follows that $\|(x - x_n)P_{\mathcal{H}_0}\|_1 \to 0$, and since $\varepsilon > 0$ was arbitrary we then have $\|x - x_n\|_1 \to 0$. ■

Theorem 2.1.7. *The map $\psi : \mathcal{B}(\mathcal{H}) \to L^1(\mathcal{B}(\mathcal{H}))^*$ given by $\psi_a(x) = \operatorname{Tr}(ax)$, for $a \in \mathcal{B}(\mathcal{H})$, $x \in L^1(\mathcal{B}(\mathcal{H}))$, is a Banach space isomorphism.*

Proof. From Theorem 2.1.5 we have that ψ is a linear contraction.

Suppose $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$, then $(\xi, \eta) \mapsto \varphi(\xi \otimes \overline{\eta})$ defines a bounded sesquilinear form on $\mathcal{H}$ and hence there exists a bounded operator $a \in \mathcal{B}(\mathcal{H})$ such that $\langle a\xi, \eta\rangle = \varphi(\xi \otimes \overline{\eta})$, for all $\xi, \eta \in \mathcal{H}$. Since the finite rank operators is dense in

$L^1(\mathcal{B}(\mathcal{H}))$, and since operators of the form $\xi \otimes \overline{\eta}$ span the finite rank operators we have $\varphi = \psi_a$, thus we see that ψ is bijective.

We also have

$$\begin{aligned} \|a\| &= \sup_{\substack{\xi,\eta\in\mathcal{H},\\ \|\xi\|,\|\eta\|\leq 1}} |\langle a\xi, \eta\rangle| \\ &= \sup_{\substack{\xi,\eta\in\mathcal{H},\\ \|\xi\|,\|\eta\|\leq 1}} |\operatorname{Tr}(a(\xi \otimes \overline{\eta}))| \leq \|\psi_a\|. \end{aligned}$$

Hence ψ is isometric. ∎

2.2 Hilbert-Schmidt operators

Given a Hilbert space $\mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$, we say that x is a Hilbert-Schmidt operator on $\mathcal{H}$ if $|x|^2 \in L^1(\mathcal{B}(\mathcal{H}))$. We define the set of Hilbert-Schmidt operators by $L^2(\mathcal{B}(\mathcal{H}))$, or $L^2(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

Lemma 2.2.1. *$L^2(\mathcal{B}(\mathcal{H}))$ is a self-adjoint ideal in $\mathcal{B}(\mathcal{H})$, and if $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ then $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$, and*

$$\operatorname{Tr}(xy) = \operatorname{Tr}(yx).$$

Proof. Since $|x+y|^2 \leq |x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$ we see that $L^2(\mathcal{B}(\mathcal{H}))$ is a linear space, also since $|ax|^2 \leq \|a\|^2|x|^2$ we have that $L^2(\mathcal{B}(\mathcal{H}))$ is a left ideal. Moreover, if $x = v|x|$ is the polar decomposition of x then we have $xx^* = v|x|^2v^*$, and thus $x^* \in L^2(\mathcal{B}(\mathcal{H}))$ and $\operatorname{Tr}(xx^*) = \operatorname{Tr}(x^*x)$. In particular, $L^2(\mathcal{B}(\mathcal{H}))$ is also a right ideal.

By the polarization identity

$$4y^*x = \sum_{k=0}^{3} i^k |x + i^k y|^2,$$

we have that $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$ for $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, and

$$\begin{aligned} 4\operatorname{Tr}(y^*x) &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x + i^k y)^*(x + i^k y)) \\ &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x + i^k y)(x + i^k y)^*) = 4\operatorname{Tr}(xy^*). \end{aligned}$$ ∎

From the previous lemma we see that the sesquilinear form on $L^2(\mathcal{B}(\mathcal{H}))$ give by

$$\langle x, y\rangle_2 = \operatorname{Tr}(y^*x)$$

is well defined and positive definite. We again have $\|axb\|_2 \leq \|a\| \, \|b\| \, \|x\|_2$, and any $x \in L^2(\mathcal{B}(\mathcal{H}))$ can be approximated in $\| \cdot \|_2$ by operators px where p is a finite rank projection. Thus, the same argument as for the trace class operators shows that the Hilbert-Schmidt operators is complete in the Hilbert-Schmidt norm.

Also, note that if $x \in L^2(\mathcal{B}(\mathcal{H}))$ then since $\|y\| \leq \|y\|_2$ for all $y \in L^2(\mathcal{B}(\mathcal{H}))$ it follows that

$$\begin{aligned} \|x\|_2 &= \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})), \\ \|y\|_2 \leq 1}} |\operatorname{Tr}(y^* x)| \\ &\leq \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})), \\ \|y\|_2 \leq 1}} \|y\| \|x\|_1 \leq \|x\|_1. \end{aligned}$$

Proposition 2.2.2. *Let $\mathcal{H}$ be a Hilbert space and suppose $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then*

$$\|xy\|_1 \leq \|x\|_2 \|y\|_2.$$

Proof. If we consider the polar decomposition $xy = v|xy|$, then by the Cauchy-Schwarz inequality we have

$$\begin{aligned} \|xy\|_1 &= |\operatorname{Tr}(v^* xy)| = |\langle y, x^* v \rangle_2| \\ &\leq \|x^* v\|_2 \|y\|_2 \leq \|x\|_2 \|y\|_2. \end{aligned}$$ ■

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then we may extend a bounded operator $x : \mathcal{H} \to \mathcal{K}$ to a bounded operator $\tilde{x} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ by $\tilde{x}(\xi \oplus \eta) = 0 \oplus x\xi$. We define $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ as the bounded operators $x : \mathcal{H} \to \mathcal{K}$ such that $\tilde{x} \in L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$. In this way $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ forms a closed subspace of $L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$.

Note that $\mathrm{HS}(\mathcal{H}, \mathbb{C})$ is the dual Banach space of $\mathcal{H}$, and is naturally anti-isomorphic to $\mathcal{H}$, we denote this isomorphism by $\xi \mapsto \overline{\xi}$. We call this the **conjugate Hilbert space** of $\mathcal{H}$, and denote it by $\overline{\mathcal{H}}$. Note that we have the natural identification $\overline{\overline{\mathcal{H}}} = \mathcal{H}$. Also, we have a natural anti-linear map $x \mapsto \overline{x}$ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\overline{\mathcal{H}})$ given by $\overline{x}\overline{\xi} = \overline{x\xi}$.

If we wish to emphasize that we are considering only the Hilbert space aspects of the Hilbert-Schmidt operators, we often use the notation $\mathcal{H} \overline{\otimes} \mathcal{K}$ for the Hilbert-Schmidt operators $\mathrm{HS}(\mathcal{H}, \overline{\mathcal{K}})$. In this setting we call $\mathcal{H} \overline{\otimes} \mathcal{K}$ the **Hilbert space tensor product** of $\mathcal{H}$ with $\mathcal{K}$. Note that if $\{\xi_i\}_i$ and $\{\eta_j\}_j$ form orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, then $\{\xi_i \otimes \eta_j\}_{i,j}$ forms an orthonormal basis for $\mathcal{H} \overline{\otimes} \mathcal{K}$. We see that the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ of $\mathcal{H}$ and $\mathcal{K}$ can be realized as the subspace of finite rank operators, i.e., we have $\mathcal{H} \otimes \mathcal{K} = \mathrm{sp}\{\xi \otimes \eta \mid \xi \in \mathcal{H}, \eta \in \mathcal{K}\}$.

If $x \in \mathcal{B}(\mathcal{H})$ and $y \in \mathcal{B}(\mathcal{K})$ then we obtain an operator $x \otimes y \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ which is given by $(x \otimes y)h = xh\overline{y}^*$. We then have that $\|x \otimes y\| \leq \|x\| \|y\|$, and $(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$ for all $\xi \in \mathcal{H}$, and $\eta \in \mathcal{K}$. We also have $(x \otimes y)^* = x^* \otimes y^*$, and the map $(x, y) \mapsto x \otimes y$ is separately linear in each variable. If $A \subset \mathcal{B}(\mathcal{H})$ and $B \subset \mathcal{B}(\mathcal{K})$ are algebras then the tensor product $A \otimes B$ is the algebra generated by operators of the form $a \otimes b$ for $a \in A$ and $b \in B$.

If (X, μ) is a measure space then we have a particularly nice description of the Hilbert-Schmidt operators on $L^2(X, \mu)$.

Theorem 2.2.3. *For each $k \in L^2(X \times X, \mu \times \mu)$ the integral operator T_k defined by*

$$T_k\xi(x) = \int k(x,y)\xi(y)d\mu(y), \qquad \xi \in L^2(X,\mu),$$

is a Hilbert-Schmidt operator on $L^2(X,\mu)$. Moreover, the map $k \mapsto T_k$ is a unitary operator from $L^2(X \times X, \mu \times \mu)$ to $L^2(\mathcal{B}(L^2(X,\mu)))$. Moreover, if we define $k^*(x,y) = \overline{k(x,y)}$ then we have $T_k^* = T_{k^*}$.

Proof. For all $\eta \in L^2(X,\mu)$, the Cauchy-Schwarz inequality gives

$$\|k(x,y)\xi(y)\eta(x)\|_1 \leq \|k\|_2 \|\xi\|_{L^2(X,\mu)} \|\eta\|_2.$$

This shows that T_k is a well defined operator on $L^2(X,\mu)$ and $\|T_k\| \leq \|k\|_2$. If $\{\xi_i\}_i$ gives an orthonormal basis for $L^2(X,\mu)$ and $k(x,y) = \sum \alpha_{i,j}\xi_i(x)\xi_j(y)$ is a finite sum then for $\eta \in L^2(X,\mu)$ we have

$$T_k\eta = \sum \alpha_{i,j}\langle \xi, \xi_j \rangle \xi_i = (\sum \alpha_{i,j}\xi_i \otimes \overline{\xi_j})\eta.$$

Thus, $\|T_k\|_2 = \|\sum \alpha_{i,j}\xi_i \otimes \overline{\xi_j}\|_2 = \|k\|_2$, which shows that $k \mapsto T_k$ is a unitary operator.

The same formula above also shows that $T_k^* = T_{k^*}$. ∎

2.3 Compact operators

We denote by $\mathcal{H}_1$ the unit ball in $\mathcal{H}$.

Theorem 2.3.1. *For $x \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent:*

(i) $x \in \overline{\mathcal{FR}(\mathcal{H})}^{\|\cdot\|}$.

(ii) *x restricted to $\mathcal{H}_1$ is continuous from the weak to the norm topology.*

(iii) *$x(\mathcal{H}_1)$ is compact in the norm topology.*

(iv) *$x(\mathcal{H}_1)$ has compact closure in the norm topology.*

Proof. (i) $\implies$ (ii) Let $\{\xi_\alpha\}_\alpha$ be net in $\mathcal{H}_1$ which weakly converges to ξ. By hypothesis for every $\varepsilon > 0$ there exists $y \in \mathcal{FR}(\mathcal{H})$ such that $\|x - y\| < \varepsilon$. We then have

$$\|x\xi - x\xi_\alpha\| \leq \|y\xi - y\xi_\alpha\| + 2\varepsilon.$$

Thus, it is enough to consider the case when $x \in \mathcal{FR}(\mathcal{H})$. This case follows easily since then the range of x is then finite dimensional where the weak and norm topologies agree.

(ii) $\implies$ (iii) $\mathcal{H}_1$ is compact in the weak topology and hence $x(\mathcal{H}_1)$ is compact being the continuous image of a compact set.

(iii) $\implies$ (iv) This implication is obvious.

(iv) $\Longrightarrow$ (i) Let P_α be a net of finite rank projections such that $\|P_\alpha \xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}$. Then $P_\alpha x$ are finite rank and if $\|P_\alpha x - x\| \not\to 0$ then there exists $\varepsilon > 0$, and $\xi_\alpha \in \mathcal{H}_1$ such that $\|x\xi_\alpha - P_\alpha x \xi_\alpha\| \geq \varepsilon$. By hypothesis we may pass to a subnet and assume that $x\xi_\alpha$ has a limit ξ in the norm topology. We then have

$$\begin{aligned}\varepsilon \leq \|x\xi_\alpha - P_\alpha x\xi_\alpha\| &\leq \|\xi - P_\alpha \xi\| + \|(1 - P_\alpha)(x\xi_\alpha - \xi)\| \\ &\leq \|\xi - P_\alpha \xi\| + \|x\xi_\alpha - \xi\| \to 0,\end{aligned}$$

which gives a contradiction. ■

If any of the above equivalent conditions are satisfied we say that x is a **compact operator**. We denote the space of compact operators by $\mathcal{K}(\mathcal{H})$. Clearly $\mathcal{K}(\mathcal{H})$ is a norm closed two sided ideal in $\mathcal{B}(\mathcal{H})$.

Exercise 2.3.2. Show that the map $\psi : L^1(\mathcal{B}(\mathcal{H})) \to \mathcal{K}(\mathcal{H})^*$ given by $\psi_x(a) = \mathrm{Tr}(ax)$ implements a Banach space isomorphism between $L^1(\mathcal{B}(\mathcal{H}))$ and $\mathcal{K}(\mathcal{H})^*$.

2.4 Locally convex topologies on the space of operators

Let $\mathcal{H}$ be a Hilbert space. On $\mathcal{B}(\mathcal{H})$ we define the following locally convex topologies:

- The **weak operator topology** (WOT) is defined by the family of seminorms $T \mapsto |\langle T\xi, \eta\rangle|$, for $\xi, \eta \in \mathcal{H}$.
- The **strong operator topology** (SOT) is defined by the family of seminorms $T \mapsto \|T\xi\|$, for $\xi \in \mathcal{H}$.

Note that the from coarsest to finest topologies we have

$$\text{WOT} \prec \text{SOT} \prec \text{Uniform}.$$

Also note that since an operator T is normal if and only if $\|T\xi\| = \|T^*\xi\|$ for all $\xi \in \mathcal{H}$, it follows that the adjoint is SOT continuous on the set of normal operators.

Lemma 2.4.1. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(i) There exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$ such that $\varphi(T) = \sum_{i=1}^n \langle T\xi_i, \eta_i\rangle$, for all $T \in \mathcal{B}(\mathcal{H})$.

(ii) φ is WOT continuous.

(iii) φ is SOT continuous.

