Notes on von Neumann algebras

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Chapter 1

Spectral theory

If A is a complex unital algebra then we denote by $G(A)$ the set of elements which have a two sided inverse. If $x \in A$, the **spectrum** of x is

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \notin G(A)\}.$$

The complement of the spectrum is called the **resolvent** and denoted $\rho_A(x)$.

Proposition 1.0.1. *Let A be a unital algebra over $\mathbb{C}$, and consider $x, y \in A$. Then $\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}$.*

Proof. If $1 - xy \in G(A)$ then we have

$$\begin{aligned}(1 - yx)(1 + y(1 - xy)^{-1}x) &= 1 - yx + y(1 - xy)^{-1}x - yxy(1 - xy)^{-1}x \\ &= 1 - yx + y(1 - xy)(1 - xy)^{-1}x = 1.\end{aligned}$$

Similarly, we have

$$(1 + y(1 - xy)^{-1}x)(1 - yx) = 1,$$

and hence $1 - yx \in G(A)$. ■

Knowing the formula for the inverse beforehand of course made the proof of the previous proposition quite a bit easier. But this formula is quite natural to consider. Indeed, if we just consider formal power series then we have

$$(1 - yx)^{-1} = \sum_{k=0}^{\infty}(yx)^k = 1 + y(\sum_{k=0}^{\infty}(xy)^k)x = 1 + y(1 - xy)^{-1}x.$$

1.1 Banach and C^*-algebras

A **Banach algebra** is a Banach space A, which is also an algebra such that

$$\|xy\| \leq \|x\|\|y\|.$$

A Banach algebra A is **involutive** if it possesses an anti-linear involution $*$, such that $\|x^*\| = \|x\|$, for all $x \in A$.

If an involutive Banach algebra A additionally satisfies

$$\|x^*x\| = \|x\|^2,$$

for all $x \in A$, then we say that A is a **C^*-algebra**. If a Banach or C^*-algebra is unital, then we further require $\|1\| = 1$.

Note that if A is a unital involutive Banach algebra, and $x \in G(A)$ then $(x^{-1})^* = (x^*)^{-1}$, and hence $\sigma_A(x^*) = \overline{\sigma_A(x)}$.

Example 1.1.1. Let K be a locally compact Hausdorff space. Then the space $C_0(K)$ of complex valued continuous functions which vanish at infinity is a C^*-algebra when given the supremum norm $\|f\|_\infty = \sup_{x \in K} |f(x)|$. This is unital if and only if K is compact.

Example 1.1.2. Let $\mathcal{H}$ be a complex Hilbert space. Then the space of all bounded operators $\mathcal{B}(\mathcal{H})$ is a C^*-algebra when endowed with the operator norm $\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|$.

Lemma 1.1.3. *Let A be a unital Banach algebra and suppose $x \in A$ such that $\|1 - x\| < 1$, then $x \in G(A)$.*

Proof. Since $\|1 - x\| < 1$, the element $y = \sum_{k=0}^\infty (1 - x)^k$ is well defined, and it is easy to see that $xy = yx = 1$. ■

Proposition 1.1.4. *Let A be a unital Banach algebra, then $G(A)$ is open, and the map $x \mapsto x^{-1}$ is a continuous map on $G(A)$.*

Proof. If $y \in G(A)$ and $\|x - y\| < \|y^{-1}\|$ then $\|1 - xy^{-1}\| < 1$ and hence by the previous lemma $xy^{-1} \in G(A)$ (hence also $x = xy^{-1}y \in G(A)$) and

$$\begin{aligned}
\|xy^{-1}\| &\leq \sum_{n=0}^\infty \|(1 - xy^{-1})\|^n \\
&\leq \sum_{n=0}^\infty \|y^{-1}\|^n \|y - x\|^n = \frac{1}{1 - \|y\|^{-1}\|y - x\|}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x^{-1} - y^{-1}\| &= \|x^{-1}(y - x)y^{-1}\| \\
&\leq \|y^{-1}(xy^{-1})^{-1}\|\|y^{-1}\|\|y - x\| \leq \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\|\|y - x\|}\|y - x\|.
\end{aligned}$$

Thus continuity follows from continuity of the map $t \mapsto \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\| t} t$, at $t = 0$. ■

Proposition 1.1.5. *Let A be a unital Banach algebra, and suppose $x \in A$, then $\sigma_A(x)$ is a non-empty compact set.*

Proof. If $\|x\| < |\lambda|$ then $\frac{x}{\lambda} - 1 \in G(A)$ by Lemma 1.1.3, also $\sigma_A(x)$ is closed by Proposition 1.1.4, thus $\sigma_A(x)$ is compact.

To see that $\sigma_A(x)$ is non-empty note that for any linear functional $\varphi \in A^*$, we have that $f(z) = \varphi((x-z)^{-1})$ is analytic on $\rho_A(x)$. Indeed, if $z, z_0 \in \rho_A(x)$ then we have

$$(x-z)^{-1} - (x-z_0)^{-1} = (x-z)^{-1}(z-z_0)(x-z_0)^{-1}.$$

Since inversion is continuous it then follows that

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \varphi((x-z_0)^{-2}).$$

We also have $\lim_{z\to\infty} f(z) = 0$, and hence if $\sigma_A(x)$ were empty then f would be a bounded entire function and we would then have $f = 0$. Since $\varphi \in A^*$ were arbitrary this would then contradict the Hahn-Banach theorem. ■

Theorem 1.1.6 (Gelfand-Mazur)**.** *Suppose A is a unital Banach algebra such that every non-zero element is invertible, then $A \cong \mathbb{C}$.*

Proof. Fix $x \in A$, and take $\lambda \in \sigma(x)$. Since $x - \lambda$ is not invertible we have that $x - \lambda = 0$, and the result then follows. ■

If $f(z) = \sum_{k=0}^n a_k z^k$ is a polynomial, and $x \in A$, a unital Banach algebra, then we define $f(x) = \sum_{k=0}^n a_k x^k \in A$.

Proposition 1.1.7. *Let A be a unital Banach algebra, $x \in A$ and f a polynomial. then $\sigma_A(f(x)) = f(\sigma_A(x))$.*

Proof. If $\lambda \in \sigma_A(x)$, and $f(z) = \sum_{k=0}^n a_k z^k$ then

$$\begin{aligned} f(x) - f(\lambda) &= \sum_{k=1}^n a_k(x^k - \lambda^k) \\ &= (x-\lambda)\sum_{k=1} a_k \sum_{j=0}^{k-1} x^j \lambda^{k-j-1}, \end{aligned}$$

hence $f(\lambda) \in \sigma_A(x)$. conversely if $\mu \notin f(\sigma_A(x))$ and we factor $f - \mu$ as

$$f - \mu = \alpha_n(x-\lambda_1)\cdots(x-\lambda_n),$$

then since $f(\lambda) - \mu \neq 0$, for all $\lambda \in \sigma_A(x)$ it follows that $\lambda_i \notin \sigma_A(x)$, for $1 \leq i \leq n$, hence $f(x) - \mu \in G(A)$. ■

If A is a unital Banach algebra and $x \in A$, the **spectral radius** of x is

$$r(x) = \sup_{\lambda\in\sigma_A(x)} |\lambda|.$$

Note that by Proposition 1.1.5 the spectral radius is finite, and the supremum is attained. Also note that by Proposition 1.0.1 we have the very useful equality $r(xy) = r(yx)$ for all x and y in a unital Banach algebra A. A priori the spectral radius depends on the Banach algebra in which x lives, but we will show now that this is not the case.

Proposition 1.1.8. *Let A be a unital Banach algebra, and suppose $x \in A$. Then $\lim_{n\to\infty} \|x^n\|^{1/n}$ exists and we have*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

Proof. By Proposition 1.1.7 we have $r(x^n) = r(x)^n$, and hence

$$r(x)^n \leq \|x^n\|,$$

showing that $r(x) \leq \liminf_{n\to\infty} \|x^n\|^{1/n}$.

To show that $r(x) \geq \limsup_{n\to\infty} \|x^n\|^{1/n}$, consider the domain $\Omega = \{z \in \mathbb{C} \mid |z| > r(x)\}$, and fix a linear functional $\varphi \in A^*$. We showed in Proposition 1.1.5 that $z \mapsto \varphi((x-z)^{-1})$ is analytic in Ω and as such we have a Laurent expansion

$$\varphi((z-x)^{-1}) = \sum_{n=0}^{\infty} \frac{a_n}{z^n},$$

for $|z| > r(x)$. However, we also know that for $|z| > \|x\|$ we have

$$\varphi((z-x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n}.$$

By uniqueness of the Laurent expansion we then have that

$$\varphi((z-x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n},$$

for $|z| > r(x)$.

Hence for $|z| > r(x)$ we have that $\lim_{n\to\infty} \frac{\varphi(x^{n-1})}{|z|^n} = 0$, for all linear functionals $\varphi \in A^*$. By the uniform boundedness principle we then have $\lim_{n\to\infty} \frac{\|x^{n-1}\|}{|z|^n} = 0$, hence $|z| > \limsup_{n\to\infty} \|x^n\|^{1/n}$, and thus

$$r(x) \geq \limsup_{n\to\infty} \|x^n\|^{1/n}. \qquad \blacksquare$$

Exercise 1.1.9. Suppose A is a unital Banach algebra, and $I \subset A$ is a closed two sided ideal, then A/I is again a unital Banach algebra, when given the norm $\|a + I\| = \inf_{y\in I} \|a + y\|$, and $(a + I)(b + I) = (ab + I)$.

Exercise 1.1.10. Let A be a unital Banach algebra and suppose $x, y \in A$ such that $xy = yx$. Show that $r(xy) \leq r(x)r(y)$.

1.2 The Gelfand transform

Let A be a abelian Banach algebra, the **spectrum** of A, denoted by $\sigma(A)$, is the set of continuous $*$-homomorphsims $\varphi : A \to \mathbb{C}$ such that $\|\varphi\| = 1$, which we endow with the weak*-topology as a subset of A^*.

Note that if A is unital, and $\varphi : A \to \mathbb{C}$ is a $*$-homomorphism, then it follows easily that $\ker(\varphi) \cap G(A) = \emptyset$. In particular, this shows that $\varphi(x) \in \sigma(x)$, since $x - \varphi(x) \in \ker(\varphi)$. Hence, for all $x \in A$ we have $|\varphi(x)| \leq r(x) \leq \|x\|$. Since, $\varphi(1) = 1$ this shows that the condition $\|\varphi\| = 1$ is automatic in the unital case.

It is also easy to see that when A is unital $\sigma(A)$ is closed and bounded, by the Banach-Alaoglu theorem it is then a compact Hausdorff space.

Proposition 1.2.1. *Let A be a unital Banach algebra. Then the association $\varphi \mapsto \ker(\varphi)$ gives a bijection between the spectrum of A and the space of maximal ideals.*

Proof. If $\varphi \in \sigma(A)$ then $\ker(\varphi)$ is clearly an ideal, and if we have a larger ideal I, then there exists $x \in I$ such that $\varphi(x) \neq 0$, hence $1 - x/\varphi(x) \in \ker(\varphi) \subset I$ and so $1 = (1 - x/\varphi(x)) + x/\varphi(x) \in I$ which implies $I = A$.

Conversely, if $I \subset A$ is a maximal ideal, then $I \cap G(A) = \emptyset$ and hence $\|1 - y\| \geq 1$ for all $y \in I$. Thus, $\overline{I}$ is also an ideal and $1 \notin I$ which shows that $I = \overline{I}$ by maximality. We then have that A/I is a unital Banach algebra, and since I is maximal we have that all non-zero elements of A/I are invertible. Thus, by the Gelfand-Mazur theorem we have $A/I \cong \mathbb{C}$ and hence the projection map $\pi : A \to A/I \cong \mathbb{C}$ gives a continuous homomorphism with kernel I. ■

Suppose A is a unital C^*-algebra which is generated (as a unital C^*-algebra) by a single element x, if $\lambda \in \sigma_A(x)$ then we can consider the closed ideal generated by $x - \lambda$ which is maximal since x generates A. This therefore induces a map from $\sigma_A(x)$ to $\sigma(A)$. We leave it to the reader to check that this map is actually a homeomorphism.

Let A be a unital abelian Banach algebra, the **Gelfand transform** is the map $\Gamma : A \to C(\sigma(A))$ defined by

$$\Gamma(x)(\varphi) = \varphi(x).$$

Theorem 1.2.2. *Let A be a unital abelian Banach algebra, then the Gelfand transform is a contractive homomorphism, and $\Gamma(x)$ is invertible in $C(\sigma(A))$ if and only if x is invertible in A.*

Proof. It is easy to see that the Gelfand transform is a contractive homomorphism. Also, if $x \in G(A)$, then $\Gamma(a)\Gamma(a^{-1}) = \Gamma(aa^{-1}) = \Gamma(1) = 1$, hence $\Gamma(x)$ is invertible. Conversely, if $x \notin G(A)$ then since A is abelian we have that the ideal generated by x is non-trivial, hence by Zorn's lemma we see that x is contained in a maximal ideal $I \subset A$, and from Proposition 1.2.1 there exists $\varphi \in \sigma(A)$ such that $\Gamma(x)(\varphi) = \varphi(x) = 0$. Hence, in this case $\Gamma(x)$ is not invertible. ■

Corollary 1.2.3. *Let A be a unital abelian Banach algebra, then $\sigma(\Gamma(x)) = \sigma(x)$, and in particular $\|\Gamma(x)\| = r(\Gamma(x)) = r(x)$, for all $x \in A$.*

1.3 Continuous functional calculus

Let A be a C^*-algebra. An element $x \in A$ is:

- **normal** if $xx^* = x^*x$.
- **self-adjoint** if $x = x^*$, and **skew-adjoint** if $x = -x^*$.
- **positive** if $x = y^*y$ for some $y \in A$.
- a **projection** if $x^* = x^2 = x$.
- **unitary** if A is unital, and $x^*x = xx^* = 1$.
- **isometric** if A is unital, and $x^*x = 1$.
- **partially isometric** if x^*x is a projection.

We denote by A_+ the set of positive elements, and $a, b \in A$ are two self-adjoint elements then we write $a \leq b$ if $b - a \in A_+$. Note that if $x \in A$ then $x^*A_+x \subset A_+$, in particular, if $a, b \in A$ are self-adjoint such that $a \leq b$, then $x^*ax \leq x^*bx$.

Proposition 1.3.1. *Let A be a C^*-algebra and $x \in A$ normal, then $\|x\| = r(x)$.*

Proof. We first show this if x is self-adjoint, in which case we have $\|x^2\| = \|x\|^2$, and by induction we have $\|x^{2^n}\| = \|x\|^{2^n}$ for all $n \in \mathbb{N}$. Therefore, $\|x\| = \lim_{n\to\infty} \|x^{2^n}\|^{2^n} = r(x)$.

If x is normal then by Exercise 1.1.10 we have

$$\|x\|^2 = \|x^*x\| = r(x^*x) \leq r(x^*)r(x) = r(x)^2 \leq \|x\|^2. \qquad \blacksquare$$

Corollary 1.3.2. *Let A and B be two unital C^*-algebras and $\Phi : A \to B$ a unital $*$-homomorphism, then Φ is contractive. If Φ is a $*$-isomorphism, then Φ is isometric.*

Proof. Since Φ is a unital $*$-homomorphism we clearly have $\Phi(G(A)) \subset G(B)$, from which it follows that $\sigma_B(\Phi(x)) \subset \sigma_A(x)$, and hence $r(\Phi(x)) \leq r(x)$, for all $x \in A$. By Proposition 1.3.1 we then have

$$\|\Phi(x)\|^2 = \|\Phi(x^*x)\| = r(\Phi(x^*x)) \leq r(x^*x) = \|x^*x\| = \|x\|^2.$$

If Φ is a $*$-isomorphism then so is Φ^{-1} which then shows that Φ is isometric. $\blacksquare$

Corollary 1.3.3. *Let A be a unital complex involutive algebra, then there is at most one norm on A which makes A into a C^*-algebra.*

Proof. If there were two norms which gave a C^*-algebra structure to A then by the previous corollary the identity map would be an isometry. $\blacksquare$

Lemma 1.3.4. *Let A be a unital C^*-algebra, if $x \in A$ is self-adjoint then $\sigma_A(x) \subset \mathbb{R}$.*

Proof. Suppose $\lambda = \alpha + i\beta \in \sigma_A(x)$ where $\alpha, \beta \in \mathbb{R}$. If we consider $y = x - \alpha + it$ where $t \in \mathbb{R}$, then we have $i(\beta + t) \in \sigma_A(y)$ and y is normal. Hence,

$$\begin{aligned}(\beta + t)^2 \leq r(y)^2 &= \|y\|^2 = \|y^*y\| \\ &= \|(x-\alpha)^2 + t^2\| \leq \|x - \alpha\|^2 + t^2,\end{aligned}$$

and since $t \in \mathbb{R}$ was arbitrary it then follows that $\beta = 0$. ■

Lemma 1.3.5. *Let A be a unital Banach algebra and suppose $x \notin G(A)$. If $x_n \in G(A)$ such that $\|x_n - x\| \to 0$, then $\|x_n^{-1}\| \to \infty$.*

Proof. If $\|x_n^{-1}\|$ were bounded then we would have that $\|1 - xx_n^{-1}\| < 1$ for some n. Thus, we would have that $xx_n^{-1} \in G(A)$ and hence also $x \in G(A)$. ■

Proposition 1.3.6. *Let B be a unital C^*-algebra and $A \subset B$ a unital C^*-subalgebra. If $x \in A$ then $\sigma_A(x) = \sigma_B(x)$.*

Proof. Note that we always have $G(A) \subset G(B)$. If $x \in A$ is self-adjoint such that $x \notin G(A)$, then by Lemma 1.3.4 we have $it \in \rho_A(x)$ for $t > 0$. By the previous lemma we then have

$$\lim_{t \to 0} \|(x - it)^{-1}\| = \infty,$$

and thus $x \notin G(B)$ since inversion is continuous in $G(B)$.

For general $x \in A$ we then have

$$x \in G(A) \Leftrightarrow x^*x \in G(A) \Leftrightarrow x^*x \in G(B) \Leftrightarrow x \in G(B).$$

In particular, we have $\sigma_A(x) = \sigma_B(x)$ for all $x \in A$. ■

Because of the previous result we will henceforth write simply $\sigma(x)$ for the spectrum of an element in a C^*-algebra.

Theorem 1.3.7. *Let A be a unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C(\sigma(A))$ gives an isometric isomorphism between A and $C(\sigma(A))$.*

Proof. If x is self-adjoint then from Lemma 1.3.4 we have $\sigma(\Gamma(x)) = \sigma(x) \subset \mathbb{R}$, and hence $\overline{\Gamma(x)} = \Gamma(x^*)$. In general, if $x \in A$ we can write x as $x = a + ib$ where $a = \frac{x + x^*}{2}$ and $b = \frac{i(x^* - x)}{2}$ are self-adjoint. Hence, $\Gamma(x^*) = \Gamma(a - ib) = \Gamma(a) - i\Gamma(b) = \overline{\Gamma(a) + i\Gamma(b)} = \overline{\Gamma(x)}$ and so Γ is a $*$-homomorphism.

By Proposition 1.3.1, if $x \in A$ we then have

$$\begin{aligned}\|x\|^2 &= \|x^*x\| = r(x^*x) \\ &= r(\Gamma(x^*x)) = \|\Gamma(x^*)\Gamma(x)\| = \|\Gamma(x)\|^2,\end{aligned}$$

and so Γ is isometric, and in particular injective.

To show that Γ is surjective note that $\Gamma(A)$ is self-adjoint, and closed since Γ is isometric. Moreover, $\Gamma(A)$ contains the constants and clearly separates points, hence $\Gamma(A) = C(\sigma(A))$ by the Stone-Weierstrauss theorem. ■

Since we have seen above that if A is generated as a unital C^*-algebra by a single normal element $x \in A$, then we have a natural homeomorphism $\sigma(x) \cong \sigma(A)$. Thus by considering the inverse Gelfand transform we obtain an isomorphism between $C(\sigma(x))$ and A which we denote by $f \mapsto f(x)$.

Theorem 1.3.8 (Continuous functional calculus)**.** *Let A and B be a unital C^*-algebras, with $x \in A$ normal, then this functional calculus satisfies the following properties:*

(i) *The map $f \mapsto f(x)$ is a homomorphism from $C(\sigma(x))$ to A, and if $f(z) = \sum_{k=0}^n a_k z^k$ is a polynomial, then $f(x) = \sum_{k=0}^n a_k x^k$.*

(ii) *For $f \in C(\sigma(x))$ we have $\sigma(f(x)) = f(\sigma(x))$.*

(iii) *If $\Phi : A \to B$ is a C^*-homomorphism then $\Phi(f(x)) = f(\Phi(x))$.*

(iv) *If $x_n \in A$ is a sequence of normal elements such that $\|x_n - x\| \to 0$, Ω is a compact neighborhood of $\sigma(x)$, and $f \in C(\Omega)$, then for large enough n we have $\sigma(x_n) \subset \Omega$ and $\|f(x_n) - f(x)\| \to 0$.*

Proof. Parts (i), and (ii) follow easily from Theorem 1.3.7. Part (iii) is obvious for polynomials and then follows for all continuous functions by approximation.

For part (iv), the fact that $\sigma(x_n) \subset \Omega$ for large n follows from continuity of inversion. If we write $C = \sup_n \|x_n\|$ and we have $\varepsilon > 0$ arbitrary, then we may take a polynomial $g : \Omega \to \mathbb{C}$ such that $\|f - g\|_\infty < \varepsilon$ and we have

$$\limsup_{n\to\infty} \|f(x_n) - f(x)\| \leq 2\|f - g\|_\infty C + \limsup_{n\to\infty} \|g(x_n) + g(x)\| < 2C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have $\lim_{n\to\infty} \|f(x_n) - f(x)\| = 0$. ∎

1.3.1 The non-unital case

If A is not a unital C^*-algebra then we may consider the space $\tilde{A} = A \oplus \mathbb{C}$ which is a $*$-algebra with multiplication

$$(x \oplus \alpha) \cdot (y \oplus \beta) = (xy + \alpha y + \beta x) \oplus \alpha\beta,$$

and involution $(x \oplus \alpha)^* = x^* \oplus \overline{\alpha}$. We may also place a norm on $\tilde{A}$ given by

$$\|x \oplus \alpha\| = \sup_{y \in A, \|y\| \leq 1} \|xy + \alpha y\|.$$

We call $\tilde{A}$ the **unitization** of A.

Proposition 1.3.9. *Let A be a non-unital C^*-algebra, then the unitization $\tilde{A}$ is again a C^*-algebra, and the map $x \mapsto x \oplus 0$ is an isometric $*$-isomorphism of A onto a maximal ideal in $\tilde{A}$.*

Proof. The map $x \mapsto x \oplus 0$ is indeed isometric since on one hand we have $\|x \oplus 0\| = \sup_{y \in A, \|y\| \leq 1} \|xy\| \leq \|x\|$, while on the other hand if $x \neq 0$, and we set $y = x^*/\|x^*\|$ then we have $\|x\| = \|xx^*\|/\|x^*\| = \|xy\| \leq \|x \oplus 0\|$.

The norm on $\tilde{A}$ is nothing but the operator norm when we view $\tilde{A}$ as acting on A by left multiplication and hence we have that this is at least a semi-norm such that $\|xy\| \leq \|x\|\|y\|$, for all $x, y \in \tilde{A}$. To see that this is actually a norm note that if $\alpha \neq 0$, but $\|x \oplus \alpha\| = 0$ then for all $y \in A$ we have $\|xy + \alpha y\| \leq \|x \oplus \alpha\|\|y\| = 0$, and hence $e = -x/\alpha$ is a left identity for A. Taking adjoints we see that e^* is a right identity for A, and then $e = ee^* = e^*$ is an identity for A which contradicts that A is non-unital. Thus, $\|\cdot\|$ is indeed a norm.

It is easy to see then that $\tilde{A}$ is then complete, and hence all that remains to be seen is the C^*-identity. Since, each for each $y \in A$, $\|y\| \leq 1$ we have $(y \oplus 0)^*(x \oplus \alpha) \in A \oplus 0 \cong A$ it follows that the C^*-identity holds here, and so

$$\begin{aligned}\|(x \oplus \alpha)^*(x \oplus \alpha)\| &\geq \|(y \oplus 0)^*(x \oplus \alpha)^*(x \oplus \alpha)(y \oplus 0)\| \\ &= \|(x \oplus \alpha)(y \oplus 0)\|^2.\end{aligned}$$

Taking the supremum over all $y \in A$, $\|y\| \leq 1$ we then have

$$\|(x \oplus \alpha)^*(x \oplus \alpha)\| \geq \|x \oplus \alpha\|^2 \geq \|(x \oplus \alpha)^*(x \oplus \alpha)\|. \qquad \blacksquare$$

Lemma 1.3.10. *If A is a non-unital abelian C^*-algebra, then any norm 1 multiplicative linear functional $\varphi \in \sigma(A)$ has a unique extension $\tilde{\varphi} \in \tilde{A}$.*

Proof. If we consider $\tilde{\varphi}(x \oplus \alpha) = \varphi(x) + \alpha$ then the result follows easily. $\blacksquare$

In particular, this shows that $\sigma(A)$ is homeomorphic to $\sigma(\tilde{A}) \setminus \{\varphi_0\}$ where φ_0 is defined by $\varphi(x, \alpha) = \alpha$. Thus, $\sigma(A)$ is locally compact.

If $x \in A$ then the **spectrum** $\sigma(x)$ of x is defined to be the spectrum of $x \oplus 0 \in \tilde{A}$. Note that for a non-unital C^*-algebra A, since $A \subset \tilde{A}$ is an ideal it follows that $0 \in \sigma(x)$ whenever $x \in A$.

By considering the embedding $A \subset \tilde{A}$ we are able to extend the spectral theorem and continuous functional calculus to the non-unital setting. We leave the details to the reader.

Theorem 1.3.11. *Let A be a non-unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C_0(\sigma(A))$ gives an isometric isomorphism between A and $C_0(\sigma(A))$.*

Theorem 1.3.12. *Let A be a C^*-algebra, and $x \in A$ a normal element, then if $f \in C(\sigma(x))$ such that $f(0) = 0$, then $f(x) \in A \subset \tilde{A}$.*

Exercise 1.3.13. Suppose K is a non-compact, locally compact Hausdorff space, and $K \cup \{\infty\}$ is the one point compactification. Show that we have a natural isomorphism $C(K \cup \{\infty\}) \cong \widetilde{C_0(K)}$.

1.4 Applications of functional calculus

Given any element x in a C^*-algebra A, we can decompose x uniquely as a sum of a self-adoint and skew-adjoint elements $\frac{x+x^*}{2}$ and $\frac{x-x^*}{2}$. We refer to the self-adjoint elements $\frac{x+x^*}{2}$ and $i\frac{x^*-x}{2}$ the **real** and **imaginary** parts of x, note that the real and imaginary parts of x have norms no grater than that of x.

Also, if $x \in A$ is self-adjoint then from above we know that $\sigma(x) \subset \mathbb{R}$, hence by considering $x_+ = (0 \vee t)(x)$ and $x_- = -(0 \wedge t)(x)$ it follows easily from functional calculus that $\sigma(x_+), \sigma(x_-) \subset [0, \infty)$, $x_+x_- = x_-x_+ = 0$, and $x = x_+ - x_-$. We call x_+ and x_- the **positive** and **negative** parts of x.