Proof. The implications (i) $\implies$ (ii) and (ii) $\implies$ (iii) are clear and so we will only show (iii) $\implies$ (i). Suppose φ is SOT continuous. Thus, the inverse image of the open ball in $\mathbb{C}$ is open in the SOT and hence by considering the semi-norms which define the topology we have that there exists a constant $K > 0$, and $\xi_1, \dots, \xi_n \in \mathcal{H}$ such that

$$|\varphi(T)|^2 \leq K \sum_{i=1}^{n} \|T\xi_i\|^2.$$

If we then consider $\{\oplus_{i=1}^n T\xi_i \mid T \in \mathcal{B}(\mathcal{H})\} \subset \mathcal{H}^{\oplus n}$, and let $\mathcal{H}_0$ be its closure, we have that

$$\oplus_{i=1}^n T\xi_i \mapsto \varphi(T)$$

extends to a well defined, continuous linear functional on $\mathcal{H}_0$ and hence by the Riesz representation theorem there exists $\eta_1, \dots, \eta_n \in \mathcal{H}$ such that

$$\varphi(T) = \sum_{i=1}^{n} \langle T\xi_i, \eta_i \rangle,$$

for all $T \in \mathcal{B}(\mathcal{H})$. ∎

Corollary 2.4.2. *Let $K \subset \mathcal{B}(\mathcal{H})$ be a convex set, then the WOT, SOT, and closures of K coincide.*

Proof. By Lemma 2.4.1 the three topologies above give rise to the same dual space, hence this follows from the the Hahn-Banach separation theorem. ∎

If $\mathcal{H}$ is a Hilbert space then the map $\mathrm{id} \otimes 1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ defined by $(\mathrm{id} \otimes 1)(x) = x \otimes 1$ need not be continuous in either of the locally convex topologies defined above even though it is an isometric C^*-homomorphism with respect to the uniform topology. Thus, on $\mathcal{B}(\mathcal{H})$ we define the following additional locally convex topologies:

- The **σ-weak operator topology** (σ-WOT) is defined by pulling back the WOT of $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ under the map $\mathrm{id} \otimes 1$.
- The **σ-strong operator topology** (σ-SOT) is defined by pulling back the SOT of $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ under the map $\mathrm{id} \otimes 1$.

Note that the σ-weak operator topology can alternately be defined by the family of semi-norms $T \mapsto |\operatorname{Tr}(Ta)|$, for $a \in L^1(\mathcal{B}(\mathcal{H}))$. Hence, under the identification $\mathcal{B}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))^*$, we have that the weak*-topology on $\mathcal{B}(\mathcal{H})$ agrees with the σ-WOT.

Lemma 2.4.3. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(*i*) *There exists a trace class operator $a \in L^1(\mathcal{B}(\mathcal{H}))$ such that $\varphi(x) = \operatorname{Tr}(xa)$ for all $x \in \mathcal{B}(\mathcal{H})$*

(ii) φ *is* σ*-WOT continuous.*

(iii) φ *is* σ*-SOT continuous.*

Proof. Again, we need only show the implication (iii) $\implies$ (i), so suppose φ is σ-SOT continuous. Then by the Hahn-Banach theorem, considering $\mathcal{B}(\mathcal{H})$ as a subspace of $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$ through the map $\mathrm{id} \otimes 1$, we may extend φ to a SOT continuous linear functional on $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$. Hence by Lemma 2.4.1 there exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}\overline{\otimes}\ell^2\mathbb{N}$ such that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\varphi(x) = \sum_{i=1}^n \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle.$$

For each $1 \leq i \leq n$ we may define $a_i, b_i \in \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$ as the operators corresponding to ξ_i, η_i in the Hilbert space isomorphism $\mathcal{H} \otimes \ell^2\mathbb{N} \cong \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$. By considering $a = \sum_{i=1}^n b_i^* a_i \in L^1(\mathcal{B}(\mathcal{H}))$, it then follows that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned} \mathrm{Tr}(xa) &= \sum_{i=1}^n \langle a_i x, b_i \rangle_2 \\ &= \sum_{i=1}^n \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle = \varphi(x). \end{aligned}$$ ■

By the Banach-Alaoglu theorem we obtain the following corollary.

Corollary 2.4.4. *The unit ball in* $\mathcal{B}(\mathcal{H})$ *is compact in the* σ*-WOT.*

Corollary 2.4.5. *The WOT and the* σ*-WOT agree on bounded sets.*

Proof. The identity map is clearly continuous from the σ-WOT to the WOT. Since both spaces are Hausdorff it follows that this is a homeomorphism from the σ-WOT compact unit ball in $\mathcal{B}(\mathcal{H})$. By scaling we therefore have that this is a homeomorphism on any bounded set. ■

Exercise 2.4.6. Show that the adjoint $T \mapsto T^*$ is continuous in the WOT, and when restricted to the space of normal operators is continuous in the SOT, but is not continuous in the SOT on the space of all bounded operators.

Exercise 2.4.7. Show that operator composition is jointly continuous in the SOT on bounded subsets.

Exercise 2.4.8. Show that the SOT agrees with the σ-SOT on bounded subsets of $\mathcal{B}(\mathcal{H})$.

Exercise 2.4.9. Show that pairing $\langle x, a \rangle = \mathrm{Tr}(a^* x)$ gives an identification between $\mathcal{K}(\mathcal{H})^*$ and $(L^1(\mathcal{B}(\mathcal{H})), \| \cdot \|_1)$.

2.5 Von Neumann algebras and the double commutant theorem

A **von Neumann algebra** (over a Hilbert space $\mathcal{H}$) is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains 1 and is closed in the weak operator topology.

Note that since subalgebras are of course convex, it follows from Corollary 2.4.2 that von Neumann algebras are also closed in the strong operator topology.

If $A \subset \mathcal{B}(\mathcal{H})$ then we denote by $W^*(A)$ the von Neumann subalgebra which is generated by A, i.e., $W^*(A)$ is the smallest von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ which contains A.

Lemma 2.5.1. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $(A)_1$ is compact in the WOT.*

Proof. This follows directly from Corollary 2.4.4. ∎

Corollary 2.5.2. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then $(A)_1$ and $A_{\text{s.a.}}$ are closed in the weak and strong operator topologies.*

Proof. Since taking adjoints is continuous in the weak operator topology it follows that $A_{\text{s.a.}}$ is closed in the weak operator topology, and by the previous result this is also the case for $(A)_1$. ∎

If $B \subset \mathcal{B}(\mathcal{H})$, the **commutant** of B is

$$B' = \{T \in \mathcal{B}(\mathcal{H}) \mid TS = ST, \text{ for all } S \in B\}.$$

We also use the notation $B'' = (B')'$ for the **double commutant**.

Theorem 2.5.3. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint set, then A' is a von Neumann algebra.*

Proof. It is easy to see that A' is a self-adjoint algebra containing 1. To see that it is closed in the weak operator topology just notice that if $x_\alpha \in A'$ is a net such that $x_\alpha \to x \in \mathcal{B}(\mathcal{H})$ then for any $a \in A$, and $\xi, \eta \in \mathcal{H}$, we have

$$\langle [x, a]\xi, \eta\rangle = \langle xa\xi, \eta\rangle - \langle x\xi, a^*\eta\rangle$$

$$= \lim_{\alpha\to\infty} \langle x_\alpha a\xi, \eta\rangle - \langle x_\alpha\xi, a^*\eta\rangle = \lim_{\alpha\to\infty} \langle [x_\alpha, a]\xi, \eta\rangle = 0.$$ ∎

Corollary 2.5.4. *A self-adjoint maximal abelian subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra.*

Proof. Since A is maximal abelian we have $A = A'$. ∎

Lemma 2.5.5. *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing 1. Then for all $\xi \in \mathcal{H}$, and $x \in A''$ there exists $x_\alpha \in A$ such that $\lim_{\alpha\to\infty} \|(x - x_\alpha)\xi\| = 0$.*

Proof. Consider the closed subspace $\mathcal{K} = \overline{A\xi} \subset \mathcal{H}$, and denote by p the projection onto this subspace. Since for all $a \in A$ we have $a\mathcal{K} \subset \mathcal{K}$, it follows that $ap = pap$. But since A is self-adjoint it then also follows that for all $a \in A$ we have $pa = (a^*p)^* = (pa^*p)^* = pap = ap$, and hence $p \in A'$.

We therefore have that $xp = xp^2 = pxp$ and hence $x\mathcal{K} \subset \mathcal{K}$. Since $1 \in A$ it follows that $\xi \in \mathcal{K}$ and hence also $x\xi \in \overline{A\xi}$. ∎

Theorem 2.5.6 (Von Neumann's double commutant theorem)**.** *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing* 1*. Then A'' is equal to the weak operator topology closure of A.*

Proof. By Theorem 2.5.3 we have that A'' is closed in the weak operator topology, and we clearly have $A \subset A''$, so we just need to show that $A \subset A''$ is dense in the weak operator topology. For this we use the previous lemma together with a matrix trick.

Let $\xi_1, \ldots, \xi_n \in \mathcal{H}$, $x \in A''$ and consider the subalgebra $\tilde{A}$ of $\mathcal{B}(\mathcal{H}^n) \cong \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$ consisting of diagonal matrices with constant diagonal coefficients contained in A. Then the diagonal matrix whose diagonal entries are all x is easily seen to be contained in $\tilde{A}''$, hence the previous lemma applies and so there exists a net $a_\alpha \in A$ such that $\lim_{\alpha \to \infty} \|(x - a_\alpha)\xi_k\| = 0$, for all $1 \leq k \leq n$. This shows that $A \subset A''$ is dense in the strong operator topology. ∎

We also have the following formulation which is easily seen to be equivalent.

Corollary 2.5.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra. Then A is a von Neumann algebra if and only if $A = A''$.*

Corollary 2.5.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $x \in A$, and consider the polar decomposition $x = v|x|$. Then $v \in A$.*

Proof. Note that $\ker(v) = \ker(|x|)$, and if $a \in A'$ then we have $a \ker(|x|) \subset \ker(|x|)$. Also, we have

$$\|(av - va)|x|\xi\| = \|ax\xi - xa\xi\| = 0,$$

for all $\xi \in \mathcal{H}$. Hence av and va agree on $\ker(|x|) + \overline{R(|x|)} = \mathcal{H}$, and so $v \in A'' = A$. ∎

Proposition 2.5.9. *Let (X, μ) be a probability space. Consider the Hilbert space $L^2(X, \mu)$, and the map $M : L^\infty(X, \mu) \to \mathcal{B}(L^2(X, \mu))$ defined by $(M_g\xi)(x) = g(x)\xi(x)$, for all $\xi \in L^2(X, \mu)$. Then M is an isometric $*$-isomorphism from $L^\infty(X, \mu)$ onto a maximal abelian von Neumann subalgebra of $\mathcal{B}(L^2(X, \mu))$.*

Proof. The fact that M is a $*$-isomorphism onto its image is clear. If $g \in L^\infty(X, \mu)$ then by definition of $\|g\|_\infty$ we can find a sequence E_n of measurable subsets of X such that $0 < \mu(E_n)$, and $|g|_{E_n} \geq \|g\|_\infty - 1/n$, for all $n \in \mathbb{N}$. We then have

$$\|M_g\| \geq \|M_g 1_{E_n}\|_2 / \|1_{E_n}\|_2 \geq \|g\|_\infty - 1/n.$$

The inequality $\|g\|_\infty \leq \|M_g\|$ is also clear and hence M is isometric.

To see that $M(L^\infty(X,\mu))$ is maximal abelian let's suppose $T \in \mathcal{B}(L^2(X,\mu))$ commutes with M_f for all $f \in L^\infty(X,\mu)$. We define $f \in L^2(X,\mu)$ by $f = T(1_X)$.

For each $g, h \in L^\infty(X,\mu)$, we have

$$|\int fg\overline{h}d\mu| = |\langle M_g T(1_X), h\rangle|$$

$$= |\langle T(g), h\rangle| \leq \|T\|\|g\|_2\|h\|_2.$$

Since $L^\infty(X,\mu) \subset L^2(X,\mu)$ is dense in $\|\cdot\|_2$, it then follows from Hölder's inequality that $f \in L^\infty(X,\mu)$, and $T = M_f$. ■

Because of the previous result we will often identify $L^\infty(X,\mu)$ with the subalgebra of $\mathcal{B}(L^2(X,\mu))$ as described above. This should not cause any confusion.

With minor modifications the previous result can be shown to hold for any measure space (X,μ) which is a disjoint union of probability spaces, e.g., if (X,μ) is σ-finite, or if X is arbitrary and μ is the counting measure.

Exercise 2.5.10. Let X be an uncountable set, $\mathcal{B}_1$ the set of all subsets of X, $\mathcal{B}_2 \subset B_1$ the set consisting of all sets which are either countable or have countable complement, and μ the counting measure on X. Show that the identity map implements a unitary operator $\mathrm{id} : L^2(X,\mathcal{B}_1,\mu) \to L^2(X,\mathcal{B}_2,\mu)$, and we have $L^\infty(X,\mathcal{B}_2,\mu) \subsetneq L^\infty(X,\mathcal{B}_2,\mu)'' = \mathrm{id}\, L^\infty(X,\mathcal{B}_1,\mu)\, \mathrm{id}^*$.

2.6 Kaplansky's density theorem

Proposition 2.6.1. *If $f \in C(\mathbb{C})$ then $x \mapsto f(x)$ is continuous in the strong operator topology on any bounded set of normal operators in $\mathcal{B}(\mathcal{H})$.*

Proof. By the Stone-Weierstrass theorem we can approximate f uniformly well by polynomials on any compact set. Since multiplication is jointly SOT continuous on bounded sets, and since taking adjoints is SOT continuous on normal operators, the result follows easily. ■

Proposition 2.6.2 (The Cayley transform)**.** *The map $x \mapsto (x-i)(x+i)^{-1}$ is strong operator topology continuous from the set of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ into the unitary operators in $\mathcal{B}(\mathcal{H})$.*

Proof. Suppose $\{x_k\}_k$ is a net of self-adjoint operators such that $x_k \to x$ in the SOT. By the spectral mapping theorem we have $\|(x_k+i)^{-1}\| \leq 1$ and hence for all $\xi \in \mathcal{H}$ we have

$$\begin{aligned} &\|(x-i)(x+i)^{-1}\xi - (x_k-i)(x_k+i)^{-1}\xi\| \\ &= \|(x_k+i)^{-1}((x_k+i)(x-i) - (x_k-i)(x+i))(x+i)^{-1}\xi\| \\ &= \|2i(x_k+i)^{-1}(x-x_k)(x+i)^{-1}\xi\| \leq 2\|(x-x_k)(x+i)^{-1}\xi\| \to 0. \end{aligned}$$ ■

Corollary 2.6.3. *If $f \in C_0(\mathbb{R})$ then $x \mapsto f(x)$ is strong operator topology continuous on the set of self-adjoint operators.*

Proof. Since f vanishes at infinity, we have that $g(t) = f\left(i\frac{1+t}{1-t}\right)$ defines a continuous function on $\mathbb{T}$ if we set $g(1) = 0$. By Proposition 2.6.1 $x \mapsto g(x)$ is then SOT continuous on the space of unitaries. If $U(z) = \frac{z-i}{z+i}$ is the Cayley transform, then by Proposition 2.6.2 it follows that $f = g \circ U$ is SOT continuous being the composition of two SOT continuous functions. ∎

Theorem 2.6.4 (Kaplansky's density theorem)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ and denote by B the strong operator topology closure of A.*

(*i*) *The strong operator topology closure of $A_{\text{s.a.}}$ is $B_{\text{s.a.}}$.*

(*ii*) *The strong operator topology closure of $(A)_1$ is $(B)_1$.*

Proof. We may assume that A is a C^*-algebra. If $\{x_k\}_k \subset A$ is a net of elements which converge in the SOT to a self-adjoint element x_k, then since taking adjoints is WOT continuous we have that $\frac{x_k+x_k^*}{2} \to x$ in the WOT. But $A_{\text{s.a.}}$ is convex and so the WOT and SOT closures coincide, showing (a). Moreover, if $\{y_k\}_k \subset A_{\text{s.a.}}$ such that $y_k \to x$ in the SOT then by considering a function $f \in C_0(\mathbb{R})$ such that $f(t) = t$ for $|t| \leq \|x\|$, and $|f(t)| \leq \|x\|$, for $t \in \mathbb{R}$, we have $\|f(y_k)\| \leq \|x\|$, for all k and $f(y_k) \to f(x)$ in the SOT by Corollary 2.6.3. Hence $(A)_1 \cap A_{\text{s.a.}}$ is SOT dense in $(B)_1 \cap B_{\text{s.a.}}$.

Note that $\mathbb{M}_2(A)$ is SOT dense in $\mathbb{M}_2(B) \subset \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Therefore if $x \in (B)_1$ then $\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (\mathbb{M}_2(B))_1$ is self-adjoint. Hence from above there exists a net of operators $\tilde{x}_n \in (\mathbb{M}_2(A))_1$ such that $\tilde{x}_n \to \tilde{x}$ in the SOT. Writing $\tilde{x}_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ we then have that $\|b_n\| \leq 1$ and $b_n \to x$ in the SOT. ∎

Corollary 2.6.5. *A self-adjoint unital subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $(A)_1$ is closed in the SOT.*

Corollary 2.6.6. *A self-adjoint unital subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if A is closed in the σ-WOT.*

2.6.1 Preduals

Proposition 2.6.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let $A_* \subset A^*$ be the subspace of σ-WOT continuous linear functionals, then $(A_*)^* = A$ and under this identification the weak*-topology on A agrees with the σ-WOT.*

Proof. By the Hahn-Banach Theorem, and Lemma 2.4.3 we can identify A_* with $L^1(\mathcal{B}(\mathcal{H}))/A_\perp$, where $A_\perp$ is the pre-annihilator

$$A_\perp = \{x \in L^1(\mathcal{B}(\mathcal{H})) \mid \mathrm{Tr}(ax) = 0, \text{ for all } a \in A\}.$$

From the general theory of Banach spaces it follows that $(L^1(\mathcal{B}(\mathcal{H}))/A_\perp)^*$ is canonically isomorphic to the weak* closure of A, which is equal to A by Corollary 2.6.6. The fact that the weak*-topology on A agrees with the σ-WOT is then obvious. ∎

If $A \subset \mathcal{B}(\mathcal{H})$ and $B \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras, then a linear map $\Phi : A \to B$ is said to be **normal** if it is continuous from the σ-WOT of A to the σ-WOT of B.

Exercise 2.6.8. Suppose $A \subset \mathcal{B}(\mathcal{H})$ and $B \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras, and $\Phi : A \to B$ is a bounded linear map. Show that Φ is normal if and only if the dual map $\Phi^* : B^* \to A^*$ given by $\Phi^*(\psi)(a) = \psi(\Phi(a))$ satisfies $\Phi^*(B_*) \subset A_*$.

2.7 Borel functional calculus

If $T \in \mathbb{M}_n(\mathbb{C})$ is a normal matrix, then there are different perspectives one can take when describing the spectral theorem for T. The first, a basis free approach, is to consider the eigenvalues $\sigma(T)$ for T, and to each eigenvalue λ associate to it the projection $E(\lambda)$ onto the corresponding eigenspace. Since T is normal we have that the $E(\lambda)$'s are pairwise orthogonal and we have

$$T = \sum_{\lambda \in \sigma(T)} \lambda E(\lambda).$$

The second approach is to use that since T is normal, it is diagonalizable. We therefore could find a unitary matrix U such that UTU^* is a diagonal matrix with diagonal entries λ_i. If we denote by $E_{i,i}$ the elementary matrix with a 1 in the (i, i) position and 0 elsewhere, then we have

$$T = U^*(\sum_{i=1}^{n} \lambda_i E_{i,i})U.$$

For bounded normal operators there are two similar approaches to the spectral theorem. The first approach is to find a substitute for the projections $E(\lambda)$ and this leads naturally to the notion of a spectral measure. For the second approach, this naturally leads to the interpretation of diagonal matrices corresponding to multiplication by essentially bounded functions on a probability space.

Lemma 2.7.1. *Let $x_\alpha \in \mathcal{B}(\mathcal{H})$ be an increasing net of positive operators such that $\sup_\alpha \|x_\alpha\| < \infty$, then there exists a bounded operator $x \in \mathcal{B}(\mathcal{H})$ such that $x_\alpha \to x$ in the SOT.*

Proof. We may define a quadratic form on $\mathcal{H}$ by $\xi \mapsto \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$. Since $\sup_\alpha \|x_\alpha\| < \infty$ we have that this quadratic form is bounded and hence there exists a bounded positive operator $x \in \mathcal{B}(\mathcal{H})$ such that $\|\sqrt{x}\xi\|^2 = \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$, for all $\xi \in \mathcal{H}$. Note that $x_\alpha \leq x$ for all α, and $\sup_\alpha \|(x - x_\alpha)^{1/2}\| < \infty$. Thus for each $\xi \in \mathcal{H}$ we have

$$\begin{aligned} \|(x - x_\alpha)\xi\|^2 &\leq \|(x - x_\alpha)^{1/2}\|^2 \|(x - x_\alpha)^{1/2}\xi\|^2 \\ &= \|(x - x_\alpha)^{1/2}\|^2 (\|\sqrt{x}\xi\|^2 - \|\sqrt{x_\alpha}\xi\|^2) \to 0. \end{aligned}$$

Hence, $x_\alpha \to x$ in the SOT. ∎

Corollary 2.7.2. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\{p_\iota\}_{\iota \in I} \subset A$ is a collection of pairwise orthogonal projections then $p = \sum_{\iota \in I} p_\iota \in A$ is well defined as a SOT limit of finite sums.*

2.7.1 Spectral measures

Let K be a compact Hausdorff space and let $\mathcal{H}$ be a Hilbert space. A **spectral measure** E on K relative to $\mathcal{H}$ is a mapping from the Borel subsets of K to the set of projections in $\mathcal{B}(\mathcal{H})$ such that

(i) $E(\emptyset) = 0, E(K) = 1$.

(ii) $E(B_1 \cap B_2) = E(B_1)E(B_2)$ for all Borel sets B_1 and B_2.

(iii) For all $\xi, \eta \in \mathcal{H}$ the function

$$B \mapsto E_{\xi,\eta}(B) = \langle E(B)\xi, \eta \rangle$$

is a finite Radon measure on K.

Example 2.7.3. If K is a compact Hausdorff space and μ is a σ-finite Radon measure on K, then the map $E(B) = 1_B \in L^\infty(K, \mu) \subset \mathcal{B}(L^2(K, \mu))$ defines a spectral measure on K relative to $L^2(K, \mu)$.

We denote by $B_\infty(K)$ the space of all bounded Borel functions on K. This is clearly a C^*-algebra with the sup norm.

For each $f \in B_\infty(K)$ it follows that the map

$$(\xi, \eta) \mapsto \int f \, dE_{\xi,\eta}$$

gives a continuous sesqui-linear form on $\mathcal{H}$ and hence it follows that there exists a bounded operator T such that $\langle T\xi, \eta \rangle = \int f \, dE_{\xi,\eta}$. We denote this operator T by $\int f \, dE$ so that we have the formula $\langle (\int f \, dE)\xi, \eta \rangle = \int f \, dE_{\xi,\eta}$, for each $\xi, \eta \in \mathcal{H}$.

Theorem 2.7.4. *Let K be a compact Hausdorff space, let $\mathcal{H}$ be a Hilbert space, and suppose that E is a spectral measure on K relative to $\mathcal{H}$. Then the association*

$$f \mapsto \int f \, dE$$

defines a continuous $*$-homomorphism from $B_\infty(K)$ to $\mathcal{B}(\mathcal{H})$. Moreover, the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, and if $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty$, then $\int f_n dE \to \int f dE$ in the SOT.

Proof. It is easy to see that this map defines a linear contraction which preserves the adjoint operation. If $A, B \subset K$ are Borel subsets, and $\xi, \eta \in \mathcal{H}$, then denoting $x = \int 1_A dE$, $y = \int 1_B dE$, and $z = \int 1_{A \cap B} dE$ we have

$$\begin{aligned} \langle xy\xi, \eta \rangle &= \langle E(A)y\xi, \eta \rangle = \langle E(B)\xi, E(A)\eta \rangle \\ &= \langle E(B \cap A)\xi, \eta \rangle = \langle z\xi, \eta \rangle. \end{aligned}$$

Hence $xy = z$, and by linearity we have that $(\int f\, dE)(\int g\, dE) = \int fg\, dE$ for all simple functions $f, g \in B_\infty(K)$. Since every function in $B_\infty(K)$ can be approximated uniformly by simple functions this shows that this is indeed a $*$-homomorphism.

To see that the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, note that if a commutes with all operators of the form $\int f\, dE$ for $f \in C(K)$ then for all $\xi, \eta \in \mathcal{H}$ we have

$$0 = \langle (a(\int f\, dE) - (\int f\, dE)a)\xi, \eta\rangle = \int f\, dE_{\xi, a^*\eta} - \int f\, dE_{a\xi,\eta}.$$

Thus $E_{\xi,a^*\eta} = E_{a\xi,\eta}$ and hence we have that a also commutes with operators of the form $\int g\, dE$ for any $g \in B_\infty(K)$. Therefore by Theorem 2.5.6 $\int g\, dE$ is contained in the von Neumann algebra generated by the image of $C(K)$.

Now suppose $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty(K)$. For each $\xi, \eta \in \mathcal{H}$ we have

$$\int f_n\, dE_{\xi,\eta} \to \int f\, dE_{\xi,\eta},$$

hence $\int f_n\, dE$ converges in the WOT to $\int f\, dE$. However, since $\int f_n\, dE$ is an increasing sequence of bounded operators with $\| \int f_n\, dE \| \leq \|f\|_\infty$, Lemma 2.7.1 shows that $\int f_n\, dE$ converges in the SOT to some operator $x \in \mathcal{B}(\mathcal{H})$ and we must then have $x = \int f\, dE$. ■

The previous theorem shows, in particular, that if A is a unital abelian C^*-algebra, and E is a spectral measure on $\sigma(A)$ relative to $\mathcal{H}$, then we obtain a unital $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ by the formula

$$\pi(x) = \int \Gamma(x) dE.$$

We next show that in fact every unital $*$-representation arises in this way.

Theorem 2.7.5. *Let A be a unital abelian C^*-algebra, $\mathcal{H}$ a Hilbert space and $\pi : A \to \mathcal{B}(\mathcal{H})$ a unital $*$-representation. Then there is a unique spectral measure E on $\sigma(A)$ relative to $\mathcal{H}$ such that for all $x \in A$ we have*

$$\pi(x) = \int \Gamma(x) dE.$$

Proof. For each $\xi, \eta \in \mathcal{H}$ we have that $f \mapsto \langle \pi(\Gamma^{-1}(f))\xi, \eta\rangle$ defines a bounded linear functional on $\sigma(A)$ and hence by the Riesz representation therorem there exists a Radon measure $E_{\xi,\eta}$ such that for all $f \in C(\sigma(A))$ we have

$$\langle \pi(\Gamma^{-1}(f))\xi, \eta\rangle = \int f dE_{\xi,\eta}.$$

Since the Gelfand transform is a $*$-homomorphism we verify easily that $f dE_{\xi,\eta} = dE_{\pi(\Gamma^{-1}(f))\xi,\eta} = dE_{\xi,\pi(\Gamma^{-1}(\overline{f}))\eta}$.

Thus for each Borel set $B \subset \sigma(A)$ we can consider the sesquilinear form $(\xi, \eta) \mapsto \int 1_B dE_{\xi,\eta}$. We have $|\int f dE_{\xi,\eta}| \leq \|f\|_\infty \|\xi\| \|\eta\|$, for all $f \in C(\sigma(A))$ and hence this sesquilinear form is bounded and there exists a bounded operator $E(B)$ such that $\langle E(B)\xi, \eta \rangle = \int 1_B dE_{\xi,\eta}$, for all $\xi, \eta \in \mathcal{H}$. For all $f \in C(\sigma(A))$ we have

$$\langle \pi(\Gamma^{-1}(f))E(B)\xi, \eta \rangle = \int 1_B dE_{\xi, \pi(\Gamma^{-1}(\overline{f}))\eta} = \int 1_B f dE_{\xi,\eta}.$$

Thus it follows that $E(B)^* = E(B)$, and $E(B')E(B) = E(B' \cap B)$, for any Borel set $B' \subset \sigma(A)$. In particular, $E(B)$ is a projection and E gives a spectral measure on $\sigma(A)$ relative to $\mathcal{H}$. The fact that for $x \in A$ we have $\pi(x) = \int \Gamma(x) dE$ follows easily from the way we constructed E. ∎

If $\mathcal{H}$ is a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$ is a normal operator, then by applying the previous theorem to the C^*-subalgebra A generated by x and 1, and using the identification $\sigma(A) = \sigma(x)$ we obtain a homomorphism from $B_\infty(\sigma(x))$ to $\mathcal{B}(\mathcal{H})$ and hence for $f \in B_\infty(\sigma(x))$ we may define

$$f(x) = \int f dE.$$

Note that it is straight forward to check that considering the function $f(z) = z$ we have

$$x = \int z dE(z).$$

We now summarize some of the properties of this functional calculus which follow easily from the previous results.