1.4.1 The positive cone

Lemma 1.4.1. *Suppose we have self-adjoint elements $x, y \in A$ such that $\sigma(x), \sigma(y) \subset [0, \infty)$ then $\sigma(x + y) \subset [0, \infty)$.*

Proof. Let $a = \|x\|$, and $b = \|y\|$. Since x is self-adjoint and $\sigma(x) \subset [0, \infty)$ we may use the spectral radius formula to see that $\|a - x\| = r(a - x) = a$. Similarly we have $\|b - y\| = b$ and since $\|x + y\| \leq a + b$ we have

$$\sup_{\lambda \in \sigma(x+y)} \{a + b - \lambda\} = r((a + b) - x) = \|(a + b) - (x + y)\|$$
$$\leq \|x - a\| + \|y - b\| = a + b.$$

Therefore, $\sigma(x + y) \subset [0, \infty)$. ■

Proposition 1.4.2. *Let A be a C^*-algebra. A normal element $x \in A$ is*

(i) *self-adjoint if and only if $\sigma(x) \subset \mathbb{R}$.*

(ii) *positive if and only if $\sigma(x) \subset [0, \infty)$.*

(iii) *unitary if and only if $\sigma(x) \subset \mathbb{T}$.*

(iv) *a projection if and only if $\sigma(x) \subset \{0, 1\}$.*

Proof. Parts (i), (iii), and (iv) all follow easily by applying continuous functional calculus. For part (ii) if x is normal and $\sigma(x) \subset [0, \infty)$ then $x = (\sqrt{x})^2 = (\sqrt{x})^*\sqrt{x}$ is positive. It also follows easily that if $x = y^*y$ where y is normal then $\sigma(x) \subset [0, \infty)$. Thus, the difficulty arises only when $x = y^*y$ where y is perhaps not normal.

Suppose $x = y^*y$ for some $y \in A$, to show that $\sigma(x) \subset [0, \infty)$, decompose x into its positive and negative parts $x = x_+ - x_-$ as described above. Set $z = yx_-$ and note that $z^*z = x_-(y^*y)x_- = -x_-^3$, and hence $\sigma(zz^*) \subset \sigma(z^*z) \subset (-\infty, 0]$.

If $z = a + ib$ where a and b are self-adjoint, then we have $zz^* + z^*z = 2a^2 + 2b^2$, hence we also have $\sigma(zz^* + z^*z) \subset [0, \infty)$ and so by Lemma 1.4.1 we have $\sigma(z^*z) = \sigma((2a^2 + 2b^2) - zz^*) \subset [0, \infty)$. Therefore $\sigma(-x_-^3) = \sigma(z^*z) \subset \{0\}$ and since x_- is normal this shows that $x_-^3 = 0$, and consequently $x_- = 0$. ■

Corollary 1.4.3. *Let A be a C^*-algebra. An element $x \in A$ is a partial isometry if and only if x^* is a partial isometry.*

Proof. Since x^*x is normal, it follows from the previous proposition that x is a partial isometry if and only if $\sigma(x^*x) \subset \{0,1\}$. Since $\sigma(x^*x) \cup \{0\} = \sigma(xx^*) \cup \{0\}$ this gives the result. ∎

Corollary 1.4.4. *Let A be a C^*-algebra, then the set of positive elements forms a closed cone. Moreover, if $a \in A$ is self-adjoint, and A is unital, then we have $a \leq \|a\|$.*

Note that if $x \in A$ is an arbitrary element of a C^*-algebra A, then from above we have that x^*x is positive and hence we may define the **absolute value** of x as the unique element $|x| \in A$ such that $|x|^2 = x^*x$.

Proposition 1.4.5. *Let A be a unital C^*-algebra, then every element is a linear combination of four unitaries.*

Proof. If $x \in A$ is self-adjoint and $\|x\| \leq 1$, then $u = x + i(1 - x^2)^{1/2}$ is a unitary and we have $x = u + u^*$. In general, we can decompose x into its real and imaginary parts and then write each as a linear combination of two unitaries. ∎

Proposition 1.4.6. *Let A be a C^*-algebra, and suppose $x, y \in A_+$ such that $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$. Moreover, if A is unital and $x, y \in A$ are invertible, then $y^{-1} \leq x^{-1}$.*

Proof. First consider the case that A is unital and x and y are invertible, then we have
$$y^{-1/2}xy^{-1/2} \leq 1,$$
hence
$$\begin{aligned} x^{1/2}y^{-1}x^{1/2} &\leq \|x^{1/2}y^{-1}x^{1/2}\| = r(x^{1/2}y^{-1}x^{1/2}) \\ &= r(y^{-1/2}xy^{-1/2}) \leq 1. \end{aligned}$$
Conjugating by $x^{-1/2}$ gives $y^{-1} \leq x^{-1}$.

We also have
$$\|y^{-1/2}x^{1/2}\|^2 = \|y^{-1/2}xy^{-1/2}\| \leq 1,$$
therefore
$$\begin{aligned} y^{-1/4}x^{1/2}y^{-1/4} &\leq \|y^{-1/4}x^{1/2}y^{-1/4}\| = r(y^{-1/4}x^{1/2}y^{-1/4}) \\ &= r(y^{-1/2}x^{1/2}) \leq \|y^{-1/2}x^{1/2}\| \leq 1. \end{aligned}$$
Conjugating by $y^{1/4}$ gives $x^{1/2} \leq y^{1/2}$.

In the general case we may consider the unitization of A, and note that if $\varepsilon > 0$, then we have $0 \leq x + \varepsilon \leq y + \varepsilon$, where $x + \varepsilon$, and $y + \varepsilon$ are invertible, hence from above we have
$$(x + \varepsilon)^{1/2} \leq (y + \varepsilon)^{1/2}.$$
Taking the limit as $\varepsilon \to 0$ we obtain the result. ∎

In general, a continuous real valued function f defined on an interval I is said to be **operator monotone** if $f(a) \leq f(b)$ whenever $\sigma(a), \sigma(b) \subset I$, and $a \leq b$. The previous proposition shows that the functions $f(t) = \sqrt{t}$, and $f(t) = -1/t$, $t > 0$ are operator monotone.

Corollary 1.4.7. *Let A be a C^*-algebra, then for $x, y \in A$ we have $|xy| \leq \|x\||y|$.*

Proof. Since $|xy|^2 = y^*x^*xy \leq \|x\|^2 y^*y$, this follows from the previous proposition. ∎

1.4.2 Extreme points

Given a involutive normed algebra A, we denote by $(A)_1$ the unit ball of A, and $A_{\text{s.a.}}$ the subspace of self-adjoint elements.

Proposition 1.4.8. *Let A be a C^*-algebra.*

(i) The extreme points of $(A_+)_1$ are the projections of A.

(ii) The extreme points of $(A_{\text{s.a.}})_1$ are the self-adjoint unitaries in A.

(iii) Every extreme point of $(A)_1$ is a partial isometry in A.

Proof. (i) If $x \in (A_+)_1$, then we have $x^2 \leq 2x$, and $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$. Hence if x is an extreme point then we have $x = x^2$ and so x is a projection. For the converse we first consider the case when A is abelian, and so we may assume $A = C_0(K)$ for some locally compact Hausdorff space K. If x is a projection then $x = 1_E$ is the characteristic function on some open and closed set $E \subset K$, hence the result follows easily from the fact that 0 and 1 are extreme points of $[0, 1]$.

For the general case, suppose $p \in A$ is a projection, if $p = \frac{1}{2}(a + b)$ then $\frac{1}{2}a = p - b \leq p$, and hence $0 \leq (1 - p)a(1 - p) \leq 0$, thus $a = ap = pa$. We therefore have that a, b, and p commute and hence the result follows from the abelian case.

(ii) First note that if A is unital then 1 is an extreme point in the unit ball. Indeed, if $1 = \frac{1}{2}(a + b)$ where $a, b \in (A)_1$, then we have the same equation when replacing a and b by their real parts. Thus, assuming a and b are self-adjoint we have $\frac{1}{2}a = 1 - \frac{1}{2}b$ and hence a and b commute. By considering the unital C^*-subalgebra generated by a and b we may assume $A = C(K)$ for some compact Hausdorff space K, and then it is an easy exercise to conclude that $a = b = 1$.

If u is a unitary in A, then the map $x \mapsto ux$ is a linear isometry of A, thus since 1 is an extreme point of $(A)_1$ it follows that u is also an extreme point. In particular, if u is self-adjoint then it is an extreme point of $(A_{\text{s.a.}})_1$.

Conversely, if $x \in (A_{\text{s.a.}})_1$ is an extreme point then if $x_+ = \frac{1}{2}(a + b)$ for $a, b \in (A_+)_1$, then $0 = x_- x_+ x_- = \frac{1}{2}(x_- a x_- + x_- b x_-) \geq 0$, hence we have $(a^{1/2}x_-)^*(a^{1/2}x_-) = x_- a x_- = 0$. We conclude that $ax_- = x_- a = 0$, and similarly $bx_- = x_- b = 0$. Thus, $a - x_-$ and $b - x_-$ are in $(A_{\text{s.a.}})_1$ and $x =$

$\frac{1}{2}((a - x_-) + (b - x_-))$. Since x is an extreme point we conclude that $x = a - x_- = b - x_-$ and hence $a = b = x_+$.

We have shown now that x_+ is an extreme point in $(A_+)_1$ and thus by part (i) we conclude that x_+ is a projection. The same argument shows that x_- is also a projection, and thus x is a self-adjoint unitary.

(iii) If $x \in (A)_1$ such that x^*x is not a projection then by applying functional calculus to x^*x we can find an element $y \in A_+$ such that $x^*xy = yx^*x \neq 0$, and $\|x(1 \pm y)\|^2 = \|x^*x(1 \pm y)^2\| \leq 1$. Since $xy \neq 0$ we conclude that $x = \frac{1}{2}((x + xy) + (x - xy))$ is not an extreme point of $(A)_1$. ∎

1.4.3 Ideals and quotients

Theorem 1.4.9. *Let A be a C^*-algebra, and let $I \subset A$ be a left ideal, then there exists an increasing net $\{a_\lambda\}_\lambda \subset I$ of positive elements such that for all $x \in I$ we have*

$$\|xa_\lambda - x\| \to 0.$$

Moreover, if A is separable then the net can be taken to be a sequence.

Proof. Consider Λ to be the set of all finite subsets of $I \subset A \subset \tilde{A}$, ordered by inclusion. If $\lambda \in \Lambda$ we consider

$$h_\lambda = \sum_{x \in \lambda} x^*x, \qquad a_\lambda = |\lambda| h_\lambda (1 + |\lambda| h_\lambda)^{-1}.$$

Then we have $a_\lambda \in I$ and $0 \leq a_\lambda \leq 1$. If $\lambda \leq \lambda'$ then we clearly have $h_\lambda \leq h_{\lambda'}$ and hence by Proposition 1.4.6 we have that

$$\frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_{\lambda}\right)^{-1}.$$

Therefore

$$a_\lambda = 1 - \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_\lambda\right)^{-1} \leq 1 - \frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} = a_{\lambda'}.$$

If $y \in \lambda$ then we have

$$(y(1 - a_\lambda))^*(y(1 - a_\lambda)) \leq \sum_{x \in \lambda} (x(1 - a_\lambda))^*(x(1 - a_\lambda)) = (1 - a_\lambda) h_\lambda (1 - a_\lambda).$$

But $\|(1 - a_\lambda) h_\lambda (1 - a_\lambda)\| = \|h_\lambda (1 + |\lambda| h_\lambda)^{-2}\| \leq \frac{1}{4|\lambda|}$, from which it follows easily that $\|y - ya_\lambda\| \to 0$, for all $y \in I$.

If A is separable then so is $\overline{I}$, hence there exists a countable subset $\{x_n\}_{n \in \mathbb{N}} \subset I$ which is dense in I. If we take $\lambda_n = \{x_1, \ldots, x_n\}$, then clearly $a_n = a_{\lambda_n}$ also satisfies

$$\|y - ya_n\| \to 0.$$

∎

We call such a net $\{a_\lambda\}$ a **right approximate identity** for I. If I is self-adjoint then we also have $\|a_\lambda x - x\| = \|x^* a_\lambda - x^*\| \to 0$ and in this case we call $\{a_\lambda\}$ an **approximate identity**. Using the fact that the adjoint is an isometry we also obtain the following corollary.

Corollary 1.4.10. *Let A be a C^*-algebra, and $I \subset A$ a closed two sided ideal. Then I is self-adjoint. In particular, I is a C^*-algebra.*

Exercise 1.4.11. Show that if A is a C^*-algebra such that $x \leq y \implies x^2 \leq y^2$, for all $x, y \in A_+$, then A is abelian.

Exercise 1.4.12. Let A be a C^*-algebra and $I \subset A$ a non-trivial closed two sided ideal. Show that A/I is again a C^*-algebra.

Chapter 2

Operators on Hilbert space

Recall that if $\mathcal{H}$ is a Hilbert space then $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators is a C^*-algebra with norm

$$\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|,$$

and involution given by the adjoint, i.e., x^* is the unique bounded linear operator such that

$$\langle \xi, x^* \eta \rangle = \langle x\xi, \eta \rangle,$$

for all $\xi, \eta \in \mathcal{H}$.

Lemma 2.0.13. *Let $\mathcal{H}$ be a Hilbert space and consider $x \in \mathcal{B}(\mathcal{H})$, then $\ker(x) = R(x^*)^\perp$.*

Proof. If $\xi \in \ker(x)$, and $\eta \in \mathcal{H}$, then $\langle \xi, x^*\eta \rangle = \langle x\xi, \eta \rangle = 0$, hence $\ker(x) \subset R(x^*)^\perp$. If $\xi \in R(x^*)^\perp$ then for any $\eta \in \mathcal{H}$ we have $\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle = 0$, hence $\xi \in \ker(x)$. ■

Lemma 2.0.14. *Let $\mathcal{H}$ be a Hilbert space, then an operator $x \in \mathcal{B}(\mathcal{H})$ is*

(*i*) *normal if and only if $\|x\xi\| = \|x^*\xi\|$, for all $\xi \in \mathcal{H}$.*

(*ii*) *self-adjoint if and only if $\langle x\xi, \xi \rangle \in \mathbb{R}$, for all $\xi \in \mathcal{H}$.*

(*iii*) *positive if and only if $\langle x\xi, \xi \rangle \geq 0$, for all $\xi \in \mathcal{H}$.*

(*iv*) *an isometry if and only if $\|x\xi\| = \|\xi\|$, for all $\xi \in \mathcal{H}$.*

(*v*) *a projection if and only if x is the orthogonal projection onto some closed subspace of $\mathcal{H}$.*

(*vi*) *a partial isometry if and only if there is a closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $x_{|\mathcal{K}}$ is an isometry while $x_{|\mathcal{K}^\perp} = 0$.*

Proof.

(i) If x is normal than for all $\xi \in \mathcal{H}$ we have $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \langle xx^*\xi, \xi\rangle = \|x^*\xi\|^2$. Conversely, is $\langle (x^*x - xx^*)\xi, \xi\rangle = 0$, for all $\xi \in \mathcal{H}$, then for all $\xi, \eta \in \mathcal{H}$, by polarization we have

$$\langle (x^*x - xx^*)\xi, \eta\rangle = \sum_{k=0}^{3} i^k \langle (x^*x - xx^*)(\xi + i^k\eta), (\xi + i^k\eta)\rangle = 0.$$

Hence $x^*x = xx^*$.

(ii) If $x = x^*$ then $\overline{\langle x\xi, \xi\rangle} = \langle \xi, x\xi\rangle = \langle x\xi, \xi\rangle$. The converse follows again by a polarization argument.

(iii) If $x = y^*y$, then $\langle x\xi, \xi\rangle = \|y\xi\|^2 \geq 0$. Conversely, if $\langle x\xi, \xi\rangle \geq 0$, for all $\xi \in \mathcal{H}$ then we know from part (b) that x is self-adjoint, and for all $a > 0$ we have $\langle (x + a)\xi, \xi\rangle \geq a\|\xi\|^2$. This shows that $x + a$ is an injective operator with dense image (since the orthogonal complement of the range is trivial). Moreover, by the Cauchy-Schwarz inequality we have

$$a\|\xi\|^2 \leq \langle (x + a)\xi, \xi\rangle \leq \|(x + a)\xi\|\|\xi\|,$$

and hence $a\|\xi\| \leq \|(x + a)\xi\|$, for all $\xi \in \mathcal{H}$. In particular this shows that the image of $x + a$ is closed since if $\{(x + a)\xi_n\}$ is Cauchy then $\{\xi_n\}$ is also Cauchy. Therefore $(x + a)$ is invertible and $a\|(x + a)^{-1}\xi\| \leq \|\xi\|$, for all $\xi \in \mathcal{H}$, showing that $(x + a)^{-1}$ is bounded. Since $a > 0$ was arbitrary this shows that $\sigma(x) \subset [0, \infty)$ and hence x is positive.

(iv) If x is an isometry then $x^*x = 1$ and hence $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \|\xi\|^2$ for all $\xi \in \mathcal{H}$. The converse again follows from the polarization identity.

(v) If x is a projection then let $\mathcal{K} = \overline{R(x)} = \ker(x)^\perp$, and note that for all $\xi \in \mathcal{K}, \eta \in \ker(x), x\zeta \in R(x)$ we have $\langle x\xi, \eta + x\zeta\rangle = \langle \xi, x\zeta\rangle$, hence $x\xi \in \mathcal{K}$, and $x\xi = \xi$. This shows that x is the orthogonal projection onto the subspace $\mathcal{K}$.

(vi) This follows directly from iv and v. ■

Proposition 2.0.15 (Polar decomposition)**.** *Let $\mathcal{H}$ be a Hilbert space, and $x \in \mathcal{B}(\mathcal{H})$, then there exists a partial isometry v such that $x = v|x|$, and* $\ker(v) = \ker(|x|) = \ker(x)$. *Moreover, this decomposition is unique, in that if $x = wy$ where $y \geq 0$, and w is a partial isometry with* $\ker(w) = \ker(y)$ *then $y = |x|$, and $w = v$.*

Proof. We define a linear operator $v_0 : R(|x|) \to R(x)$ by $v_0(|x|\xi) = x\xi$, for $\xi \in \mathcal{H}$. Since $\||x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$ it follows that v_0 is well defined and extends to a partial isometry v from $\overline{R(|x|)}$ to $\overline{R(x)}$, and we have $v|x| = x$. We also have $\ker(v) = R(|x|)^\perp = \ker(|x|) = \ker(x)$.

To see the uniqueness of this decomposition suppose $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$. Then $|x|^2 = x^*x = yw^*wy = y^2$, and hence $|x| = (|x|^2)^{1/2} = (y^2)^{1/2} = y$. We then have $\ker(w) = \overline{R(|x|)}^\perp$, and $\|w|x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$, hence $w = v$. ■

2.1 Trace class operators

Given a Hilbert space $\mathcal{H}$, an operator $x \in \mathcal{B}(\mathcal{H})$ has finite rank if $\overline{R(x)} = \ker(x^*)^\perp$ is finite dimensional, the **rank** of x is $\dim(\overline{R(x)})$. We denote the space of finite rank operators by $\mathcal{FR}(\mathcal{H})$. If x is finite rank than $R(x^*) = R(x^*_{|\ker(x^*)^\perp})$ is also finite dimensional being the image of a finite dimensional space, hence we see that x^* also has finite rank. If $\xi, \eta \in \mathcal{H}$ are vectors we denote by $\xi \otimes \overline{\eta}$ the operator given by

$$(\xi \otimes \overline{\eta})(\zeta) = \langle \zeta, \eta \rangle \xi.$$

Note that $(\xi \otimes \overline{\eta})^* = \eta \otimes \overline{\xi}$, and if $\|\xi\| = \|\eta\| = 1$ then $\xi \otimes \overline{\eta}$ is a rank one partial isometry from $C\eta$ to $\mathbb{C}\xi$. Also note that if $x, y \in \mathcal{B}(\mathcal{H})$, then we have $x(\xi \otimes \overline{\eta})y = (x\xi) \otimes (\overline{y^*\eta})$.

From above we see that any finite rank operator is of the form pxq where $p, q \in \mathcal{B}(\mathcal{H})$ are projections onto finite dimensional subspaces. In particular this shows that $\mathcal{FR}(\mathcal{H}) = \mathrm{sp}\{\xi \otimes \overline{\eta} \mid \xi, \eta \in \mathcal{H}\}$

Lemma 2.1.1. *Suppose $x \in \mathcal{B}(\mathcal{H})$ has polar decomposition $x = v|x|$. Then for all $\xi \in \mathcal{H}$ we have*

$$2|\langle x\xi, \xi \rangle| \leq \langle |x|\xi, \xi \rangle + \langle |x|v^*\xi, v^*\xi \rangle.$$

Proof. If $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, then we have

$$\begin{aligned} 0 &\leq \|(|x|^{1/2} - \lambda |x|^{1/2} v^*)\xi\|^2 \\ &= \||x|^{1/2}\xi\|^2 - 2\mathrm{Re}(\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi \rangle) + \||x|^{1/2}v^*\xi\|^2. \end{aligned}$$

Taking λ such that $\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi \rangle \geq 0$, the inequality follows directly. ■

If $\{\xi_i\}$ is an orthonormal basis for $\mathcal{H}$, and $x \in \mathcal{B}(\mathcal{H})$ is positive, then we define the trace of x to be

$$\mathrm{Tr}(x) = \sum_i \langle x\xi_i, \xi_i \rangle.$$

Lemma 2.1.2. *If $x \in \mathcal{B}(\mathcal{H})$ then* $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$.

Proof. By Parseval's identity and Fubini's theorem we have

$$\begin{aligned} \sum_i \langle x^*x\xi_i, \xi_i \rangle &= \sum_i \sum_j \langle x\xi_i, \xi_j \rangle \overline{\langle \xi_j, x\xi_i \rangle} \\ &= \sum_j \sum_i \langle \xi_i, x^*\xi_j \rangle \overline{\langle \xi_i, x^*\xi_j \rangle} = \sum_j \langle xx^*\xi_j, \xi_j \rangle. \end{aligned}$$ ■

Corollary 2.1.3. *If $x \in \mathcal{B}(\mathcal{H})$ is positive and u is a unitary, then* $\mathrm{Tr}(u^*xu) = \mathrm{Tr}(x)$. *In particular, the trace is independent of the chosen orthonormal basis.*

Proof. If we write $x = y^*y$, then from the previous lemma we have

$$\mathrm{Tr}(y^*y) = \mathrm{Tr}(yy^*) = \mathrm{Tr}((yu)(u^*y^*)) = \mathrm{Tr}(u^*(y^*y)u).$$ ■

An operator $x \in \mathcal{B}(\mathcal{H})$ is said to be of **trace class** if $\|x\|_1 := \mathrm{Tr}(|x|) < \infty$. We denote the set of trace class operators by $L^1(\mathcal{B}(\mathcal{H}))$ or $L^1(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$.

Given an orthonormal basis $\{\xi_i\}$, and $x \in L^1(\mathcal{B}(\mathcal{H}))$ we define the **trace** of x by

$$\mathrm{Tr}(x) = \Sigma_i \langle x\xi_i, \xi_i \rangle.$$

By Lemma 2.1.1 this is absolutely summable, and

$$2|\,\mathrm{Tr}(x)| \leq \mathrm{Tr}(|x|) + \mathrm{Tr}(v|x|v^*) \leq 2\|x\|_1.$$

Lemma 2.1.4. *$L^1(\mathcal{B}(\mathcal{H}))$ is a two sided self-adjoint ideal in $\mathcal{B}(\mathcal{H})$ which coincides with the span of the positive operators with finite trace. The trace is independent of the chosen basis, and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.*

Proof. If $x, y \in L^1(\mathcal{B}(\mathcal{H}))$ and we let $x+y = w|x+y|$ be the polar decomposition, then we have $w^*x, w^*y \in L^1(\mathcal{B}(\mathcal{H}))$, therefore $\sum_i \langle |x+y|\xi_i, \xi_i\rangle = \sum_i \langle w^*x\xi_i, \xi_i\rangle + \langle w^*y\xi_i, \xi_i\rangle$ is absolutely summable. Thus $x + y \in L^1(\mathcal{B}(\mathcal{H}))$ and

$$\|x + y\|_1 \leq \|w^*x\|_1 + \|w^*y\|_1 \leq \|x\|_1 + \|y\|_1.$$

Thus, it follows that $L^1(\mathcal{B}(\mathcal{H}))$ is a linear space which contains the span of the positive operators with finite trace, and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.

If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ then

$$4a|x| = \sum_{k=0}^{3} i^k (a + i^k)|x|(a + i^k)^*,$$

and for each k we have

$$\mathrm{Tr}((a + i^k)|x|(a + i^k)^*) = \mathrm{Tr}(|x|^{1/2}|a + i^k|^2|x|^{1/2}) \leq \|a + i^k\|^2\,\mathrm{Tr}(|x|).$$

Thus if we take a to be the partial isometry in the polar decomposition of x we see that x is a linear combination of positive operators with finite trace, (in particular, the trace is independent of the basis). This also shows that $L^1(\mathcal{B}(\mathcal{H}))$ is a self-adjoint left ideal, and hence is also a right ideal. ∎

Theorem 2.1.5. *If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a, b \in \mathcal{B}(\mathcal{H})$ then*

$$\|x\| \leq \|x\|_1$$

$$\|axb\|_1 \leq \|a\|\ \|b\|\ \|x\|_1,$$

and

$$\mathrm{Tr}(ax) = \mathrm{Tr}(xa).$$

Proof. Since the trace is independent of the basis, and $\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|$ it follows easily that $\|x\| \leq \|x\|_1$.

Since for $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ we have $|ax| \leq \|a\||x|$ it follows that $\|ax\|_1 \leq \|a\|\|x\|_1$. Since $\|x\|_1 = \|x^*\|_1$ we also have $\|xb\|_1 \leq \|b\|\|x\|_1$.

Since the definition of the trace is independent of the chosen basis, if $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $u \in \mathcal{U}(\mathcal{H})$ we have

$$\mathrm{Tr}(xu) = \sum_i \langle xu\xi_i, \xi_i \rangle = \sum_i \langle uxu\xi_i, u\xi_i \rangle = \mathrm{Tr}(ux).$$

Since every operator $a \in \mathcal{B}(\mathcal{H})$ is a linear combination of four unitaries this also gives

$$\mathrm{Tr}(xa) = \mathrm{Tr}(ax).$$ ■

We also remark that for all $\xi, \eta \in \mathcal{H}$, the operators $\xi \otimes \overline{\eta}$ satisfy $\mathrm{Tr}(\xi \otimes \overline{\eta}) = \langle \xi, \eta \rangle$. Also, it's easy to check that $\mathcal{FR}(\mathcal{H})$ is a dense subspace of $L^1(\mathcal{B}(\mathcal{H}))$, endowed with the norm $\| \cdot \|_1$.