Theorem 2.7.6 (Borel functional calculus)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose $x \in A$ is a normal operator, then the Borel functional calculus defined by $f \mapsto f(x)$ satisfies the following properties:*

(i) *$f \mapsto f(x)$ is a continuous unital $*$-homomorphism from $B_\infty(\sigma(x))$ into A.*

(ii) *If $f \in B_\infty(\sigma(x))$ then $\sigma(f(x)) \subset f(\sigma(x))$.*

(iii) *If $f \in C(\sigma(x))$ then $f(x)$ agrees with the definition given by continuous functional calculus.*

Corollary 2.7.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then A is the uniform closure of the span of its projections.*

Proof. By decomposing an operator into its real and imaginary parts it is enough to check this for self-adjoint operators in the unit ball, and this follows from the previous theorem by approximating the function $f(t) = t$ uniformly by simple functions on $[-1, 1]$. ∎

Corollary 2.7.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then the unitary group $\mathcal{U}(A)$ is path connected in the uniform topology.*

Proof. If $u \in \mathcal{U}(A)$ is a unitary and we consider a branch of the log function $f(z) = \log z$, then from Borel functional calculus we have $u = e^{ix}$ where $x = -if(u)$ is self-adjoint. We then have that $u_t = e^{itx}$ is a uniform norm continuous path of unitaries such that $u_0 = 1$ and $u_1 = u$. ■

Corollary 2.7.9. *If $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then $\mathcal{K}(\mathcal{H})$ is the unique non-zero proper norm closed two sided ideal in $\mathcal{B}(\mathcal{H})$.*

Proof. If $I \subset \mathcal{B}(\mathcal{H})$ is a norm closed two sided ideal and $x \in I \setminus \{0\}$, then for any $\xi \in R(x^*x)$, $\|\xi\| = 1$ we can consider $y = (\xi \otimes \overline{\xi})x^*x(\xi \otimes \overline{\xi}) \in I$ which is a rank one self-adjoint operator with $R(y) = \mathbb{C}\xi$. Thus y is a multiple of $(\xi \otimes \overline{\xi})$ and hence $(\xi \otimes \overline{\xi}) \in I$. For any $\zeta, \eta \in \mathcal{H}$, we then have $\zeta \otimes \overline{\eta} = (\zeta \otimes \overline{\xi})(\xi \otimes \overline{\xi})(\xi \otimes \overline{\eta}) \in I$ and hence I contains all finite rank operators. Since I is closed we then have that $\mathcal{K}(\mathcal{H}) \subset I$.

If $x \in I$ is not compact then for some $\varepsilon > 0$ we have that $\dim(1_{[\varepsilon,\infty)}(x^*x)\mathcal{H}) = \infty$. If we let $u \in \mathcal{B}(\mathcal{H})$ be an isometry from $\mathcal{H}$ onto $1_{[\varepsilon,\infty)}(x^*x)\mathcal{H}$, then we have that $\sigma(u^*x^*xu) \subset [\varepsilon, \infty)$. Hence, $u^*x^*xu \in I$ is invertible which shows that $I = \mathcal{B}(\mathcal{H})$. ■

Exercise 2.7.10. Suppose that K is a compact Hausdorff space and E is a spectral measure for K relative to a Hilbert space $\mathcal{H}$, show that if $f \in B_\infty(K)$, and we have a decomposition of K into a countable union of pairwise disjoint Borel sets $K = \cup_{n\in\mathbb{N}} B_n$ then we have that

$$\int f dE = \sum_{n\in\mathbb{N}} \int_{B_n} f dE,$$

where the convergence of the sum is in the weak operator topology.

2.8 Abelian von Neumann algebras

Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $\xi \in \mathcal{H}$ is a non-zero vector. Then ξ is said to be **cyclic** for A if $A\xi$ is dense in $\mathcal{H}$. We say that ξ is **separating** for A if $x\xi \neq 0$, for all $x \in A$, $x \neq 0$.

Proposition 2.8.1. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then a non-zero vector $\xi \in \mathcal{H}$ is cyclic for A if and only if ξ is separating for A'.*

Proof. Suppose ξ is cyclic for A, and $x \in A'$ such that $x\xi = 0$. Then $xa\xi = ax\xi = 0$ for all $a \in A$, and since $A\xi$ is dense in $\mathcal{H}$ it follows that $x\eta = 0$ for all $\eta \in \mathcal{H}$. Conversely, if $A\xi$ is not dense, then the orthogonal projection p onto its complement is a nonzero operator in A' such that $p\xi = 0$. ■

Corollary 2.8.2. *If $A \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann algebra and $\xi \in \mathcal{H}$ is cyclic, then ξ is also separating.*

Proof. Since ξ being separating passes to von Neumann subalgebras and $A \subset A'$ this follows. ■

Infinite dimensional von Neumann algebras are never separable in the norm topology. For this reason we will say that a von Neumann algebra A is **separable** if A is separable in the SOT. Equivalently, A is separable if its predual A_* is separable.

Proposition 2.8.3. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable von Neumann algebra. Then there exists a separating vector for A.*

Proof. Since A is separable, it follows that there exists a countable collection of vectors $\{\xi_k\}_k \subset \mathcal{H}$ such that $x\xi_k = 0$ for all k only if $x = 0$. Also, since A is separable we have that $\mathcal{H}_0 = \overline{\text{sp}}(A\{\xi_k\}_k)$ is also separable. Thus, restricting A to $\mathcal{H}_0$ we may assume that $\mathcal{H}$ is separable.

By Zorn's lemma we can find a maximal family of non-zero unit vectors $\{\xi_\alpha\}_\alpha$ such that $A\xi_\alpha \perp A\xi_\beta$, for all $\alpha \neq \beta$. Since $\mathcal{H}$ is separable this family must be countable and so we may enumerate it $\{\xi_n\}_n$, and by maximality we have that $\{A\xi_n\}_n$ is dense in $\mathcal{H}$.

If we denote by p_n the orthogonal projection onto the closure of $A\xi_n$ then we have that $p_n \in A'$, hence, setting $\xi = \sum_n \frac{1}{2^n}\xi$ if $x \in A$ such that $x\xi = 0$, then for every $n \in \mathbb{N}$ we have $0 = 2^n p_n x\xi = 2^n x p_n \xi = x\xi_n$ and so $x = 0$ showing that ξ is a separating vector for A. ∎

Corollary 2.8.4. *Suppose $\mathcal{H}$ is separable, if $A \subset \mathcal{B}(\mathcal{H})$ is a maximal abelian self-adjoint subalgebra (masa), then there exists a cyclic vector for A.*

Proof. By Propostion 2.8.3 there exists a non-zero vector $\xi \in \mathcal{H}$ which is separating for A, and hence by Proposition 2.8.1 is cyclic for $A' = A$. ∎

The converse of the previous corollary also holds (without the separability hypothesis), which follows from Proposition 2.5.9, together with the following theorem.

Theorem 2.8.5. *Let $A \subset \mathcal{B}(\mathcal{H})$ be an abelian von Neumann algebra and suppose $\xi \in \mathcal{H}$ is a cyclic vector. Then for any SOT dense C^*-subalgebra $A_0 \subset A$ there exists a Radon probability measure μ on $K = \sigma(A_0)$ with* $\text{supp}(\mu) = K$*, and an unitary $U : L^2(K, \mu) \to \mathcal{H}$ such that $U^*AU = L^\infty(K, \mu) \subset \mathcal{B}(L^2(X, \mu))$.*

Proof. Fix a SOT dense C^*-algebra $A_0 \subset A$, then by the Riesz representation theorem we obtain a finite Radon measure μ on $K = \sigma(A_0)$ such that $\langle \Gamma(f)\xi, \xi\rangle = \int f d\mu$ for all $f \in C(K)$. Since the Gelfand transform takes positive operator to positive functions we see that μ is a probability measure.

We define a map $U_0 : C(K) \to \mathcal{H}$ by $f \mapsto \Gamma(f)\xi$, and note that $\|U_0(f)\|^2 = \langle \Gamma(\overline{f}f)\xi, \xi\rangle = \int \overline{f} f d\mu = \|f\|_2$. Hence U_0 extends to an isometry $U : L^2(K, \mu) \to \mathcal{H}$. Since ξ is cyclic we have that $A_0\xi \subset U(L^2(K, \mu))$ is dense and hence U is a unitary. If the support of μ were not K then there would exist a non-zero continuous function $f \in C(K)$ such that $0 = \int |f^2| d\mu = \|\Gamma(f)\xi\|^2$, but since by Corollary 2.8.2 we know that ξ is separating and hence this cannot happen.

If $f \in C(K) \subset \mathcal{B}(L^2(K, \mu))$, and $g \in C(K) \subset L^2(K, \mu)$ then we have

$$U^*\Gamma(f)Ug = U^*\Gamma(f)\Gamma(g)\xi = fg = M_f g.$$

Since $C(K)$ is $\|\cdot\|_2$-dense in $L^2(K, \mu)$ it then follows that $U^*\Gamma(f)U = M_f$, for all $f \in C(K)$ and thus $U^*A_0U \subset L^\infty(K, \mu)$. Since A_0 is SOT dense in A we then have that $U^*AU \subset L^\infty(K, \mu)$. But since $x \mapsto U^*xU$ is WOT continuous and $(A)_1$ is compact in the WOT it follows that $U^*(A)_1U = (L^\infty(K, \mu))_1$ and hence $U^*AU = L^\infty(K, \mu)$. ■

In general, if $A \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann algebra and $\xi \in \mathcal{H}$ is a non-zero vector, then we can consider the projection p onto the $\mathcal{K} = \overline{A\xi}$. We then have $p \in A'$, and $Ap \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann for which ξ is a cyclic vector, thus by the previous result Ap is $*$-isomorphic to $L^\infty(X, \mu)$ for some probability space (X, μ). An application of Zorn's Lemma can then be used to show that A is $*$-isomorphic to $L^\infty(Y, \nu)$ were (Y, ν) is a measure space which is a disjoint union of probability spaces. In the case when A is separable an even more concrete classification will be given below.

Theorem 2.8.6. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there exists a separable compact Hausdorff space K with a Radon probability measure μ on K such that A and $L^\infty(K, \mu)$ are $*$-isomorphic.*

Proof. By Proposition 2.8.3 there exists a non-zero vector $\xi \in \mathcal{H}$ which is separating for A. Thus if we consider $\mathcal{K} = \overline{A\xi}$ we have that restricting each operator $x \in A$ to $\mathcal{K}$ is a C^*-algebra isomorphism and $\xi \in \mathcal{K}$ is then cyclic. Thus, the result follows from Theorem 2.8.5. ■

If $x \in \mathcal{B}(\mathcal{H})$ is normal such that $A = W^*(x)$ is separable (e.g., if $\mathcal{H}$ is separable), then we may let A_0 be the C^*-algebra generated by x. We then obtain the following alternate version of the spectral theorem.

Corollary 2.8.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $x \in A$ is normal such that $W^*(x)$ is separable, then there exists a Radon probability measure μ on $\sigma(x)$ and a $*$-homomorphism $f \mapsto f(x)$ from $L^\infty(\sigma(x), \mu)$ into A which agrees with Borel functional calculus. Moreover, we have that $\sigma(f(x))$ is the essential range of f.*

Note that $W^*(x)$ need not be separable in general. For example, $\ell^\infty([0,1]) \subset \mathcal{B}(\ell^2([0,1]))$ is generated by the multiplication operator corresponding to the function $t \mapsto t$.

Lemma 2.8.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there exists a self-adjoint operator $x \in A$ such that $A = \{x\}''$.*

Proof. Since A is separable we have that A is countably generated as a von Neumann algebra. Indeed, just take a countable family in A which is dense in the SOT. By functional calculus we can approximate any self-adjoint element by a linear combination of projections and thus A is generated by a countable collection of projections $\{p_k\}_{k=0}^\infty$.

Define a sequence of self adjoint elements $x_n = \sum_{k=0}^n 4^{-k}p_k$, and let $x = \sum_{k=0}^\infty 4^{-k}p_k$. We denote by $A_0 = \{x\}''$. Define a continuous function $f : [-1, 2] \to \mathbb{R}$ such that $f(t) = 1$ if $t \in [1 - \frac{1}{3}, 1 + \frac{1}{3}]$ and $f(t) = 0$ if $t \leq \frac{1}{3}$, then we have that $f(x_n) = p_0$ for every n and hence by continuity of continuous functional calculus we have $p_0 = f(x) \in A_0$. The same argument shows that $p_1 = f(4(x - p_0)) \in A_0$ and by induction it follows easily that $p_k \in A_0$ for all $k \geq 0$, thus $A_0 = A$. ■

Theorem 2.8.9. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there is a countable (possibly empty) set K such that either A is $*$-isomorphic to $\ell^\infty K$, or else A is $*$-isomorphic to $L^\infty([0,1], \lambda) \oplus \ell^\infty K$ where λ is Lebesgue measure.*

Proof. Since A is separable we have from Lemma 2.8.8 that as a von Neumann algebra A is generated by a single self-adjoint element $x \in A$.

We define $K = \{a \in \sigma(x) \mid 1_{\{a\}}(x) \neq 0\}$. Since the projections corresponding to elements in K are pairwise orthogonal it follows that K is countable. Further, if we denote by $p_K = \sum_{a \in K} 1_{\{a\}}$ then we have that $Ap_K \cong \ell^\infty K$. Thus, all that remains is to show that if $(1 - p_K) \neq 0$ then $(1 - p_K)A = \{(1 - p_K)x\}'' \cong L^\infty([0,1], \lambda)$.

Set $x_0 = (1 - p_K)x \neq 0$. By our definition of K above we have that $\sigma(x_0)$ has no isolated points. Thus, we can inductively define a sequence of partitions $\{A_k^n\}_{k=1}^{2^n}$ of $\sigma(x_0)$ such that $A_k^n = A_{2k-1}^{n+1} \cup A_{2k}^{n+1}$, and A_k^n has non-empty interior, for all $n > 0$, $1 \leq k \leq 2^n$. If we then consider the elements $y_n = \sum_{k=1}^\infty \frac{k}{2^n} 1_{A_k}(x_0)$ then we have that $y_n \to y$ where $0 \leq y \leq 1$, $\{x_0\}'' = \{y\}''$ and every dyadic rational is contained in the spectrum of y (since the space of invertible operators is open in the norm topology), hence $\sigma(y) = [0,1]$.