Proposition 2.1.6. *The space of trace class operators $L^1(\mathcal{B}(\mathcal{H}))$, with the norm $\| \cdot \|_1$ is a Banach space.*

Proof. From Lemma 2.1.4 we know that $\| \cdot \|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$ and hence we need only show that $L^1(\mathcal{B}(\mathcal{H}))$ is complete. Suppose x_n is Cauchy in $L^1(\mathcal{B}(\mathcal{H}))$. Since $\|x_n - x_m\| \leq \|x_n - x_m\|_1$ it follows that x_n is also Cauchy in $\mathcal{B}(\mathcal{H})$, therefore we have $\|x - x_n\| \to 0$, for some $x \in \mathcal{B}(\mathcal{H})$, and by continuity of functional calculus we also have $\| |x| - |x_n| \| \to 0$. Thus for any finite orthonormal set $\eta_1, \ldots, \eta_k$ we have

$$\begin{aligned}\sum_{i=1}^k \langle |x|\eta_i, \eta_i \rangle &= \lim_{n\to\infty} \sum_{i=1}^k \langle |x_n|\eta_i, \eta_i \rangle \\ &\leq \lim_{n\to\infty} \|x_n\|_1 < \infty.\end{aligned}$$

Hence $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $\|x\|_1 \leq \lim_{n\to\infty} \|x_n\|_1$.

If we let $\varepsilon > 0$ be given and consider $N \in \mathbb{N}$ such that for all $n > N$ we have $\|x_n - x_N\|_1 < \varepsilon/3$, and then take $\mathcal{H}_0 \subset \mathcal{H}$ a finite dimensional subspace such that $\|x_N P_{\mathcal{H}_0^\perp}\|_1, \|x P_{\mathcal{H}_0^\perp}\|_1 < \varepsilon/3$. Then for all $n > N$ we have

$$\begin{aligned}&\|x - x_n\|_1 \\ &\quad \leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|xP_{\mathcal{H}_0^\perp} - x_N P_{\mathcal{H}_0^\perp}\|_1 + \|(x_N - x_n)P_{\mathcal{H}_0^\perp}\|_1 \\ &\quad \leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \varepsilon.\end{aligned}$$

Since $\|x - x_n\| \to 0$ it follows that $\|(x - x_n)P_{\mathcal{H}_0}\|_1 \to 0$, and since $\varepsilon > 0$ was arbitrary we then have $\|x - x_n\|_1 \to 0$. ■

Theorem 2.1.7. *The map $\psi : \mathcal{B}(\mathcal{H}) \to L^1(\mathcal{B}(\mathcal{H}))^*$ given by $\psi_a(x) = \mathrm{Tr}(ax)$, for $a \in \mathcal{B}(\mathcal{H})$, $x \in L^1(\mathcal{B}(\mathcal{H}))$, is a Banach space isomorphism.*

Proof. From Theorem 2.1.5 we have that ψ is a linear contraction.

Suppose $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$, then $(\xi, \eta) \mapsto \varphi(\xi \otimes \overline{\eta})$ defines a bounded sesquilinear form on $\mathcal{H}$ and hence there exists a bounded operator $a \in \mathcal{B}(\mathcal{H})$ such that $\langle a\xi, \eta \rangle = \varphi(\xi \otimes \overline{\eta})$, for all $\xi, \eta \in \mathcal{H}$. Since the finite rank operators is dense in

$L^1(\mathcal{B}(\mathcal{H}))$, and since operators of the form $\xi \otimes \overline{\eta}$ span the finite rank operators we have $\varphi = \psi_a$, thus we see that ψ is bijective.

We also have

$$\begin{aligned}\|a\| &= \sup_{\substack{\xi,\eta\in\mathcal{H},\\ \|\xi\|,\|\eta\|\leq 1}} |\langle a\xi, \eta\rangle| \\ &= \sup_{\substack{\xi,\eta\in\mathcal{H},\\ \|\xi\|,\|\eta\|\leq 1}} |\operatorname{Tr}(a(\xi\otimes\overline{\eta}))| \leq \|\psi_a\|.\end{aligned}$$

Hence ψ is isometric. ∎

2.2 Hilbert-Schmidt operators

Given a Hilbert space $\mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$, we say that x is a Hilbert-Schmidt operator on $\mathcal{H}$ if $|x|^2 \in L^1(\mathcal{B}(\mathcal{H}))$. We define the set of Hilbert-Schmidt operators by $L^2(\mathcal{B}(\mathcal{H}))$, or $L^2(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

Lemma 2.2.1. *$L^2(\mathcal{B}(\mathcal{H}))$ is a self-adjoint ideal in $\mathcal{B}(\mathcal{H})$, and if $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ then $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$, and*

$$\operatorname{Tr}(xy) = \operatorname{Tr}(yx).$$

Proof. Since $|x+y|^2 \leq |x+y|^2 + |x-y|^2 = 2(|x|^2+|y|^2)$ we see that $L^2(\mathcal{B}(\mathcal{H}))$ is a linear space, also since $|ax|^2 \leq \|a\|^2|x|^2$ we have that $L^2(\mathcal{B}(\mathcal{H}))$ is a left ideal. Moreover, if $x = v|x|$ is the polar decomposition of x then we have $xx^* = v|x|^2v^*$, and thus $x^* \in L^2(\mathcal{B}(\mathcal{H}))$ and $\operatorname{Tr}(xx^*) = \operatorname{Tr}(x^*x)$. In particular, $L^2(\mathcal{B}(\mathcal{H}))$ is also a right ideal.

By the polarization identity

$$4y^*x = \sum_{k=0}^{3} i^k |x + i^k y|^2,$$

we have that $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$ for $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, and

$$\begin{aligned}4\operatorname{Tr}(y^*x) &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x+i^ky)^*(x+i^ky)) \\ &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x+i^ky)(x+i^ky)^*) = 4\operatorname{Tr}(xy^*).\end{aligned}$$ ∎

From the previous lemma we see that the sesquilinear form on $L^2(\mathcal{B}(\mathcal{H}))$ give by

$$\langle x, y\rangle_2 = \operatorname{Tr}(y^*x)$$

is well defined and positive definite. We again have $\|axb\|_2 \leq \|a\|\,\|b\|\,\|x\|_2$, and any $x \in L^2(\mathcal{B}(\mathcal{H}))$ can be approximated in $\|\cdot\|_2$ by operators px where p is a finite rank projection. Thus, the same argument as for the trace class operators shows that the Hilbert-Schmidt operators is complete in the Hilbert-Schmidt norm.

Also, note that if $x \in L^2(\mathcal{B}(\mathcal{H}))$ then since $\|y\| \leq \|y\|_2$ for all $y \in L^2(\mathcal{B}(\mathcal{H}))$ it follows that

$$\begin{aligned}\|x\|_2 &= \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})),\\ \|y\|_2 \leq 1}} |\operatorname{Tr}(y^* x)| \\ &\leq \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})),\\ \|y\|_2 \leq 1}} \|y\|\|x\|_1 \leq \|x\|_1.\end{aligned}$$

Proposition 2.2.2. *Let $\mathcal{H}$ be a Hilbert space and suppose $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then*

$$\|xy\|_1 \leq \|x\|_2 \|y\|_2.$$

Proof. If we consider the polar decomposition $xy = v|xy|$, then by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\|xy\|_1 &= |\operatorname{Tr}(v^* xy)| = |\langle y, x^* v\rangle_2| \\ &\leq \|x^* v\|_2 \|y\|_2 \leq \|x\|_2 \|y\|_2.\end{aligned}$$ ∎

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then we may extend a bounded operator $x : \mathcal{H} \to \mathcal{K}$ to a bounded operator $\tilde{x} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ by $\tilde{x}(\xi \oplus \eta) = 0 \oplus x\xi$. We define $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ as the bounded operators $x : \mathcal{H} \to \mathcal{K}$ such that $\tilde{x} \in L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$. In this way $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ forms a closed subspace of $L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$.

Note that $\mathrm{HS}(\mathcal{H}, \mathbb{C})$ is the dual Banach space of $\mathcal{H}$, and is naturally anti-isomorphic to $\mathcal{H}$, we denote this isomorphism by $\xi \mapsto \overline{\xi}$. We call this the **conjugate Hilbert space** of $\mathcal{H}$, and denote it by $\overline{\mathcal{H}}$. Note that we have the natural identification $\overline{\overline{\mathcal{H}}} = \mathcal{H}$. Also, we have a natural anti-linear map $x \mapsto \overline{x}$ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\overline{\mathcal{H}})$ given by $\overline{x}\overline{\xi} = \overline{x\xi}$.

If we wish to emphasize that we are considering only the Hilbert space aspects of the Hilbert-Schmidt operators, we often use the notation $\mathcal{H} \overline{\otimes} \mathcal{K}$ for the Hilbert-Schmidt operators $\mathrm{HS}(\mathcal{H}, \overline{\mathcal{K}})$. In this setting we call $\mathcal{H} \overline{\otimes} \mathcal{K}$ the **Hilbert space tensor product** of $\mathcal{H}$ with $\mathcal{K}$. Note that if $\{\xi_i\}_i$ and $\{\eta_j\}_j$ form orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, then $\{\xi_i \otimes \eta_j\}_{i,j}$ forms an orthonormal basis for $\mathcal{H} \overline{\otimes} \mathcal{K}$. We see that the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ of $\mathcal{H}$ and $\mathcal{K}$ can be realized as the subspace of finite rank operators, i.e., we have $\mathcal{H} \otimes \mathcal{K} = \mathrm{sp}\{\xi \otimes \eta \mid \xi \in \mathcal{H}, \eta \in \mathcal{K}\}$.

If $x \in \mathcal{B}(\mathcal{H})$ and $y \in \mathcal{B}(\mathcal{K})$ then we obtain an operator $x \otimes y \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ which is given by $(x \otimes y)h = xh\overline{y}^*$. We then have that $\|x \otimes y\| \leq \|x\|\|y\|$, and $(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$ for all $\xi \in \mathcal{H}$, and $\eta \in \mathcal{K}$.

If (X, μ) is a measure space then we have a particularly nice description of the Hilbert-Schmidt operators on $L^2(X, \mu)$.

Theorem 2.2.3. *For each $k \in L^2(X \times X, \mu \times \mu)$ the integral operator T_k defined by*

$$T_k \xi(x) = \int k(x, y)\xi(y) d\mu(y), \qquad \xi \in L^2(X, \mu),$$

is a Hilbert-Schmidt operator on $L^2(X, \mu)$. Moreover, the map $k \mapsto T_k$ is a unitary operator from $L^2(X \times X, \mu \times \mu)$ to $L^2(\mathcal{B}(L^2(X, \mu)))$. Moreover, if we define $k^*(x, y) = \overline{k(x, y)}$ then we have $T_k^* = T_{k^*}$.

Proof. For all $\eta \in L^2(X, \mu)$, the Cauchy-Schwarz inequality gives

$$\|k(x, y)\xi(y)\eta(x)\|_1 \leq \|k\|_2 \|\xi\|_{L^2(X,\mu)} \|\eta\|_2.$$

This shows that T_k is a well defined operator on $L^2(X, \mu)$ and $\|T_k\| \leq \|k\|_2$. If $\{\xi_i\}_i$ gives an orthonormal basis for $L^2(X, \mu)$ and $k(x, y) = \sum \alpha_{i,j} \xi_i(x)\xi_j(y)$ is a finite sum then for $\eta \in L^2(X, \mu)$ we have

$$T_k \eta = \sum \alpha_{i,j} \langle \xi, \xi_j \rangle \xi_i = (\sum \alpha_{i,j} \xi_i \otimes \overline{\xi_j})\eta.$$

Thus, $\|T_k\|_2 = \| \sum \alpha_{i,j} \xi_i \otimes \overline{\xi_j} \|_2 = \|k\|_2$, which shows that $k \mapsto T_k$ is a unitary operator.

The same formula above also shows that $T_k^* = T_{k^*}$. ∎

2.3 Compact operators

We denote by $\mathcal{H}_1$ the unit ball in $\mathcal{H}$.

Theorem 2.3.1. *For $x \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent:*

(i) $x \in \overline{\mathcal{FR}(\mathcal{H})}^{\|\cdot\|}$.

(ii) *x restricted to $\mathcal{H}_1$ is continuous from the weak to the norm topology.*

(iii) *$x(\mathcal{H}_1)$ is compact in the norm topology.*

(iv) *$x(\mathcal{H}_1)$ has compact closure in the norm topology.*

Proof. (i) $\implies$ (ii) Let $\{\xi_\alpha\}_\alpha$ be net in $\mathcal{H}_1$ which weakly converges to ξ. By hypothesis for every $\varepsilon > 0$ there exists $y \in \mathcal{FR}(\mathcal{H})$ such that $\|x - y\| < \varepsilon$. We then have

$$\|x\xi - x\xi_\alpha\| \leq \|y\xi - y\xi_\alpha\| + 2\varepsilon.$$

Thus, it is enough to consider the case when $x \in \mathcal{FR}(\mathcal{H})$. This case follows easily since then the range of x is then finite dimensional where the weak and norm topologies agree.

(ii) $\implies$ (iii) $\mathcal{H}_1$ is compact in the weak topology and hence $x(\mathcal{H}_1)$ is compact being the continuous image of a compact set.

(iii) $\implies$ (iv) This implication is obvious.

(iv) $\implies$ (i) Let P_α be a net of finite rank projections such that $\|P_\alpha \xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}$. Then $P_\alpha x$ are finite rank and if $\|P_\alpha x - x\| \not\to 0$ then there exists $\varepsilon > 0$, and $\xi_\alpha \in \mathcal{H}_1$ such that $\|x\xi_\alpha - P_\alpha x \xi_\alpha\| \geq \varepsilon$. By hypothesis we may pass to a subnet and assume that $x\xi_\alpha$ has a limit ξ in the norm topology. We then have

$$\begin{aligned}\varepsilon &\leq \|x\xi_\alpha - P_\alpha x\xi_\alpha\| \leq \|\xi - P_\alpha\xi\| + \|(1-P_\alpha)(x\xi_\alpha - \xi)\| \\ &\leq \|\xi - P_\alpha\xi\| + \|x\xi_\alpha - \xi\| \to 0,\end{aligned}$$

which gives a contradiction. ■

If any of the above equivalent conditions are satisfied we say that x is a **compact operator**. We denote the space of compact operators by $\mathcal{K}(\mathcal{H})$. Clearly $\mathcal{K}(\mathcal{H})$ is a norm closed two sided ideal in $\mathcal{B}(\mathcal{H})$.