By Theorem 2.8.6 it then follows that $\{x_0\}'' = \{y\}'' \cong L^\infty([0,1], \mu)$ for some Radon measure μ on $[0,1]$ which has full support and no atoms. If we define the function $\theta : [0,1] \to [0,1]$ by $\theta(t) = \mu([0,t])$ then θ gives a continuous bijection of $[0,1]$, and we have $\theta_* \mu = \lambda$, since both are Radon probability measures such that for intervals $[a,b]$ we have $\theta_*\mu([a,b]) = \mu([\theta^{-1}(a), \theta^{-1}(b)]) = \lambda([a,b])$. The map $\theta^* : L^\infty([0,1], \lambda) \to L^\infty([0,1], \mu)$ given by $\theta^*(f) = f \circ \theta^{-1}$ is then easily seen to be a $*$-isomorphism. ■

Chapter 3

Unbounded operators

3.1 Definitions and examples

Let $\mathcal{H}$, and $\mathcal{K}$ Hilbert spaces. An **linear operator** $T : \mathcal{H} \to \mathcal{K}$ consists of a linear subspace $D(T) \subset \mathcal{H}$ together with a linear map from $D(T)$ to $\mathcal{K}$ (which will also be denoted by T). A linear operator $T : \mathcal{H} \to \mathcal{K}$ is bounded if there exists $K \geq 0$ such that $\|T\xi\| \leq K\|\xi\|$ for all $\xi \in D(T)$.

The **graph** of T is the subspace

$$\mathcal{G}(T) = \{\xi \oplus T\xi \mid \xi \in \mathcal{H}\} \subset \mathcal{H} \oplus \mathcal{K},$$

T is said to be **closed** if its graph $\mathcal{G}(T)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{K}$, and T is said to be **closable** if there exists an unbounded closed operator $S : \mathcal{H} \to \mathcal{K}$ such that $\overline{\mathcal{G}(T)} = \mathcal{G}(S)$. If T is closable we denote the operator S by $\overline{T}$ and call it the **closure** of T. A linear operator T is **densely defined** if $D(T)$ is a dense subspace. We denote by $\mathcal{C}(\mathcal{H}, \mathcal{K})$ the set of closed, densely defined linear operators from $\mathcal{H}$ to $\mathcal{K}$, and we also write $\mathcal{C}(\mathcal{H})$ for $\mathcal{C}(\mathcal{H}, \mathcal{H})$. Note that we may consider $\mathcal{B}(\mathcal{H}, \mathcal{K}) \subset \mathcal{C}(\mathcal{H}, \mathcal{K})$.

If $T, S : \mathcal{H} \to \mathcal{K}$ are two linear operators, then we say that S is an **extension** of T and write $S \sqsubseteq T$ if $D(S) \subset D(T)$ and $T_{|D(S)} = S$. Also, if $T : \mathcal{H} \to \mathcal{K}$, and $S : \mathcal{K} \to \mathcal{L}$ are linear operators, then the composition $ST : \mathcal{H} \to \mathcal{L}$ is the linear operator with domain

$$D(ST) = \{\xi \in D(T) \mid T\xi \in D(S)\},$$

defined by $(ST)(\xi) = S(T(\xi))$, for all $\xi \in D(ST)$. We may similarly define addition of linear operators as

$$D(S + T) = D(S) \cap D(T),$$

and $(S + T)\xi = S\xi + T\xi$, for all $\xi \in D(S + T)$. Even if S and T are both densely defined this need not be the case for ST or $S + T$. Both composition and addition are associative operations, and we still have the right distributive property $(R+S)T = (RT) + (ST)$, although note that in general we only have $T(R+S) \sqsupseteq (TR) + (TS)$.

If $S \in \mathcal{C}(\mathcal{H})$, and $T \in \mathcal{B}(\mathcal{H})$ then ST is still closed, although it may not be densely defined. Similarly, TS will be densely defined, although it may not be closed. If T also has a bounded inverse, then both ST and TS will be closed and densely defined.

If $T : \mathcal{H} \to \mathcal{K}$ is a densely defined linear operator, and $\eta \in \mathcal{K}$ such that the linear functional $\xi \mapsto \langle T\xi, \eta \rangle$ is bounded on $D(T)$, then by the Riesz representation theorem there exists a unique vector $T^*\eta \in \mathcal{H}$ such that for all $\xi \in D(T)$ we have

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle.$$

We denote by $D(T^*)$ the linear subspace of all vectors η such that $\xi \mapsto \langle T\xi, \eta \rangle$ is bounded, and we define the linear operator $\eta \mapsto T^*\eta$ to be the **adjoint** of T. Note that T^* is only defined for operators T which are densely defined.

A densely defined operator $T : \mathcal{H} \to \mathcal{H}$ is **symmetric** if $T \sqsubseteq T^*$, and is **self-adjoint** if $T = T^*$.

Example 3.1.1. Let $A = (a_{i,j}) \in M_{\mathbb{N}}(\mathbb{C})$ be a matrix, for each $n \in \mathbb{N}$ we consider the finite rank operator $T_n = \sum_{i,j \leq n} a_{i,j} \delta_i \otimes \overline{\delta_j}$, so that we may think of T_n as changing the entries of A to 0 whenever $i > n$, or $j > n$.

We set $D = \{\xi \in \ell^2\mathbb{N} \mid \lim_{n\to\infty} T_n\xi \text{ exists.}\}$, and we define $T_A : D \to \ell^2\mathbb{N}$ by $T_A\xi = \lim_{n\to\infty} T_n\xi$.

Suppose now that for each $j \in \mathbb{N}$ we have $\{a_{i,j}\}_i \in \ell^2\mathbb{N}$. Then we have $\mathbb{C}\mathbb{N} \subset D$ and so T_A is densely defined. If $\eta \in D(T_A^*)$ then it is easy to see that if we denote by P_n the projection onto the span of $\{\delta_i\}_{i \leq n}$, then we have $P_n = T_n^*\eta$, hence $\eta \in D(T_{A^*})$ where A^* is the Hermitian transpose of the matrix A. Conversely, it is also easy to see that $D(T_{A^*}) \subset D(T_A^*)$, and so $T_A^* = T_{A^*}$.

In particular, if $\{a_{i,j}\}_i \in \ell^2\mathbb{N}$, for every $j \in \mathbb{N}$, and if $\{a_{i,j}\}_j \in \ell^2\mathbb{N}$, for every $i \in \mathbb{N}$, then $T_A \in \mathcal{C}(\ell^2 N)$.

Example 3.1.2. Let (X, μ) be a σ-finite measure space and $f \in \mathcal{M}(X, \mu)$ a measurable function. We define the linear operator $M_f : L^2(X, \mu) \to L^2(X, \mu)$ by setting $D(M_f) = \{g \in L^2(X, \mu) \mid fg \in L^2(X, \mu)\}$, and $M_f(g) = fg$ for $g \in D(M_f)$. It's easy to see that each M_f is a closed operator, and that $f \mapsto M_f$ preserves the $*$-algebraic structure of $\mathcal{M}(X, \mu)$.

Example 3.1.3. Let $D \subset L^2[0,1]$ denote the space of absolutely continuous functions $f : [0,1] \to \mathbb{C}$, such that $f(0) = f(1) = 0$, and $f' \in L^2(0,1)$. It's not hard to check that D is dense in $L^2[0,1]$, and we may consider the densely defined operator $T : L^2(0,1) \to L^2(0,1)$ with domain D, given by $T(f) = if'$.

If $g \in D(T^*)$, and $h = T^*g$, then set $H(x) = \int_0^x h(t)\,\mathrm{d}t$, and note that $H(1) = \int_0^1 h(t)\,\mathrm{d}t = \langle 1, T^*g \rangle = \langle T(1), g \rangle = 0 = H(0)$. For every $f \in D$, integration by parts gives

$$i\int_0^1 f'\overline{g} = \langle Tf, g \rangle = \langle f, h \rangle = \int_0^1 f\overline{H'} = -\int_0^1 f'\overline{H}.$$

Thus, $\langle f', H - ig\rangle = 0$ for all $f \in D$, so that $H - ig \in \{f' \mid f \in D\}^\perp = \{1\}^{\perp\perp}$, and so $H - ig$ is a constant function. In particular, we see that g is absolutely continuous, and $g' = ih \in L^2[0,1]$. Conversely, if $g : [0,1] \to \mathbb{C}$ is absolutely continuous and $g' \in L^2[0,1]$ then it is equally easy to see that $g \in D(T^*)$, and $T^*g = ig'$.

In particular, this shows that T is symmetric, but not self-adjoint, and the same argument shows that $T^{**} = T$ (We'll see in Proposition 3.1.6 below that this implies that T is closed). If we consider instead the space $\tilde{D}$ consisting of all absolutely continuous functions $f : [0,1] \to \mathbb{C}$, such that $f(0) = f(1)$, and if we define $S : \tilde{D} \to L^2[0,1]$ by $S(f) = if'$, then a similar argument shows that S is self-adjoint. Thus, we have the following sequence of extensions:

$$T^{**} = T \sqsubseteq S = S^* \sqsubseteq T^*.$$

Lemma 3.1.4. *Let $T : \mathcal{H} \to \mathcal{K}$ be a densely defined operator, and denote by $J : \mathcal{H} \oplus \mathcal{K} \to \mathcal{K} \oplus \mathcal{H}$ the isometry defined by $J(\xi \oplus \eta) = -\eta \oplus \xi$. Then we have $\mathcal{G}(T^*) = J(\mathcal{G}(T))^\perp$.*

Proof. If $\eta, \zeta \in \mathcal{K}$, the $\eta \oplus \zeta \in J(\mathcal{G}(T))^\perp$ if and only if for all $\xi \in D(T)$ we have

$$0 = \langle -T\xi \oplus \xi, \eta \oplus \zeta\rangle = \langle \xi, \zeta\rangle - \langle T\xi, \eta\rangle.$$

Which, since $\mathcal{H} = \overline{D(T)}$, is also if and only if $\eta \in D(T^*)$ and $\zeta = T^*\eta$. ■

Corollary 3.1.5. *For any densely defined operator $T : \mathcal{H} \to \mathcal{K}$, the operator T^* is closed. In particular, self-adjoint operators are closed, and symmetric operators are closable.*

Proposition 3.1.6. *A densely defined operator $T : \mathcal{H} \to \mathcal{K}$ is closable if and only if T^* is densely defined, and if this is the case then we have $\overline{T} = (T^*)^*$.*

Proof. Suppose first that T^* is densely defined. Then by Lemma 3.1.4 we have

$$\mathcal{G}((T^*)^*) = -J^*(J(\mathcal{G}(T))^\perp)^\perp = (\mathcal{G}(T)^\perp)^\perp = \overline{\mathcal{G}(T)},$$

hence T is closable and $(T^*)^* = \overline{T}$.

Conversely, if T is closable then take $\zeta \in D(T^*)^\perp$.

For all $\eta \in D(T^*)$ we have

$$0 = \langle \zeta, \eta\rangle = \langle 0 \oplus \zeta, -T^*\eta \oplus \eta\rangle,$$

and hence $0 \oplus \zeta \in (-J^*\mathcal{G}(T^*))^\perp = \overline{\mathcal{G}(T)}$. Since T is closable we then have $\zeta = 0$. ■

We leave the proof of the following lemma to the reader.

Lemma 3.1.7. *Suppose $T : \mathcal{H} \to \mathcal{K}$, and $R, S : \mathcal{K} \to \mathcal{L}$ are densely defined operators such that ST (resp. $R+S$) is also densely defined, then $T^*S^* \sqsubseteq (ST)^*$ (resp. $R^* + S^* \sqsubseteq (R+S)^*$).*

3.1.1 The spectrum of a linear operator

Let $T : \mathcal{H} \to \mathcal{K}$ be an injective linear operator. The **inverse** of T is the linear operator $T^{-1} : \mathcal{K} \to \mathcal{H}$ with domain $D(T^{-1}) = R(T)$, such that $T^{-1}(T\xi) = \xi$, for all $\xi \in D(T^{-1})$.

The **resolvent** set of an operator $T : \mathcal{H} \to \mathcal{H}$ is

$$\rho(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is injective and } (T - \lambda)^{-1} \in \mathcal{B}(\mathcal{H})\}.$$

The **spectrum** of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

If $\sigma \in \mathcal{U}(\mathcal{H} \oplus \mathcal{K})$ is given by $\sigma(\xi \oplus \eta) = \eta \oplus \xi$, and if $T : \mathcal{H} \to \mathcal{K}$ is injective then we have that $\mathcal{G}(T^{-1}) = \sigma(\mathcal{G}(T))$. Hence, if $T : \mathcal{H} \to \mathcal{H}$ is not closed then $\sigma(T) = \mathbb{C}$. Also, note that if $T \in \mathcal{C}(\mathcal{H})$ then by the closed graph theorem shows that $\lambda \in \rho(T)$ if and only if $T - \lambda$ gives a bijection between $D(T)$ and $\mathcal{H}$.

Lemma 3.1.8. *Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ be injective with dense range, then $(T^*)^{-1} = (T^{-1})^*$. In particular, for $T \in \mathcal{C}(\mathcal{H})$ we have $\sigma(T^*) = \overline{\sigma(T)}$.*

Proof. If we consider the unitary operators J, and σ from above then we have

$$\begin{aligned} \mathcal{G}((T^*)^{-1}) &= \sigma(\mathcal{G}(T^*)) = \sigma J(\mathcal{G}(T))^{\perp} \\ &= J^*(\sigma\mathcal{G}(T))^{\perp} = J^*(\mathcal{G}(T^{-1}))^{\perp} = \mathcal{G}((T^{-1})^*). \end{aligned}$$ ■

Lemma 3.1.9. *If $T \in \mathcal{C}(\mathcal{H})$, then $\sigma(T)$ is a closed subset of $\mathbb{C}$.*

Proof. We will show that $\rho(T)$ is open by showing that whenever $\lambda \in \rho(T)$ with $|\alpha - \lambda| < \|(T - \lambda)^{-1}\|^{-1}$, then $\alpha \in \rho(T)$. Thus, suppose $\lambda \in \rho(T)$ and $\alpha \in \mathbb{C}$ such that $|\lambda - \alpha| < \|(T - \lambda)^{-1}\|^{-1}$. Then for all $\xi \in D(T)$ we have

$$\|\xi - (T - \lambda)^{-1}(T - \alpha)\xi\| = \|(T - \lambda)^{-1}(\alpha - \lambda)\xi\| < \|\xi\|.$$

Hence, by Lemma 1.1.3, $S_0 = (T - \lambda)^{-1}(T - \alpha)$, is bounded and its closure $S \in \mathcal{B}(\mathcal{H})$ has a bounded inverse. We then have $S^{-1}(T - \lambda)^{-1}(T - \alpha)\xi = \xi$, for all $\xi \in D(T)$, so that $(T - \alpha)$ is injective and $S^{-1}(T - \lambda)^{-1} \sqsupseteq (T - \alpha)^{-1}$. Note that $S_0(D(T)) = D(T)$ and hence we also have $S(D(T)) = D(T)$. Thus, $R(S^{-1}(T - \lambda)^{-1}) = D(T) = R((T - \alpha)^{-1}(\mathcal{H}))$, since both maps are injective we then have $(T - \alpha)^{-1} = S^{-1}(T - \lambda)^{-1} \in \mathcal{B}(\mathcal{H})$. ■

Note that an unbounded operator may have empty spectrum. Indeed, if $S \in \mathcal{B}(\mathcal{H})$ has a densely defined inverse, then for each $\lambda \in \sigma(S^{-1}) \setminus \{0\}$ we have $(S - \lambda^{-1})\lambda(\lambda - S)^{-1}S^{-1} = S(\lambda - S^{-1})(\lambda - S)^{-1}S^{-1} = \text{id}$. Hence $\sigma(S^{-1}) \setminus \{0\} \subset (\sigma(S) \setminus \{0\})^{-1}$. Thus, it is enough to find a bounded operator $S \in \mathcal{B}(\mathcal{H})$ such that S is injective but not surjective, and has dense range $\sigma(S) = \{0\}$. For example, the compact operator $S \in \mathcal{B}(\ell^2\mathbb{Z})$ given by $(S\delta_n) = \frac{1}{|n|+1}\delta_{n+1}$ is injective with dense range, but is not surjective, and $\|S^{2n}\| \leq 1/n!$, so that $r(S) = 0$ and hence $\sigma(S) = \{0\}$.