Exercise 2.3.2. Show that the map $\psi : L^1(\mathcal{B}(\mathcal{H})) \to \mathcal{K}(\mathcal{H})^*$ given by $\psi_x(a) = \mathrm{Tr}(ax)$ implements a Banach space isomorphism between $L^1(\mathcal{B}(\mathcal{H}))$ and $\mathcal{K}(\mathcal{H})^*$.

2.4 Locally convex topologies on the space of operators

Let $\mathcal{H}$ be a Hilbert space. On $\mathcal{B}(\mathcal{H})$ we define the following locally convex topologies:

- The **weak operator topology** (WOT) is defined by the family of seminorms $T \mapsto |\langle T\xi, \eta\rangle|$, for $\xi, \eta \in \mathcal{H}$.
- The **strong operator topology** (SOT) is defined by the family of seminorms $T \mapsto \|T\xi\|$, for $\xi \in \mathcal{H}$.

Note that the from coarsest to finest topologies we have

$$\text{WOT} \prec \text{SOT} \prec \text{Uniform}.$$

Also note that since an operator T is normal if and only if $\|T\xi\| = \|T^*\xi\|$ for all $\xi \in \mathcal{H}$, it follows that the adjoint is SOT continuous on the set of normal operators.

Lemma 2.4.1. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(*i*) *There exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$ such that $\varphi(T) = \sum_{i=1}^n \langle T\xi_i, \eta_i\rangle$, for all $T \in \mathcal{B}(\mathcal{H})$.*

(*ii*) *φ is WOT continuous.*

(*iii*) *φ is SOT continuous.*

Proof. The implications (i) $\implies$ (ii) and (ii) $\implies$ (iii) are clear and so we will only show (iii) $\implies$ (i). Suppose φ is SOT continuous. Thus, the inverse image of the open ball in $\mathbb{C}$ is open in the SOT and hence by considering the semi-norms which define the topology we have that there exists a constant $K > 0$, and $\xi_1, \ldots, \xi_n \in \mathcal{H}$ such that

$$|\varphi(T)|^2 \leq K \sum_{i=1}^{n} \|T\xi_i\|^2.$$

If we then consider $\{\oplus_{i=1}^{n} T\xi_i \mid T \in \mathcal{B}(\mathcal{H})\} \subset \mathcal{H}^{\oplus n}$, and let $\mathcal{H}_0$ be its closure, we have that

$$\oplus_{i=1}^{n} T\xi_i \mapsto \varphi(T)$$

extends to a well defined, continuous linear functional on $\mathcal{H}_0$ and hence by the Riesz representation theorem there exists $\eta_1, \ldots, \eta_n \in \mathcal{H}$ such that

$$\varphi(T) = \sum_{i=1}^{n} \langle T\xi_i, \eta_i \rangle,$$

for all $T \in \mathcal{B}(\mathcal{H})$. ■

Corollary 2.4.2. *Let $K \subset \mathcal{B}(\mathcal{H})$ be a convex set, then the WOT, SOT, and closures of K coincide.*

Proof. By Lemma 2.4.1 the three topologies above give rise to the same dual space, hence this follows from the the Hahn-Banach separation theorem. ■

If $\mathcal{H}$ is a Hilbert space then the map $\mathrm{id} \otimes 1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ defined by $(\mathrm{id} \otimes 1)(x) = x \otimes 1$ need not be continuous in either of the locally convex topologies defined above even though it is an isometric C^*-homomorphism with respect to the uniform topology. Thus, on $\mathcal{B}(\mathcal{H})$ we define the following additional locally convex topologies:

- The **σ-weak operator topology** (σ-WOT) is defined by pulling back the WOT of $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ under the map $\mathrm{id} \otimes 1$.
- The **σ-strong operator topology** (σ-SOT) is defined by pulling back the SOT of $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ under the map $\mathrm{id} \otimes 1$.

Note that the σ-weak operator topology can alternately be defined by the family of semi-norms $T \mapsto |\operatorname{Tr}(Ta)|$, for $a \in L^1(\mathcal{B}(\mathcal{H}))$. Hence, under the identification $\mathcal{B}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))^*$, we have that the weak*-topology on $\mathcal{B}(\mathcal{H})$ agrees with the σ-WOT.

Lemma 2.4.3. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(i) There exists a trace class operator $a \in L^1(\mathcal{B}(\mathcal{H}))$ such that $\varphi(x) = \operatorname{Tr}(xa)$ for all $x \in \mathcal{B}(\mathcal{H})$

(ii) φ is σ-WOT continuous.

(iii) φ *is* σ*-SOT continuous.*

Proof. Again, we need only show the implication (iii) $\implies$ (i), so suppose φ is σ-SOT continuous. Then by the Hahn-Banach theorem, considering $\mathcal{B}(\mathcal{H})$ as a subspace of $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$ through the map $\mathrm{id} \otimes 1$, we may extend φ to a SOT continuous linear functional on $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$. Hence by Lemma 2.4.1 there exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}\overline{\otimes}\ell^2\mathbb{N}$ such that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\varphi(x) = \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle.$$

For each $1 \leq i \leq n$ we may define $a_i, b_i \in \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$ as the operators corresponding to ξ_i, η_i in the Hilbert space isomorphism $\mathcal{H} \otimes \ell^2\mathbb{N} \cong \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$. By considering $a = \sum_{i=1}^{n} b_i^* a_i \in L^1(\mathcal{B}(\mathcal{H}))$, it then follows that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned} \mathrm{Tr}(xa) &= \sum_{i=1}^{n} \langle a_i x, b_i \rangle_2 \\ &= \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle = \varphi(x). \end{aligned}$$ ■

By the Banach-Alaoglu theorem we obtain the following corollary.

Corollary 2.4.4. *The unit ball in* $\mathcal{B}(\mathcal{H})$ *is compact in the* σ*-WOT.*

Corollary 2.4.5. *The WOT and the* σ*-WOT agree on bounded sets.*

Proof. The identity map is clearly continuous from the σ-WOT to the WOT. Since both spaces are Hausdorff it follows that this is a homeomorphism from the σ-WOT compact unit ball in $\mathcal{B}(\mathcal{H})$. By scaling we therefore have that this is a homeomorphism on any bounded set. ■

Exercise 2.4.6. Show that the adjoint $T \mapsto T^*$ is continuous in the WOT, and when restricted to the space of normal operators is continuous in the SOT, but is not continuous in the SOT on the space of all bounded operators.

Exercise 2.4.7. Show that operator composition is jointly continuous in the SOT on bounded subsets.

Exercise 2.4.8. Show that the SOT agrees with the σ-SOT on bounded subsets of $\mathcal{B}(\mathcal{H})$.

Exercise 2.4.9. Show that pairing $\langle x, a \rangle = \mathrm{Tr}(a^* x)$ gives an identification between $\mathcal{K}(\mathcal{H})^*$ and $(L^1(\mathcal{B}(\mathcal{H})), \| \cdot \|_1)$.

2.5 Von Neumann algebras and the double commutant theorem

A **von Neumann algebra** (over a Hilbert space $\mathcal{H}$) is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains 1 and is closed in the weak operator topology.

Note that since subalgebras are of course convex, it follows from Corollary 2.4.2 that von Neumann algebras are also closed in the strong operator topology.

If $A \subset \mathcal{B}(\mathcal{H})$ then we denote by $W^*(A)$ the von Neumann subalgebra which is generated by A, i.e., $W^*(A)$ is the smallest von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ which contains A.

Lemma 2.5.1. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $(A)_1$ is compact in the WOT.*

Proof. This follows directly from Corollary 2.4.4. ∎

Corollary 2.5.2. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then $(A)_1$ and $A_{\text{s.a.}}$ are closed in the weak and strong operator topologies.*

Proof. Since taking adjoints is continuous in the weak operator topology it follows that $A_{\text{s.a.}}$ is closed in the weak operator topology, and by the previous result this is also the case for $(A)_1$. ∎

If $B \subset \mathcal{B}(\mathcal{H})$, the **commutant** of B is

$$B' = \{T \in \mathcal{B}(\mathcal{H}) \mid TS = ST, \text{ for all } S \in B\}.$$

We also use the notation $B'' = (B')'$ for the **double commutant**.

Theorem 2.5.3. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint set, then A' is a von Neumann algebra.*

Proof. It is easy to see that A' is a self-adjoint algebra containing 1. To see that it is closed in the weak operator topology just notice that if $x_\alpha \in A'$ is a net such that $x_\alpha \to x \in \mathcal{B}(\mathcal{H})$ then for any $a \in A$, and $\xi, \eta \in \mathcal{H}$, we have

$$\langle [x,a]\xi, \eta\rangle = \langle xa\xi, \eta\rangle - \langle x\xi, a^*\eta\rangle$$

$$= \lim_{\alpha\to\infty} \langle x_\alpha a\xi, \eta\rangle - \langle x_\alpha \xi, a^*\eta\rangle = \lim_{\alpha\to\infty} \langle [x_\alpha, a]\xi, \eta\rangle = 0.$$ ∎

Corollary 2.5.4. *A self-adjoint maximal abelian subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra.*

Proof. Since A is maximal abelian we have $A = A'$. ∎

Lemma 2.5.5. *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing 1. Then for all $\xi \in \mathcal{H}$, and $x \in A''$ there exists $x_\alpha \in A$ such that $\lim_{\alpha\to\infty} \|(x-x_\alpha)\xi\| = 0$.*

Proof. Consider the closed subspace $\mathcal{K} = \overline{A\xi} \subset \mathcal{H}$, and denote by p the projection onto this subspace. Since for all $a \in A$ we have $a\mathcal{K} \subset \mathcal{K}$, it follows that $ap = pap$. But since A is self-adjoint it then also follows that for all $a \in A$ we have $pa = (a^*p)^* = (pa^*p)^* = pap = ap$, and hence $p \in A'$.

We therefore have that $xp = xp^2 = pxp$ and hence $x\mathcal{K} \subset \mathcal{K}$. Since $1 \in A$ it follows that $\xi \in \mathcal{K}$ and hence also $x\xi \in \overline{A\xi}$. ∎

Theorem 2.5.6 (Von Neumann's double commutant theorem)**.** *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing 1. Then A'' is equal to the weak operator topology closure of A.*

Proof. By Theorem 2.5.3 we have that A'' is closed in the weak operator topology, and we clearly have $A \subset A''$, so we just need to show that $A \subset A''$ is dense in the weak operator topology. For this we use the previous lemma together with a matrix trick.

Let $\xi_1, \ldots, \xi_n \in \mathcal{H}$, $x \in A''$ and consider the subalgebra $\tilde{A}$ of $\mathcal{B}(\mathcal{H}^n) \cong \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$ consisting of diagonal matrices with constant diagonal coefficients contained in A. Then the diagonal matrix whose diagonal entries are all x is easily seen to be contained in $\tilde{A}''$, hence the previous lemma applies and so there exists a net $a_\alpha \in A$ such that $\lim_{\alpha\to\infty} \|(x - a_\alpha)\xi_k\| = 0$, for all $1 \leq k \leq n$. This shows that $A \subset A''$ is dense in the strong operator topology. ∎

We also have the following formulation which is easily seen to be equivalent.

Corollary 2.5.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra. Then A is a von Neumann algebra if and only if $A = A''$.*

Corollary 2.5.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $x \in A$, and consider the polar decomposition $x = v|x|$. Then $v \in A$.*

Proof. Note that $\ker(v) = \ker(|x|)$, and if $a \in A'$ then we have $a\ker(|x|) \subset \ker(|x|)$. Also, we have

$$\|(av - va)|x|\xi\| = \|ax\xi - xa\xi\| = 0,$$

for all $\xi \in \mathcal{H}$. Hence av and va agree on $\ker(|x|) + \overline{R(|x|)} = \mathcal{H}$, and so $v \in A'' = A$. ∎

Proposition 2.5.9. *Let (X, μ) be a probability space. Consider the Hilbert space $L^2(X, \mu)$, and the map $M : L^\infty(X, \mu) \to \mathcal{B}(L^2(X, \mu))$ defined by $(M_g\xi)(x) = g(x)\xi(x)$, for all $\xi \in L^2(X, \mu)$. Then M is an isometric $*$-isomorphism from $L^\infty(X, \mu)$ onto a maximal abelian von Neumann subalgebra of $\mathcal{B}(L^2(X, \mu))$.*

Proof. The fact that M is a $*$-isomorphism onto its image is clear. If $g \in L^\infty(X, \mu)$ then by definition of $\|g\|_\infty$ we can find a sequence E_n of measurable subsets of X such that $0 < \mu(E_n)$, and $|g|_{E_n} \geq \|g\|_\infty - 1/n$, for all $n \in \mathbb{N}$. We then have

$$\|M_g\| \geq \|M_g 1_{E_n}\|_2 / \|1_{E_n}\|_2 \geq \|g\|_\infty - 1/n.$$

The inequality $\|g\|_\infty \leq \|M_g\|$ is also clear and hence M is isometric.

To see that $M(L^\infty(X,\mu))$ is maximal abelian let's suppose $T \in \mathcal{B}(L^2(X,\mu))$ commutes with M_f for all $f \in L^\infty(X,\mu)$. We define $f \in L^2(X,\mu)$ by $f = T(1_X)$.

For each $g, h \in L^\infty(X,\mu)$, we have

$$|\int fg\overline{h}d\mu| = |\langle M_g T(1_X), h\rangle|$$

$$= |\langle T(g), h\rangle| \leq \|T\|\|g\|_2\|h\|_2.$$

Since $L^\infty(X,\mu) \subset L^2(X,\mu)$ is dense in $\|\cdot\|_2$, it then follows from Hölder's inequality that $f \in L^\infty(X,\mu)$, and $T = M_f$. ■

Because of the previous result we will often identify $L^\infty(X,\mu)$ with the subalgebra of $\mathcal{B}(L^2(X,\mu))$ as described above. This should not cause any confusion.

With minor modifications the previous result can be shown to hold for any measure space (X,μ) which is a disjoint union of probability spaces, e.g., if (X,μ) is σ-finite, or if X is arbitrary and μ is the counting measure.

Exercise 2.5.10. Let X be an uncountable set, $\mathcal{B}_1$ the set of all subsets of X, $\mathcal{B}_2 \subset B_1$ the set consisting of all sets which are either countable or have countable complement, and μ the counting measure on X. Show that the identity map implements a unitary operator $\mathrm{id} : L^2(X,\mathcal{B}_1,\mu) \to L^2(X,\mathcal{B}_2,\mu)$, and we have $L^\infty(X,\mathcal{B}_2,\mu) \subsetneq L^\infty(X,\mathcal{B}_2,\mu)'' = \mathrm{id}\, L^\infty(X,\mathcal{B}_1,\mu)\,\mathrm{id}^*$.

2.6 Kaplansky's density theorem

Proposition 2.6.1. *If $f \in C(\mathbb{C})$ then $x \mapsto f(x)$ is continuous in the strong operator topology on any bounded set of normal operators in $\mathcal{B}(\mathcal{H})$.*

Proof. By the Stone-Weierstrass theorem we can approximate f uniformly well by polynomials on any compact set. Since multiplication is jointly SOT continuous on bounded sets, and since taking adjoints is SOT continuous on normal operators, the result follows easily. ■

Proposition 2.6.2 (The Cayley transform)**.** *The map $x \mapsto (x-i)(x+i)^{-1}$ is strong operator topology continuous from the set of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ into the unitary operators in $\mathcal{B}(\mathcal{H})$.*

Proof. Suppose $\{x_k\}_k$ is a net of self-adjoint operators such that $x_k \to x$ in the SOT. By the spectral mapping theorem we have $\|(x_k+i)^{-1}\| \leq 1$ and hence for all $\xi \in \mathcal{H}$ we have

$$\begin{aligned} &\|(x-i)(x+i)^{-1}\xi - (x_k-i)(x_k+i)^{-1}\xi\| \\ &= \|(x_k+i)^{-1}((x_k+i)(x-i) - (x_k-i)(x+i))(x+i)^{-1}\xi\| \\ &= \|2i(x_k+i)^{-1}(x-x_k)(x+i)^{-1}\xi\| \leq 2\|(x-x_k)(x+i)^{-1}\xi\| \to 0. \end{aligned}$$ ■

Corollary 2.6.3. *If $f \in C_0(\mathbb{R})$ then $x \mapsto f(x)$ is strong operator topology continuous on the set of self-adjoint operators.*

Proof. Since f vanishes at infinity, we have that $g(t) = f\left(i\frac{1+t}{1-t}\right)$ defines a continuous function on $\mathbb{T}$ if we set $g(1) = 0$. By Proposition 2.6.1 $x \mapsto g(x)$ is then SOT continuous on the space of unitaries. If $U(z) = \frac{z-i}{z+i}$ is the Cayley transform, then by Proposition 2.6.2 it follows that $f = g \circ U$ is SOT continuous being the composition of two SOT continuous functions. ∎

Theorem 2.6.4 (Kaplansky's density theorem)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ and denote by B the strong operator topology closure of A.*

(*i*) *The strong operator topology closure of $A_{\text{s.a.}}$ is $B_{\text{s.a.}}$.*

(*ii*) *The strong operator topology closure of $(A)_1$ is $(B)_1$.*

Proof. We may assume that A is a C^*-algebra. If $\{x_k\}_k \subset A$ is a net of elements which converge in the SOT to a self-adjoint element x_k, then since taking adjoints is WOT continuous we have that $\frac{x_k+x_k^*}{2} \to x$ in the WOT. But $A_{\text{s.a.}}$ is convex and so the WOT and SOT closures coincide, showing (a). Moreover, if $\{y_k\}_k \subset A_{\text{s.a.}}$ such that $y_k \to x$ in the SOT then by considering a function $f \in C_0(\mathbb{R})$ such that $f(t) = t$ for $|t| \leq \|x\|$, and $|f(t)| \leq \|x\|$, for $t \in \mathbb{R}$, we have $\|f(y_k)\| \leq \|x\|$, for all k and $f(y_k) \to f(x)$ in the SOT by Corollary 2.6.3. Hence $(A)_1 \cap A_{\text{s.a.}}$ is SOT dense in $(B)_1 \cap B_{\text{s.a.}}$.

Note that $\mathbb{M}_2(A)$ is SOT dense in $\mathbb{M}_2(B) \subset \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Therefore if $x \in (B)_1$ then $\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (\mathbb{M}_2(B))_1$ is self-adjoint. Hence from above there exists a net of operators $\tilde{x}_n \in (\mathbb{M}_2(A))_1$ such that $\tilde{x}_n \to \tilde{x}$ in the SOT. Writing $\tilde{x}_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ we then have that $\|b_n\| \leq 1$ and $b_n \to x$ in the SOT. ∎

Corollary 2.6.5. *A self-adjoint unital subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $(A)_1$ is closed in the SOT.*

Corollary 2.6.6. *A self-adjoint unital subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if A is closed in the σ-WOT.*

2.6.1 Preduals

Proposition 2.6.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let $A_* \subset A^*$ be the subspace of σ-WOT continuous linear functionals, then $(A_*)^* = A$ and under this identification the weak*-topology on A agrees with the σ-WOT.*

Proof. By the Hahn-Banach Theorem, and Lemma 2.4.3 we can identify A_* with $L^1(\mathcal{B}(\mathcal{H}))/A_\perp$, where $A_\perp$ is the pre-annihilator

$$A_\perp = \{x \in L^1(\mathcal{B}(\mathcal{H})) \mid \mathrm{Tr}(ax) = 0, \text{ for all } a \in A\}.$$

From the general theory of Banach spaces it follows that $(L^1(\mathcal{B}(\mathcal{H}))/A_\perp)^*$ is canonically isomorphic to the weak* closure of A, which is equal to A by Corollary 2.6.6. The fact that the weak*-topology on A agrees with the σ-WOT is then obvious. ∎

If $A \subset \mathcal{B}(\mathcal{H})$ and $B \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras, then a linear map $\Phi : A \to B$ is said to be **normal** if it is continuous from the σ-WOT of A to the σ-WOT of B.

Exercise 2.6.8. Suppose $A \subset \mathcal{B}(\mathcal{H})$ and $B \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras, and $\Phi : A \to B$ is a bounded linear map. Show that Φ is normal if and only if the dual map $\Phi^* : B^* \to A^*$ given by $\Phi^*(\psi)(a) = \psi(\Phi(a))$ satisfies $\Phi^*(B_*) \subset A_*$.

2.7 Borel functional calculus

If $T \in \mathbb{M}_n(\mathbb{C})$ is a normal matrix, then there are different perspectives one can take when describing the spectral theorem for T. The first, a basis free approach, is to consider the eigenvalues $\sigma(T)$ for T, and to each eigenvalue λ associate to it the projection $E(\lambda)$ onto the corresponding eigenspace. Since T is normal we have that the $E(\lambda)$'s are pairwise orthogonal and we have

$$T = \sum_{\lambda \in \sigma(T)} \lambda E(\lambda).$$

The second approach is to use that since T is normal, it is diagonalizable. We therefore could find a unitary matrix U such that UTU^* is a diagonal matrix with diagonal entries λ_i. If we denote by $E_{i,i}$ the elementary matrix with a 1 in the (i, i) position and 0 elsewhere, then we have

$$T = U^*(\sum_{i=1}^{n} \lambda_i E_{i,i})U.$$

For bounded normal operators there are two similar approaches to the spectral theorem. The first approach is to find a substitute for the projections $E(\lambda)$ and this leads naturally to the notion of a spectral measure. For the second approach, this naturally leads to the interpretation of diagonal matrices corresponding to multiplication by essentially bounded functions on a probability space.

Lemma 2.7.1. *Let $x_\alpha \in \mathcal{B}(\mathcal{H})$ be an increasing net of positive operators such that $\sup_\alpha \|x_\alpha\| < \infty$, then there exists a bounded operator $x \in \mathcal{B}(\mathcal{H})$ such that $x_\alpha \to x$ in the SOT.*

Proof. We may define a quadratic form on $\mathcal{H}$ by $\xi \mapsto \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$. Since $\sup_\alpha \|x_\alpha\| < \infty$ we have that this quadratic form is bounded and hence there exists a bounded positive operator $x \in \mathcal{B}(\mathcal{H})$ such that $\|\sqrt{x}\xi\|^2 = \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$, for all $\xi \in \mathcal{H}$. Note that $x_\alpha \leq x$ for all α, and $\sup_\alpha \|(x - x_\alpha)^{1/2}\| < \infty$. Thus for each $\xi \in \mathcal{H}$ we have

$$\begin{aligned}\|(x - x_\alpha)\xi\|^2 &\leq \|(x - x_\alpha)^{1/2}\|^2 \|(x - x_\alpha)^{1/2}\xi\|^2 \\ &= \|(x - x_\alpha)^{1/2}\|^2 (\|\sqrt{x}\xi\|^2 - \|\sqrt{x_\alpha}\xi\|^2) \to 0.\end{aligned}$$

Hence, $x_\alpha \to x$ in the SOT. ∎

Corollary 2.7.2. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\{p_\iota\}_{\iota \in I} \subset A$ is a collection of pairwise orthogonal projections then $p = \sum_{\iota \in I} p_\iota \in A$ is well defined as a SOT limit of finite sums.*

2.7.1 Spectral measures

Let K be a compact Hausdorff space and let $\mathcal{H}$ be a Hilbert space. A **spectral measure** E on K relative to $\mathcal{H}$ is a mapping from the Borel subsets of K to the set of projections in $\mathcal{B}(\mathcal{H})$ such that

(i) $E(\emptyset) = 0, E(K) = 1$.

(ii) $E(B_1 \cap B_2) = E(B_1)E(B_2)$ for all Borel sets B_1 and B_2.

(iii) For all $\xi, \eta \in \mathcal{H}$ the function

$$B \mapsto E_{\xi,\eta}(B) = \langle E(B)\xi, \eta \rangle$$

is a finite Radon measure on K.

Example 2.7.3. If K is a compact Hausdorff space and μ is a σ-finite Radon measure on K, then the map $E(B) = 1_B \in L^\infty(K, \mu) \subset \mathcal{B}(L^2(K, \mu))$ defines a spectral measure on K relative to $L^2(K, \mu)$.

We denote by $B_\infty(K)$ the space of all bounded Borel functions on K. This is clearly a C^*-algebra with the sup norm.

For each $f \in B_\infty(K)$ it follows that the map

$$(\xi, \eta) \mapsto \int f \, dE_{\xi,\eta}$$

gives a continuous sesqui-linear form on $\mathcal{H}$ and hence it follows that there exists a bounded operator T such that $\langle T\xi, \eta \rangle = \int f \, dE_{\xi,\eta}$. We denote this operator T by $\int f \, dE$ so that we have the formula $\langle (\int f \, dE)\xi, \eta \rangle = \int f \, dE_{\xi,\eta}$, for each $\xi, \eta \in \mathcal{H}$.

Theorem 2.7.4. *Let K be a compact Hausdorff space, let $\mathcal{H}$ be a Hilbert space, and suppose that E is a spectral measure on K relative to $\mathcal{H}$. Then the association*

$$f \mapsto \int f \, dE$$

defines a continuous $*$-homomorphism from $B_\infty(K)$ to $\mathcal{B}(\mathcal{H})$. Moreover, the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, and if $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty$, then $\int f_n dE \to \int f dE$ in the SOT.

Proof. It is easy to see that this map defines a linear contraction which preserves the adjoint operation. If $A, B \subset K$ are Borel subsets, and $\xi, \eta \in \mathcal{H}$, then denoting $x = \int 1_A dE$, $y = \int 1_B dE$, and $z = \int 1_{A \cap B} dE$ we have

$$\begin{aligned} \langle xy\xi, \eta \rangle &= \langle E(A)y\xi, \eta \rangle = \langle E(B)\xi, E(A)\eta \rangle \\ &= \langle E(B \cap A)\xi, \eta \rangle = \langle z\xi, \eta \rangle. \end{aligned}$$

Hence $xy = z$, and by linearity we have that $(\int f\,dE)(\int g\,dE) = \int fg\,dE$ for all simple functions $f, g \in B_\infty(K)$. Since every function in $B_\infty(K)$ can be approximated uniformly by simple functions this shows that this is indeed a $*$-homomorphism.

To see that the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, note that if a commutes with all operators of the form $\int f\,dE$ for $f \in C(K)$ then for all $\xi, \eta \in \mathcal{H}$ we have

$$0 = \langle (a(\int f\,dE) - (\int f\,dE)a)\xi, \eta\rangle = \int f\,dE_{\xi, a^*\eta} - \int f\,dE_{a\xi,\eta}.$$

Thus $E_{\xi,a^*\eta} = E_{a\xi,\eta}$ and hence we have that a also commutes with operators of the form $\int g\,dE$ for any $g \in B_\infty(K)$. Therefore by Theorem 2.5.6 $\int g\,dE$ is contained in the von Neumann algebra generated by the image of $C(K)$.