3.1.2 Quadratic forms

A **quadratic form** $q : \mathcal{H} \to \mathbb{C}$ on a Hilbert space $\mathcal{H}$ consists of a linear subspace $D(q) \subset \mathcal{H}$, together with a sesquilinear form $q : D(q) \times D(q) \to \mathbb{C}$. We say that q is **densely defined** if $D(q)$ is dense. If $\xi \in D(q)$ then we write $q(\xi)$ for $q(\xi, \xi)$; note that we have the polarization identity $q(\xi, \eta) = \frac{1}{4} \sum_{k=0}^{3} i^k q(\xi + i^k \eta)$, and in general, a function $q : D \to \mathcal{H}$ defines a sesquilinear from through the polarization identity if and only if it satisfies the parallelogram identity $q(\xi + \eta) + q(\xi - \eta) = 2q(\xi) + 2q(\eta)$ for all $\xi, \eta \in D(q)$. A quadratic form q is **non-negative definite** if $q(\xi) \geq 0$ for all $\xi \in \mathcal{H}$.

If q is a non-negative definite quadratic form and we denote by $\mathcal{H}_q$ the separation and completion of $D(q)$ with respect to q, then we may consider the identity map $I : D(q) \to \mathcal{H}_q$, and note that for $\xi, \eta \in D(q)$ we have $\langle \xi, \eta \rangle_q := \langle \xi, \eta \rangle + q(\xi, \eta)$ coincides with the inner-product coming from the graph of I. The quadratic form q is **closed** if I is closed, i.e., $(D(q), \langle \cdot, \cdot \rangle_q)$ is complete. We'll say that q is **closable** if I is closable, and in this case we denote by $\overline{q}$ the closed quadratic form given by $D(\overline{q}) = D(\overline{I})$, and $\overline{q}(\xi, \eta) = \langle \overline{I}\xi, \overline{I}\eta \rangle$.

Theorem 3.1.10. *Let $q : \mathcal{H} \to [0, \infty)$ be a non-negative definite quadratic form, then the following conditions are equivalent:*

(*i*) *q is closed.*

(*ii*) *There exists a Hilbert space $\mathcal{K}$, and a closed linear operator $T : \mathcal{H} \to \mathcal{K}$ with $D(T) = D(q)$ such that $q(\xi, \eta) = \langle T\xi, T\eta \rangle$ for all $\xi, \eta \in D(T)$.*

(*iii*) *q is lower semi-continuous, i.e., for any sequence $\xi_n \in D(q)$, such that $\xi_n \to \xi$, and $\liminf_{n\to\infty} q(\xi_n) < \infty$, we have $\xi \in D(q)$ and $q(\xi) \leq \liminf_{n\to\infty} q(\xi_n)$.*

Proof. The implication (i) $\implies$ (ii) follows from the discussion preceding the theorem. For (ii) $\implies$ (iii) suppose that $T : \mathcal{H} \to \mathcal{K}$ is a closed linear operator such that $D(T) = D(q)$, and $q(\xi, \eta) = \langle T\xi, T\eta \rangle$ for all $\xi, \eta \in D(T)$. If $\xi_n \in D(T)$ is a sequence such that $\xi_n \to \xi \in \mathcal{H}$, and $K = \liminf_{n\to\infty} \|T\xi_n\|^2 < \infty$, then by taking a subsequence we may assume that $K = \lim_{n\to\infty} \|T\xi_n\|^2$, and $T\xi_n \to \eta$ weakly for some $\eta \in \mathcal{H}$. Taking convex combinations we may then find a sequence ξ_n' such that $\xi_n' \to \xi \in \mathcal{H}$, $T\xi_n' \to \eta$ strongly, and $\|\eta\| = \lim_{n\to\infty} \|T\xi_n'\|^2 \leq K$. Since T is closed we then have $\xi \in D(T)$, and $T\xi = \eta$, so that $\|T\xi\|^2 \leq K$.

We show (iii) $\implies$ (i) by contraposition, so suppose that $\mathcal{H}_q$ is the separation and completion of $D(q)$ with respect to q, and that $I : D(q) \to \mathcal{H}_q$ is not closed. If I were closable, then there would exist a sequence $\xi_n \in D(q)$ such that $\xi_n \to \eta \in \mathcal{H}$, and $I(\xi_n)$ is Cauchy, but $\eta \notin D(q)$. However, if $I(\xi_n)$ is Cauchy then in particular we have that $q(\xi_n)$ is bounded, hence this sequence would show that (iii) does not hold.

Thus, we may assume that I is not closable, so that there exists a sequence $\xi_n \in D(q)$ such that $\|\xi_n\| \to 0$, and $I(\xi_n) \to \eta \neq 0$. Since, $D(q)$ is dense in $\mathcal{H}_q$ there exists $\eta_0 \in D(q)$ such that $q(\eta_0 - \eta) < q(\eta_0)$. We then have that $\eta_0 - \xi_n \to$

$\eta_0 \in \mathcal{H}$, and by the triangle inequality, $\lim_{n\to\infty} q(\eta_0 - \xi_n) = q(\eta_0 - \eta) \leq q(\eta_0)$. Thus, the sequence $\eta_0 - \xi_n$ shows that (iii) does not hold in this case also. ∎

Corollary 3.1.11. *Let $q_n : \mathcal{H} \to [0, \infty)$ be a sequence of closed non-negative definite quadratic forms, and assume that this sequence is increasing, i.e., $q_n(\xi)$ is an increasing sequence for all $\xi \in \cap_{n\in\mathbb{N}} D(q_n)$. Then there exists a closed quadratic form $q : \mathcal{H} \to [0, \infty)$ with domain*

$$D(q) = \{\xi \in \cap_{n\in\mathbb{N}} D(q_n) \mid \lim_{n\to\infty} q_n(\xi) < \infty\}$$

such that $q(\xi) = \lim_{n\to\infty} q_n(\xi)$, for all $\xi \in D(q)$.

Proof. If we define q as above then note that since each q_n satisfies the parallelogram identity then so does q, and hence q has a unique sesquilinear extension on $D(q)$. That q is closed follows easily from condition (iii) of Theorem 3.1.10. ∎

3.2 Symmetric operators and extensions

Lemma 3.2.1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a densely defined operator, then T is symmetric if and only if $\langle T\xi, \xi\rangle \in \mathbb{R}$, for all $\xi \in D(T)$.*

Proof. If T is symmetric then for all $\xi \in D(T)$ we have $\langle T\xi, \xi\rangle = \langle \xi, T\xi\rangle = \overline{\langle T\xi, \xi\rangle}$. Conversely, if $\langle T\xi, \xi\rangle = \langle \xi, T\xi\rangle$ for all $\xi \in D(T)$, then the polarization identity shows that $D(T) \subset D(T^*)$ and $T^*\xi = T\xi$ for all $\xi \in D(T)$. ∎

Proposition 3.2.2. *Let $T \in \mathcal{C}(\mathcal{H})$ be a symmetric operator, then for all $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$, we have $\ker(T - \lambda) = \{0\}$, and $R(T - \lambda)$ is closed.*

Proof. Fix $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, and set $\lambda = \alpha + i\beta$. For $\xi \in D(T)$ we have

$$\begin{aligned} \|(T-\lambda)\xi\|^2 &= \|(T-\alpha)\xi\|^2 + \|\beta\xi\|^2 - 2\mathrm{Re}(\langle (T-\alpha)\xi, i\beta\xi\rangle) \\ &= \|(T-\alpha)\xi\|^2 + \|\beta\xi\|^2 \geq \beta^2\|\xi\|^2. \end{aligned} \tag{3.1}$$

Thus, $\ker(T - \lambda) = \{0\}$, and if $\xi_n \in D(T)$ such that $(T - \lambda)\xi_n$ is Cauchy, then so is ξ_n, and hence $\xi_n \to \eta$ for some $\eta \in \mathcal{H}$. Since T is closed we have $\eta \in D(T)$ and $(T - \lambda)\eta = \lim_{n\to\infty}(T - \lambda)\xi_n$. Hence, $R(T - \lambda)$ is closed. ∎

Lemma 3.2.3. *Let $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{H}$ be two closed subspaces such that $\mathcal{K}_1 \cap \mathcal{K}_2^\perp = \{0\}$, then $\dim \mathcal{K}_1 \leq \dim \mathcal{K}_2$.*

Proof. Let P_i be the orthogonal projection onto $\mathcal{K}_i$. Then by hypothesis we have that P_2 is injective when viewed as an operator from $\mathcal{K}_1$ to $\mathcal{K}_2$, hence if we let v be the partial isometry in the polar decomposition of $P_{2|\mathcal{K}_1}$ then v is an isometry and so $\dim \mathcal{K}_1 \leq \dim \mathcal{K}_2$. ∎

Theorem 3.2.4. *If $T \in \mathcal{C}(\mathcal{H})$ is symmetric, then $\dim \ker(T^* - \lambda)$ is a constant function for $\Im\lambda > 0$, and for $\Im\lambda < 0$.*

Proof. Note that the result will follow easily if we show that for all $\lambda, \alpha \in \mathbb{C}$ such that $|\alpha-\lambda| < |\Im\lambda|/2$, then we have $\dim\ker(T^*-\lambda) = \dim\ker(T^*-\alpha)$. And this in turn follows easily if we show that for all $\lambda, \alpha \in \mathbb{C}$ such that $|\alpha - \lambda| < |\Im\lambda|$, then we have $\dim\ker(T^* - \alpha) \leq \dim\ker(T^* - \alpha)$.

Towards this end, suppose we have such $\alpha, \lambda \in \mathbb{C}$. If $\xi \in \ker(T^* - \alpha) \cap (\ker(T^* - \lambda))^\perp$ such that $\|\xi\| = 1$, then since $R(T - \overline{\lambda})$ is closed we have $\xi \in (\ker(T^* - \lambda))^\perp = R(T - \overline{\lambda})$ and so $\xi = (T - \overline{\lambda})\eta$ for some $\eta \in D(T)$. Since, $\xi \in \ker(T^* - \alpha)$ we then have

$$0 = \langle (T^* - \alpha)\xi, \eta\rangle = \langle \xi, (T - \overline{\lambda})\eta\rangle + \langle \xi, \overline{\lambda - \alpha}\eta\rangle = \|\xi\|^2 + (\lambda - \alpha)\langle \xi, \eta\rangle.$$

Hence, $1 = \|\xi\|^2 = |\lambda - \alpha||\langle \xi, \eta\rangle| < |\Im\lambda|\|\eta\|$. However, by (3.1) we have $|\Im\lambda|^2\|\eta\|^2 \leq \|(T - \overline{\lambda})\eta\|^2 = 1$, which gives a contradiction.

Thus, we conclude that $\ker(T^* - \alpha) \cap (\ker(T^* - \lambda))^\perp = \{0\}$, and hence $\dim\ker(T^* - \alpha) \leq \dim\ker(T^* - \lambda)$ by Lemma 3.2.3. ■

Corollary 3.2.5. *If $T \in \mathcal{C}(\mathcal{H})$ is symmetric, then one of the following occurs:*

(*i*) $\sigma(T) = \mathbb{C}$.

(*ii*) $\sigma(T) = \{\lambda \in \mathbb{C} \mid \Im\lambda \geq 0\}$.

(*iii*) $\sigma(T) = \{\lambda \in \mathbb{C} \mid \Im\lambda \leq 0\}$.

(*iv*) $\sigma(T) \subset \mathbb{R}$.

Proof. For $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$ then by (3.1) we have that $T - \lambda$ is injective with closed range. Thus, $\lambda \in \rho(T)$ if and only if $T - \lambda$ is surjective, or equivalently, if $T^* - \overline{\lambda}$ is injective. By the previous theorem if $T^* - \overline{\lambda}$ is injective for some λ with $\Im\lambda > 0$, then $T^* - \overline{\lambda}$ is injective for all λ with $\Im\lambda > 0$. Hence, either $\sigma(T) \subset \{\lambda \in \mathbb{C} \mid \Im\lambda \leq 0\}$ or $\{\lambda \in \mathbb{C} \mid \Im\lambda > 0\} \subset \sigma(T)$.

Since $\sigma(T)$ is closed, it is then easy to see that only one of the four possibilities can occur. ■

Theorem 3.2.6. *If $T \in \mathcal{C}(\mathcal{H})$ is symmetric, then the following are equivalent:*

(*i*) *T is self-adjoint.*

(*ii*) $\ker(T^* - i) = \ker(T^* + i) = \{0\}$.

(*iii*) $\sigma(T) \subset \mathbb{R}$.

Proof. (i) $\implies$ (ii) follows from Proposition 3.2.2, while (ii) $\Leftrightarrow$ (iii) follows from the previous corollary. To see that (ii) $\implies$ (i) suppose that $\ker(T^* - i) = \ker(T^* + i) = \{0\}$. Then by Proposition 3.2.2 we have that $R(T + i) = \ker(T^* - i)^\perp = \mathcal{H}$. Thus, $T + i$ is the only injective extension of $T + i$. Since $T^* + i$ is an injective extension of $T + i$ we conclude that $T^* + i = T + i$ and hence $T^* = T$. ■

The subspaces $\mathcal{L}_+ = \ker(T^* - i) = R(T + i)^\perp$ and $\mathcal{L}_- = \ker(T^* + i) = R(T - i)^\perp$ are the **deficiency subspaces** of the symmetric operator $T \in \mathcal{C}(\mathcal{H})$, and $n_\pm = \dim \mathcal{L}_\pm$ is the **deficiency indices**.

3.2.1 The Cayley transform

Recall from Section 2.6 that the Cayley transform $t \mapsto (t-i)(t+i)^{-1}$ and its inverse $t \mapsto i(1+t)(1-t)^{-1}$ give a bijection between self-adjoint operators $x = x^* \in \mathcal{B}(\mathcal{H})$ and unitary operators $u \in \mathcal{U}(\mathcal{H})$ such that $1 \notin \sigma(u)$. Here, we extend this correspondence to the setting of unbounded operators.

If $T \in C(\mathcal{H})$ is symmetric with deficiency subspaces $\mathcal{L}_\pm$, then the **Cayley transform** of T is the operator $U : \mathcal{H} \to \mathcal{H}$ given by $U_{|\mathcal{L}_+} = 0$, and

$$U\xi = (T-i)(T+i)^{-1}\xi$$

for all $\xi \in \mathcal{L}_+^\perp = R(T+i)$. If $\eta \in D(T)$ then by (3.1) we have that $\|(T+i)\eta\|^2 = \|T\eta\|^2 + \|\eta\|^2 = \|(T-i)\eta\|^2$, hence it follows that U is a partial isometry with initial space $\mathcal{L}_+^\perp$ and final space $\mathcal{L}_-^\perp$. Moreover, if $\xi \in D(T)$ then $(1-U)(T+i)\xi = (T+i)\xi - (T-i)\xi = 2i\xi$. Since $R(T+i) = \mathcal{L}_+^\perp$ it follows that $(1-U)(\mathcal{L}_+^\perp) = D(T)$ is dense.

Conversely, if $U \in \mathcal{B}(\mathcal{H})$ is a partial isometry with $(1-U)(U^*U\mathcal{H})$ dense, then we also have that $(1-U)$ is injective. Indeed, if $\xi \in \ker(1-U)$ then $\|\xi\| = \|U\xi\|$ so that $\xi \in UU^*\mathcal{H}$. Hence, $\xi = U^*U\xi = U^*\xi$ and so $\xi \in \ker(1-U^*) = R(1-U)^\perp = \{0\}$.

We define the **inverse Cayley transform** of U to be the densely defined operator with domain $D(T) = (1-U)(U^*U\mathcal{H})$ given by

$$T = i(1+U)(1-U)^{-1}.$$

Note that T is densely defined, and

$$\mathcal{G}(T) = \{(1-U)\xi \oplus i(1+U)\xi \mid \xi \in U^*U\mathcal{H}\}.$$

If $\xi_n \in U^*U\mathcal{H}$ such that $(1-U)\xi_n \oplus i(1+U)\xi_n$ is Cauchy, then both $(1-U)\xi_n$ and $(1+U)\xi_n$ is Cauchy and hence so is ξ_n. Thus, $\xi_n \to \xi$ for some $\xi \in U^*U\mathcal{H}$, and we have $(1-U)\xi_n \oplus i(1+U)\xi_n \to (1-U)\xi \oplus i(1+U)\xi \in \mathcal{G}(T)$. Hence, T is a closed operator.

Note also that for all $\xi, \zeta \in U^*U\mathcal{H}$ we have

$$\begin{aligned}
&\langle (1-U)\xi \oplus i(1+U)\xi, -i(1+U)\zeta \oplus (1-U)\zeta \rangle \\
&= i\langle (1-U)\xi, (1+U)\zeta \rangle + i\langle (1+U)\xi, (1-U)\zeta \rangle \\
&= 2i\langle \xi, \zeta \rangle - 2i\langle U\xi, U\zeta \rangle = 0
\end{aligned}$$

Thus, by Lemma 3.1.4 we have $\mathcal{G}(T) \subset J(\mathcal{G}(T))^\perp = \mathcal{G}(T^*)$, and hence T is symmetric.

Theorem 3.2.7. *The Cayley transform and its inverse give a bijective correspondence between densely defined closed symmetric operators $T \in \mathcal{C}(\mathcal{H})$, and partial isometries $U \in \mathcal{B}(\mathcal{H})$ such that $(1-U)(U^*U\mathcal{H})$ is dense. Moreover, self-adjoint operators correspond to unitary operators.*

Also, if $S, T \in \mathcal{C}(\mathcal{H})$ are symmetric, and $U, V \in \mathcal{B}(\mathcal{H})$ their respective Cayley transforms then we have $S \sqsubseteq T$ if and only if $U^*U\mathcal{H} \subset V^*V\mathcal{H}$ and $V\xi = U\xi$ for all $\xi \in U^*U\mathcal{H}$.

Proof. We've already seen above that the Cayley transform of a densely defined closed symmetric operator T is a partial isometry U with $(1-U)(U^*U\mathcal{H})$ dense. And conversely, the inverse Cayley transform of a partial isometry U with $(1-U)(U^*U\mathcal{H})$ dense, is a densely defined closed symmetric operator. Moreover, it is easy to see from construction that these are inverse operations.

We also see from construction that the deficiency subspaces of T are $\mathcal{L}_+ = \ker(U)$ and $\mathcal{L}_- = \ker(U^*)$ respectively. By Theorem 3.2.6 T is self-adjoint if and only if $\mathcal{L}_+ = \mathcal{L}_- = \{0\}$, which is if and only if U is a unitary.

Suppose now that $S, T \in \mathcal{C}(\mathcal{H})$ are symmetric and $U, V \in \mathcal{B}(\mathcal{H})$ are the corresponding Cayley transforms. If $S \sqsubseteq T$ then for all $\xi \in D(S) \subset D(T)$ we have $(S+i)\xi = (T+i)\xi$ and hence

$$U(S+i)\xi = (S-i)\xi = (T-i)\xi = V(S+i)\xi.$$

Therefore, $U^*U\mathcal{H} = R(S+i) \subset R(T+i) = V^*V\mathcal{H}$ and $V\xi = U\xi$ for all $\xi \in U^*U\mathcal{H}$. Conversely, if $U^*U\mathcal{H} \subset V^*V\mathcal{H}$ and $V\xi = U\xi$ for all $\xi \in U^*U\mathcal{H}$, then

$$D(S) = R((1-U)(U^*U)) = R((1-V)(U^*U)) \subset R((1-V)(V^*V)) = D(T),$$

and for all $\xi \in U^*U\mathcal{H}$ we have

$$S(1-U)\xi = i(1+U)\xi = i(1+V)\xi = T(1-V)\xi = T(1-U)\xi,$$

hence $S \sqsubseteq T$. ∎

The previous theorem in particular shows us that if $T \in \mathcal{C}(\mathcal{H})$ is a symmetric operator, and U its Cayley transform, then symmetric extensions of T are in bijective correspondence with partial isometries which extend U. Since the latter are in bijective correspondence with partial isometries from $(UU^*\mathcal{H})^\perp$ to $(U^*U\mathcal{H})^\perp$, simply translating this via the inverse Cayley transform gives the following, whose details we leave to the reader.

Theorem 3.2.8. *Let $T \in \mathcal{C}(\mathcal{H})$ be a symmetric operator, and $\mathcal{L}_\pm$ its deficiency spaces. For each partial isometry $W : \mathcal{L}_+ \to \mathcal{L}_-$, denote the operator T_W by*

$$D(T_W) = \{\xi + \eta + W\eta \mid \xi \in D(T), \eta \in W^*W(\mathcal{L}_+)\},$$

and

$$T_W(\xi + \eta + W\eta) = T\xi + i\eta - iW\eta.$$

Then T_W is a symmetric extension of T with

$$\mathcal{G}(T_W^*) = \mathcal{G}(T_W) + (\mathcal{L}_+ \ominus W^*W(\mathcal{L}_+)) + (\mathcal{L}_- \ominus WW^*(\mathcal{L}_-)).$$

Moreover, every symmetric extension arises in this way, and T_W is self-adjoint if and only if W is unitary.

Corollary 3.2.9. *If $T \in \mathcal{C}(\mathcal{H})$ is symmetric, then T has a self-adjoint extension if and only if $n_+ = n_-$.*

Exercise 3.2.10. show that for any pair $(n_+, n_-) \in (\mathbb{N} \cup \{0\} \cup \{\infty\})^2$ there exists a densely defined closed symmetric operator $T \in \mathcal{C}(\ell^2\mathbb{N})$ such that n_+ and n_- are the deficiency indices for T.

3.3 Functional calculus for normal operators

3.3.1 Positive operators

Theorem 3.3.1. *Suppose $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, then*

(i) T^*T is densely defined and $\sigma(T^*T) \subset [0, \infty)$.

(ii) T^*T is self-adjoint.

(iii) $D(T^*T)$ is a core for T.

Proof. We first show that $1+T^*T$ is onto and injective. Since $\mathcal{K} \oplus \mathcal{H} \cong J\mathcal{G}(T) \oplus \mathcal{G}(T^*)$, if $\xi \in \mathcal{H}$ then there exists $\eta \in \mathcal{H}$, $\zeta \in \mathcal{K}$ such that

$$0 \oplus \xi = -T\eta \oplus \eta + \zeta \oplus T^*\zeta.$$

Hence, $\zeta = T\eta$ and $\xi = \eta + T^*\zeta = (1 + T^*T)\eta$, showing that $(1 + T^*T)$ is onto.

If $\xi \in D(T^*T)$ then

$$\|\xi + T^*T\xi\|^2 = \|\xi\|^2 + 2\|T\xi\|^2 + \|T^*T\xi\|^2.$$

Hence, we see that $1 + T^*T$ is injective.

To see that T^*T is densely defined suppose that $\xi \in (D(T^*T))^\perp$. Since $(1+T^*T)$ is onto we can write $\xi = \eta+T^*T\eta$ for some $\eta \in \mathcal{H}$. For all $\zeta \in D(T^*T)$ we then have

$$0 = \langle (1 + T^*T)\eta, \zeta\rangle = \langle \eta, (1 + T^*T)\zeta\rangle.$$

Since $(1 + T^*T)$ is onto we then have $\eta = 0$ and hence $\xi = 0$.

Thus, T^*T is densely defined and by multiplying by scalars we see that $(-\infty, 0) \subset \rho(T)$. If $\xi = (1 + T^*T)\eta$ for $\eta \in D(T^*T)$ then we have

$$\langle (1 + T^*T)^{-1}\xi, \xi\rangle = \langle \eta, (1 + T^*T)\eta\rangle = \|\eta\|^2 + \|T\eta\|^2 \geq 0.$$

Thus $(1+T^*T)^{-1} \geq 0$ and hence it follows from Lemma 3.1.8 that $1+T^*T$ and hence also T^*T is self-adjoint. By Theorem 3.2.6 this shows that $\sigma(T^*T) \subset \mathbb{R}$, and hence $\sigma(T^*T) \subset [0, \infty)$.

Finally, to see that $D(T^*T)$ is a core for T consider $\xi \oplus T\xi \in \mathcal{G}(T)$ such that $\xi \oplus T\xi \perp \{\eta \oplus T\eta \mid \eta \in D(T^*T)\}$. Then for all $\eta \in D(T^*T)$ we have

$$0 = \langle \xi \oplus T\xi, \eta \oplus T\eta\rangle = \langle \xi, \eta\rangle + \langle T\xi, T\eta\rangle = \langle \xi, (1 + T^*T)\eta\rangle.$$

Since $(1 + T^*T)$ is onto, this shows that $\xi = 0$. ■

An operator $T \in \mathcal{C}(\mathcal{H})$ is **positive** if $T = S^*S$ for some densely defined closed operator $S : \mathcal{H} \to \mathcal{H}$.

3.3.2 Borel functional calculus

Suppose K is a locally compact Hausdorff space, and E is a spectral measure on K relative to $\mathcal{H}$. We let $B(K)$ denote the space of Borel functions on K. For each $f \in B(K)$ we define a linear operator $T = \int f \, \mathrm{d}E$ by setting

$$D(T) = \{\xi \in \mathcal{H} \mid \eta \mapsto \int f \, \mathrm{d}E_{\xi,\eta} \text{ is bounded.}\},$$

and letting $T\xi$ be the unique vector such that $\int f \, \mathrm{d}E_{\xi,\eta} = \langle T\xi, \eta \rangle$, for all $\eta \in \mathcal{H}$.

If $B \subset K$ is any Borel set such that $f_{|B}$ is bounded, then we have for all $\xi, \eta \in \mathcal{H}$ we have that $1_B E_{\xi,\eta} = E_{E(B)\xi,\eta}$ and hence $|\int f \, \mathrm{d}E_{E(B)\xi,\eta}| = |\int f_{|B} \, \mathrm{d}E_{\xi,\eta}| \leq \|f_{|B}\|_\infty \|\xi\| \|\eta\|$, and so $E(B)\mathcal{H} \subset D(T)$. Taking $B_n = \{x \in K \mid |f(x)| \leq n\}$ we then have that $\cup_{n \in \mathbb{N}} E(B_n)\mathcal{H} \subset D(T)$ and this is dense since $E(B_n)$ converges strongly to 1. Thus T is densely defined.

If $S = \int \overline{f} \, \mathrm{d}E$, then for all $\xi \in D(T)$ and $\eta \in D(S)$ we have

$$\langle T\xi, \eta \rangle = \int f \, \mathrm{d}E_{\xi,\eta} = \overline{\int \overline{f} \, \mathrm{d}E_{\eta,\xi}} = \overline{\langle S\eta, \xi \rangle} = \langle \xi, S\eta \rangle.$$

A similar argument shows that $D(T^*) \subset D(S)$, so that in fact we have $S = T^*$ and $T^* = S$. In particular, T is a closed operator, and is self-adjoint if f is real valued. It is equally easy to see that $T^*T = TT^* = \int |f|^2 \, \mathrm{d}E$.

If $T : \mathcal{H} \to \mathcal{K}$ is a closed operator, then a subspace $D \subset D(T)$ is a **core** for T if $\mathcal{G}(T) = \overline{\mathcal{G}(T_{|D})}$.

It is easy to see that if $f, g \in B(K)$ then $\int f \, \mathrm{d}E + \int g \, \mathrm{d}E \sqsubseteq \int (f+g) \, \mathrm{d}E$, and $(\int f \, \mathrm{d}E)(\int g \, \mathrm{d}E) \sqsubseteq \int fg \, \mathrm{d}E$, and in both cases the domains on the left are cores for the operators on the right. A similar result holds when considering more than two functions. In particular, on the set of all operators of the form $\int f \, \mathrm{d}E$ we may consider the operations $\hat{+}$, and $\hat{\circ}$ given by $S \hat{+} T = \overline{S + T}$, and $S \hat{\circ} T = \overline{S \circ T}$, and under these operations we have that $f \mapsto \int f \, \mathrm{d}E$ is a unital $*$-homomorphism from $B(K)$ into $\mathcal{C}(\mathcal{H})$.

We also note that $\sigma(\int f \, \mathrm{d}E)$ is contained in the range of f, for each $f \in B(K)$.

An operator $T \in \mathcal{C}(\mathcal{H})$ is **normal** if $T^*T = TT^*$. Note that equality here implies also $D(T^*T) = D(TT^*)$. We would like to associate a spectral measure for each normal operator as we did for bounded normal operators. However, our approach for bounded operators, Theorem 2.7.5, does not immediately apply since we used there that a bounded normal operator generated an abelian C^*-algebra. Our approach therefore will be to reduce the problem to the case of bounded operators.

Lemma 3.3.2. *Suppose $T \in \mathcal{C}(\mathcal{H})$, then $R = T(1 + T^*T)^{-1}$ and $S = (1 + T^*T)^{-1}$ are bounded contractions. If T is normal then we have $SR = RS$.*

Proof. If $\xi \in \mathcal{H}$, fix $\eta \in D(T^*T)$ such that $(1 + T^*T)\eta = \xi$. Then

$$\|\xi\|^2 = \|(1 + T^*T)\eta\|^2 = \|\eta\|^2 + 2\|T\eta\|^2 + \|T^*T\eta\|^2 \geq \|\eta\|^2 = \|(1 + T^*T)^{-1}\xi\|^2.$$

Hence $\|S\| \leq 1$. Similarly, $\|\xi\|^2 \geq \|T\eta\|^2 = \|R\xi\|^2$, hence also $\|R\| \leq 1$.

Suppose now that T is normal and $\xi \in D(T)$. Since $\eta \in D(T^*T)$ and $\xi = (1 + T^*T)\eta \in D(T)$ we have that $T\eta \in D(TT^*) = D(T^*T)$. Hence, $T\xi = T(1 + T^*T)\eta = (1 + TT^*)T\eta = (1 + T^*T)T\eta$. Thus, $ST\xi = TS\xi$ for all $\xi \in D(T)$.

Suppose now that $\xi \in \mathcal{H}$ is arbitrary. Since $\eta \in D(T^*T) \subset D(T)$, we have $SR\xi = ST\eta = TS\eta = RS\xi$. ■

Theorem 3.3.3. *Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator, then $\sigma(T) \neq \emptyset$ and there exists a unique spectral measure E for $\sigma(T)$ relative to $\mathcal{H}$ such that*

$$T = \int t \, \mathrm{d}E(t).$$

Proof. Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator. For each $n \in \mathbb{N}$ we denote by $P_n = 1_{(\frac{1}{n+1}, \frac{1}{n}]}(S)$, where $S = (1 + T^*T)^{-1}$. Notice that since S is a positive contraction which is injective, we have that P_n are pairwise orthogonal projections and $\sum_{n \in \mathbb{N}} P_n = 1$, where the convergence of the sum is in the strong operator topology. Note, also that if $\mathcal{H}_n = R(P_n)$ then we have $B\mathcal{H}_n = \mathcal{H}_n$ and restricting B to $\mathcal{H}_n$ we have that $\frac{1}{n+1} \leq B_{|\mathcal{H}_n} \leq \frac{1}{n}$. In particular, we have that $\mathcal{H}_n \subset R(B) = D(T^*T)$, $(1+T^*T)$ maps $\mathcal{H}_n$ onto itself for each $n \in \mathbb{N}$, and $\sigma((1 + T^*T)_{|\mathcal{H}_n}) \subset \{\lambda \in \mathbb{C} \mid n \leq |\lambda| \leq n + 1\}$.

By Lemma 3.3.2 $C = T(1 + T^*T)^{-1}$ commutes with B and since B is self-adjoint we then have that C commutes with any of the spectral projections P_n. Since we've already established that $(1 + T^*T)$ give a bijection on $\mathcal{H}_n$ it then follows that $T\mathcal{H}_n \subset \mathcal{H}_n$ for all $n \in \mathbb{N}$. Note that since T is normal, by symmetry we also have that $T^*\mathcal{H}_n \subset \mathcal{H}_n$ for all $n \in \mathbb{N}$. Hence $(TP_n)^*(TP_n) = P_n(T^*T)P_n = P_n(TT^*)P_n = (TP_n)(TP_n)^*$ for all $n \in \mathbb{N}$.

Let $I = \{n \in \mathbb{N} \mid P_n \neq 0\}$, and note that $I \neq \emptyset$ since $\sum_{n \in I} P_n = 1$. For $n \in I$, restricting to $\mathcal{H}_n$, we have that TP_n is a bounded normal operator with spectrum $\sigma(TP_n) \subset \{\lambda \in \mathbb{C} \mid n - 1 \leq |\lambda| \leq n\}$. Let E_n denote the unique spectral measure on $\sigma(TP_n)$ so that $T_{|\mathcal{H}_n} = \int t \, \mathrm{d}E_n(t)$.

We let E be the spectral measure on $\overline{\cup_{n \in I}\sigma(TP_n)} = \cup_{n \in I}\sigma(TP_n)$ which is given by $E(F) = \sum_{n \in I} E_n(F)$ for each Borel subset $F \subset \cup_{n \in I}\sigma(TP_n)$. Since the $E_n(F)$ are pairwise orthogonal it is easy to see that E is indeed a spectral measure. We set $\tilde{T}$ to be the operator $\tilde{T} = \int t \, \mathrm{d}E(t)$.

We claim that $\tilde{T} = T$. To see this, first note that if $\xi \in \mathcal{H}_n$ then $\tilde{T}\xi = TP_n\xi = T\xi$. Hence, $\tilde{T}$ and T agree on $\mathcal{K}_0 = \cup_{n \in I}\mathcal{H}_n$. Since both operators are closed, and since $\mathcal{K}_0$ is clearly a core for $\tilde{T}$, to see that they are equal it is then enough to show that $\mathcal{K}_0$ is also core for T. If we suppose that $\xi \in D(T^*T)$, and write $\xi_n = P_n\xi$ for $n \in I$, then $\lim_{N \to \infty} \sum_{n \leq N} \xi_n = \xi$, and setting $\eta =$

$(1 + T^*T)\xi$ we have

$$\begin{aligned}\sum_{n\in I} \|T\xi_n\|^2 &= \sum_{n\in I} \langle T^*T\xi_n, \xi_n\rangle \\ &= -\|\xi\|^2 + \sum_{n\in I} \langle (1 + T^*T)\xi_n, \xi_n\rangle \\ &\leq \|\xi\|\|\eta\| < \infty.\end{aligned}$$

Since T is closed we therefore have $\lim_{N\to\infty} T(\sum_{n\leq N} \xi_n) = T\xi$. Thus, $\overline{\mathcal{G}(T_{|\mathcal{K}_0})} = \overline{\mathcal{G}(T_{|D(T^*T)})} = \mathcal{G}(T)$.

Since $\sigma(T) = \sigma(\tilde{T}) = \cup_{n\in I}\sigma(TP_n)$, this completes the existence part of the proof. For the uniqueness part, if $\tilde{E}$ is a spectral measure on $\sigma(T)$ such that $T = \int t\,\mathrm{d}\tilde{E}(t)$ then by uniqueness of the spectral measure for bounded normal operators it follows that for every $F \subset \sigma(T)$ Borel, and $n \in I$, we have $P_nE(F) = P_n\tilde{E}(F)$, and hence $E = \tilde{E}$. ■

If $T = \int t\,\mathrm{d}E(t)$ as above, then for any $f \in B(\sigma(T))$ we define $f(T)$ to be the operator $f(T) = \int f(t)\,\mathrm{d}E(t)$.

Corollary 3.3.4. *Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator. Then for any $*$-polynomial $p \in \mathbb{C}[t, t^*]$ we have that $p(T)$ is densely defined and closable, and in fact $D(p(T))$ is a core for T.*

Proposition 3.3.5. *Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator, and consider the abelian von Neumann algebra $W^*(T) = \{f(T) \mid f \in B_\infty(\sigma(T))\}'' \subset \mathcal{B}(\mathcal{H})$. If $u \in \mathcal{U}(\mathcal{H})$, then $u \in \mathcal{U}(W^*(T)')$ if and only if $uTu^* = T$.*

Proof. Suppose that $u \in \mathcal{U}(\mathcal{H})$ and $T \in \mathcal{C}(\mathcal{H})$ is normal. We let E be the spectral measure on $\sigma(T)$ such that $T = \int t\,\mathrm{d}E(t)$, and consider the spectral measure $\tilde{E}$ given by $\tilde{E}(F) = uE(F)u^*$ for all $F \subset \sigma(T)$ Borel. We then clearly have $uTu^* = \int t\,\mathrm{d}\tilde{E}(t)$ from which the result follows easily. ■

If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $T : \mathcal{H} \to \mathcal{H}$ is a linear operator, then we say that T is **affiliated with** M and write $T\eta M$ if $uTu^* = T$ for all $u \in \mathcal{U}(M')$, (note that this implies $uD(T) = D(T)$ for all $u \in U(M')$). The previous proposition shows that any normal linear operator is affiliated with an abelian von Neumann algebra.

Corollary 3.3.6. *If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $T \in \mathcal{C}(\mathcal{H})$ is normal, then $T\eta M$ if and only if $f(T) \in M$ for all $f \in B_\infty(\sigma(T))$.*

Proposition 3.3.7. *Suppose M is a von Neumann algebra and $T, S : \mathcal{H} \to \mathcal{H}$ are linear operators such that $T, S\eta M$. Then $TS, (T + S)\eta M$. Moreover, if T is densely defined then $T^*\eta M$, and if S is closable then $\overline{S}\eta M$.*

Proof. Since $T, S\eta M$, for all $u \in \mathcal{U}(M')$ we have

$$\begin{aligned}uD(TS) &= \{\xi \in \mathcal{H} \mid u^*\xi \in D(S), S(u^*\xi) \in D(T)\} \\ &= \{\xi \in \mathcal{H} \mid \xi \in D(S), u^*S\xi \in D(T)\} \\ &= \{\xi \in \mathcal{H} \mid \xi \in D(S), S\xi \in D(T)\} = D(TS).\end{aligned}$$

Also, for $\xi \in D(TS)$ we have $u^*TSu\xi = (u^*Tu)(u^*Su)\xi = TS\xi$, hence $TS\eta M$. The proof that $(T+S)\eta M$ is similar.

If T is densely defined, then for all $u \in \mathcal{U}(M')$ we have

$$\begin{aligned} uD(T^*) &= \{\xi \in \mathcal{H} \mid \eta \mapsto \langle T\eta, u^*\xi\rangle \text{ is bounded.}\} \\ &= \{\xi \in \mathcal{H} \mid \eta \mapsto \langle T(u\eta), \xi\rangle \text{ is bounded.}\} = D(T^*), \end{aligned}$$

and for $\xi \in D(T^*)$, and $\eta \in D(T)$ we have $\langle T\eta, u^*\xi\rangle = \langle Tu, \xi\rangle$, from which it follows that $T^*u^*\eta = u^*T^*\eta$, and hence $T^*\eta M$.

If S is closable, then in particular we have that $u\overline{D(S)} = \overline{D(S)}$ for all $u \in \mathcal{U}(M')$. Hence if p denotes the orthogonal projection onto $\overline{D(S)}$ then $p \in M'' = M$, and $S\eta pMp \subset \mathcal{B}(p\mathcal{H})$. Hence, we may assume that S is densely defined in which case we have $S\eta M \implies S^*\eta M \implies \overline{S} = S^{**}\eta M$. ∎

3.3.3 Polar decomposition

For $T \in \mathcal{C}(\mathcal{H})$ the **absolute value** of T is the positive operator $|T| = \sqrt{T^*T} \in \mathcal{C}(\mathcal{H})$.

Theorem 3.3.8 (Polar decomposition). *Fix $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$. Then $D(|T|) = D(T)$, and there exists a unique partial isometry $v \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that* $\ker(v) = \ker(T) = \ker(|T|)$*, and $T = v|T|$.*

Proof. By Theorem 3.3.1 we have that $D(T^*T)$ is a core for both $|T|$ and T. We define the map $V_0 : \mathcal{G}(|T|_{|D(T^*T)}) \to \mathcal{G}(T)$ by $V_0(\xi \oplus |T|\xi) = \xi \oplus T\xi$. Since, for $\xi \in D(T^*T)$ we have $\|\xi\|^2 + \||T|\xi\|^2 = \|\xi\|^2 + \|T\xi\|^2$ this shows that V_0 is isometric, and since $D(T^*T)$ is a core for both $|T|$ and T we then have that V_0 extends to an isometry from $\mathcal{G}(|T|)$ onto $\mathcal{G}(T)$, and we have $D(|T|) = P_{\mathcal{H}}(\mathcal{G}(|T|)) = P_{\mathcal{H}}(V\mathcal{G}(|T|)) = P_{\mathcal{H}}(\mathcal{G}(T)) = D(T)$.

Moreover, this also shows that the map $v_0 : R(|T|) \to R(T)$ given by $v_0(|T|\xi) = T\xi$, is well defined and extends to a partial isometry $v \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\ker(v) = R(T)^\perp = \ker(T)$. From the definition of v we clearly have that $T = v|T|$. Uniqueness follows from the fact that any other partial isometry w which satisfies $T = w|T|$ must agree with v on $\overline{R(|T|)} = \ker(|T|)^\perp = \ker(T)^\perp$. ∎

Proposition 3.3.9. *If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $T \in \mathcal{C}(\mathcal{H})$ has polar decomposition $T = v|T|$, then $T\eta M$ if and only if $v \in M$ and $|T|\eta M$.*

Proof. If $T\eta M$, then $T^*T\eta M$ by Proposition 3.3.7. By Corollary 3.3.6 we then have that $|T|\eta M$. Hence, for any $u \in M'$ if $\xi \in R(|T|)$ say $\xi = |T|\eta$ for $\eta \in D(|T|) = D(T)$, then $uv\xi = uv|T|\eta = uT\eta = Tu\eta = v|T|u\eta = vu\xi$, hence $v \in M'' = M$.

Conversely, if $v \in M$ and $|T|\eta M$, then $T = (v|T|)\eta M$ by Proposition 3.3.7. ∎

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