Now suppose $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty(K)$. For each $\xi, \eta \in \mathcal{H}$ we have

$$\int f_n\,dE_{\xi,\eta} \to \int f\,dE_{\xi,\eta},$$

hence $\int f_n\,dE$ converges in the WOT to $\int f\,dE$. However, since $\int f_n\,dE$ is an increasing sequence of bounded operators with $\| \int f_n\,dE\| \leq \|f\|_\infty$, Lemma 2.7.1 shows that $\int f_n\,dE$ converges in the SOT to some operator $x \in \mathcal{B}(\mathcal{H})$ and we must then have $x = \int f\,dE$. ∎

Theorem 2.7.5. *Let $\mathcal{H}$ be a Hilbert space and suppose $A \subset \mathcal{B}(\mathcal{H})$ is an abelian C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains the identity* 1. *Then there is a unique spectral measure E on $\sigma(A)$ relative to $\mathcal{H}$ such that for all $x \in A$ we have*

$$x = \int \Gamma(x) dE.$$

Proof. For each $\xi, \eta \in \mathcal{H}$ we have that $f \mapsto \langle \Gamma^{-1}(f)\xi, \eta\rangle$ defines a bounded linear functional on $\sigma(A)$ and hence by the Riesz representation therorem there exists a Radon measure $E_{\xi,\eta}$ such that for all $f \in C(\sigma(A))$ we have

$$\langle \Gamma^{-1}(f)\xi, \eta\rangle = \int f dE_{\xi,\eta}.$$

Since the Gelfand transform is a $*$-homomorphism we verify easily that $f dE_{\xi,\eta} = dE_{\Gamma^{-1}(f)\xi,\eta} = dE_{\xi,\Gamma^{-1}(\overline{f})\eta}$.

Thus for each Borel set $B \subset \sigma(A)$ we can consider the sesquilinear form $(\xi, \eta) \mapsto \int 1_B dE_{\xi,\eta}$. We have $|\int f dE_{\xi,\eta}| \leq \|f\|_\infty \|\xi\| \|\eta\|$, for all $f \in C(\sigma(A))$ and hence this sesquilinear form is bounded and there exists a bounded operator $E(B)$ such that $\langle E(B)\xi, \eta\rangle = \int 1_B dE_{\xi,\eta}$, for all $\xi, \eta \in \mathcal{H}$. For all $f \in C(\sigma(A))$ we have

$$\langle \Gamma^{-1}(f)E(B)\xi, \eta\rangle = \int 1_B dE_{\xi,\Gamma^{-1}(\overline{f})\eta} = \int 1_B f dE_{\xi,\eta}.$$

Thus it follows that $E(B)^* = E(B)$, and $E(B')E(B) = E(B' \cap B)$, for any Borel set $B' \subset \sigma(A)$. In particular, $E(B)$ is a projection and E gives a spectral measure on $\sigma(A)$ relative to $\mathcal{H}$. The fact that for $x \in A$ we have $x = \int \Gamma(x) dE$ follows easily from the way we constructed E. ∎

If $\mathcal{H}$ is a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$ is a normal operator, then by applying the previous theorem to the C^*-subalgebra A generated by x and 1, and using the identification $\sigma(A) = \sigma(x)$ we obtain a homomorphism from $B_\infty(\sigma(x))$ to $\mathcal{B}(\mathcal{H})$ and hence for $f \in B_\infty(\sigma(x))$ we may define

$$f(x) = \int f dE.$$

Note that it is straight forward to check that for the function $f(t) = t$ we have

$$x = \int z dE(z).$$

We now summarize some of the properties of this functional calculus which follow easily from the previous results.

Theorem 2.7.6 (Borel functional calculus)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose $x \in A$ is a normal operator, then the Borel functional calculus defined by $f \mapsto f(x)$ satisfies the following properties:*

(i) $f \mapsto f(x)$ is a continuous homomorphism from $B_\infty(\sigma(x))$ into A.

(ii) If $f \in B_\infty(\sigma(x))$ then $\sigma(f(x)) \subset f(\sigma(x))$.

(iii) If $f \in C(\sigma(x))$ then $f(x)$ agrees with the definition given by continuous functional calculus.

Corollary 2.7.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then A is the uniform closure of the span of its projections.*

Proof. By decomposing an operator into its real and imaginary parts it is enough to check this for self-adjoint operators in the unit ball, and this follows from the previous theorem by approximating the function $f(t) = t$ uniformly by simple functions on $[-1, 1]$. ∎

Corollary 2.7.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then the unitary group $\mathcal{U}(A)$ is path connected in the uniform topology.*

Proof. If $u \in \mathcal{U}(A)$ is a unitary and we consider a branch of the log function $f(z) = \log z$, then from Borel functional calculus we have $u = e^{ix}$ where $x = -if(u)$ is self-adjoint. We then have that $u_t = e^{itx}$ is a uniform norm continuous path of unitaries such that $u_0 = 1$ and $u_1 = u$. ∎

Corollary 2.7.9. *If $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then $\mathcal{K}(\mathcal{H})$ is the unique non-zero proper norm closed two sided ideal in $\mathcal{B}(\mathcal{H})$.*

Proof. If $I \subset \mathcal{B}(\mathcal{H})$ is a norm closed two sided ideal and $x \in I \setminus \{0\}$, then for any $\xi \in R(x^*x)$, $\|\xi\| = 1$ we can consider $y = (\xi \otimes \overline{\xi})x^*x(\xi \otimes \overline{\xi}) \in I$ which is a rank one self-adjoint operator with $R(y) = \mathbb{C}\xi$. Thus y is a multiple of $(\xi \otimes \overline{\xi})$ and hence $(\xi \otimes \overline{\xi}) \in I$. For any $\zeta, \eta \in \mathcal{H}$, we then have $\zeta \otimes \overline{\eta} = (\zeta \otimes \overline{\xi})(\xi \otimes \overline{\xi})(\xi \otimes \overline{\eta}) \in I$ and hence I contains all finite rank operators. Since I is closed we then have that $\mathcal{K}(\mathcal{H}) \subset I$.

If $x \in I$ is not compact then for some $\varepsilon > 0$ we have that $\dim(1_{[\varepsilon,\infty)}(x^*x)\mathcal{H}) = \infty$. If we let $u \in \mathcal{B}(\mathcal{H})$ be an isometry from $\mathcal{H}$ onto $1_{[\varepsilon,\infty)}(x^*x)\mathcal{H}$, then we have that $\sigma(u^*x^*xu) \subset [\varepsilon, \infty)$. Hence, $u^*x^*xu \in I$ is invertible which shows that $I = \mathcal{B}(\mathcal{H})$. ∎

Exercise 2.7.10. Suppose that K is a compact Hausdorff space and E is a spectral measure for K relative to a Hilbert space $\mathcal{H}$, show that if $f \in B_\infty(K)$, and we have a decomposition of K into a countable union of pairwise disjoint Borel sets $K = \cup_{n \in \mathbb{N}} B_n$ then we have that

$$\int f dE = \sum_{n \in \mathbb{N}} \int_{B_n} f dE,$$

where the convergence of the sum is in the weak operator topology.

2.8 Abelian von Neumann algebras

Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $\xi \in \mathcal{H}$ is a non-zero vector. Then ξ is said to be **cyclic** for A if $A\xi$ is dense in $\mathcal{H}$. We say that ξ is **separating** for A if $x\xi \neq 0$, for all $x \in A$, $x \neq 0$.

Proposition 2.8.1. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then a non-zero vector $\xi \in \mathcal{H}$ is cyclic for A if and only if ξ is separating for A'.*

Proof. Suppose ξ is cyclic for A, and $x \in A'$ such that $x\xi = 0$. Then $xa\xi = ax\xi = 0$ for all $a \in A$, and since $A\xi$ is dense in $\mathcal{H}$ it follows that $x\eta = 0$ for all $\eta \in \mathcal{H}$. Conversely, if $A\xi$ is not dense, then the orthogonal projection p onto its complement is a nonzero operator in A' such that $p\xi = 0$. ∎

Corollary 2.8.2. *If $A \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann algebra and $\xi \in \mathcal{H}$ is cyclic, then ξ is also separating.*

Proof. Since ξ being separating passes to von Neumann subalgebras and $A \subset A'$ this follows. ∎

Infinite dimensional von Neumann algebras are never separable in the norm topology. For this reason we will say that a von Neumann algebra A is **separable** if A is separable in the SOT. Equivalently, A is separable if its predual A_* is separable.

Proposition 2.8.3. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable von Neumann algebra. Then there exists a separating vector for A.*

Proof. Since A is separable, it follows that there exists a countable collection of vectors $\{\xi_k\}_k \subset \mathcal{H}$ such that $x\xi_k = 0$ for all k only if $x = 0$. Also, since A is separable we have that $\mathcal{H}_0 = \overline{\mathrm{sp}}(A\{\xi_k\}_k)$ is also separable. Thus, restricting A to $\mathcal{H}_0$ we may assume that $\mathcal{H}$ is separable.

By Zorn's lemma we can find a maximal family of non-zero unit vectors $\{\xi_\alpha\}_\alpha$ such that $A\xi_\alpha \perp A\xi_\beta$, for all $\alpha \neq \beta$. Since $\mathcal{H}$ is separable this family must be countable and so we may enumerate it $\{\xi_n\}_n$, and by maximality we have that $\{A\xi_n\}_n$ is dense in $\mathcal{H}$.

If we denote by p_n the orthogonal projection onto the closure of $A\xi_n$ then we have that $p_n \in A'$, hence, setting $\xi = \sum_n \frac{1}{2^n}\xi$ if $x \in A$ such that $x\xi = 0$, then for every $n \in \mathbb{N}$ we have $0 = 2^n p_n x\xi = 2^n x p_n \xi = x\xi_n$ and so $x = 0$ showing that ξ is a separating vector for A. ∎

Corollary 2.8.4. *Suppose $\mathcal{H}$ is separable, if $A \subset \mathcal{B}(\mathcal{H})$ is a maximal abelian self-adjoint subalgebra (masa), then there exists a cyclic vector for A.*

Proof. By Propostion 2.8.3 there exists a non-zero vector $\xi \in \mathcal{H}$ which is separating for A, and hence by Proposition 2.8.1 is cyclic for $A' = A$. ∎

The converse of the previous corollary also holds (without the separability hypothesis), which follows from Proposition 2.5.9, together with the following theorem.

Theorem 2.8.5. *Let $A \subset \mathcal{B}(\mathcal{H})$ be an abelian von Neumann algebra and suppose $\xi \in \mathcal{H}$ is a cyclic vector. Then for any SOT dense C^*-subalgebra $A_0 \subset A$ there exists a Radon probability measure μ on $K = \sigma(A_0)$ with $\mathrm{supp}(\mu) = K$, and an unitary $U : L^2(K,\mu) \to \mathcal{H}$ such that $U^*AU = L^\infty(K,\mu) \subset \mathcal{B}(L^2(X,\mu))$.*

Proof. Fix a SOT dense C^*-algebra $A_0 \subset A$, then by the Riesz representation theorem we obtain a finite Radon measure μ on $K = \sigma(A_0)$ such that $\langle \Gamma(f)\xi, \xi\rangle = \int f d\mu$ for all $f \in C(K)$. Since the Gelfand transform takes positive operator to positive functions we see that μ is a probability measure.

We define a map $U_0 : C(K) \to \mathcal{H}$ by $f \mapsto \Gamma(f)\xi$, and note that $\|U_0(f)\|^2 = \langle \Gamma(\overline{f}f)\xi, \xi\rangle = \int \overline{f}f d\mu = \|f\|_2$. Hence U_0 extends to an isometry $U : L^2(K,\mu) \to \mathcal{H}$. Since ξ is cyclic we have that $A_0\xi \subset U(L^2(K,\mu))$ is dense and hence U is a unitary. If the support of μ were not K then there would exist a non-zero continuous function $f \in C(K)$ such that $0 = \int |f^2| d\mu = \|\Gamma(f)\xi\|^2$, but since by Corollary 2.8.2 we know that ξ is separating and hence this cannot happen.

If $f \in C(K) \subset \mathcal{B}(L^2(K,\mu))$, and $g \in C(K) \subset L^2(K,\mu)$ then we have

$$U^*\Gamma(f)Ug = U^*\Gamma(f)\Gamma(g)\xi = fg = M_f g.$$

Since $C(K)$ is $\|\cdot\|_2$-dense in $L^2(K,\mu)$ it then follows that $U^*\Gamma(f)U = M_f$, for all $f \in C(K)$ and thus $U^*A_0U \subset L^\infty(K,\mu)$. Since A_0 is SOT dense in A we then have that $U^*AU \subset L^\infty(K,\mu)$. But since $x \mapsto U^*xU$ is WOT continuous and $(A)_1$ is compact in the WOT it follows that $U^*(A)_1U = (L^\infty(K,\mu))_1$ and hence $U^*AU = L^\infty(K,\mu)$. ∎

In general, if $A \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann algebra and $\xi \in \mathcal{H}$ is a non-zero vector, then we can consider the projection p onto the $\mathcal{K} = \overline{A\xi}$. We then have $p \in A'$, and $Ap \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann for which ξ is a cyclic vector, thus by the previous result Ap is $*$-isomorphic to $L^\infty(X, \mu)$ for some probability space (X, μ). An application of Zorn's Lemma can then be used to show that A is $*$-isomorphic to $L^\infty(Y, \nu)$ were (Y, ν) is a measure space which is a disjoint union of probability spaces. In the case when A is separable an even more concrete classification will be given below.

Theorem 2.8.6. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there exists a separable compact Hausdorff space K with a Radon probability measure μ on K such that A and $L^\infty(K, \mu)$ are $*$-isomorphic.*

Proof. By Proposition 2.8.3 there exists a non-zero vector $\xi \in \mathcal{H}$ which is separating for A. Thus if we consider $\mathcal{K} = \overline{A\xi}$ we have that restricting each operator $x \in A$ to $\mathcal{K}$ is a C^*-algebra isomorphism and $\xi \in \mathcal{K}$ is then cyclic. Thus, the result follows from Theorem 2.8.5. ■

If $x \in \mathcal{B}(\mathcal{H})$ is normal such that $A = W^*(x)$ is separable, then we may let A_0 be the C^*-algebra generated by x. We then obtain the following alternate version of the spectral theorem.

Corollary 2.8.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $x \in A$ is normal such that $W^*(x)$ is separable, then there exists a Radon measure μ on $\sigma(x)$ and a $*$-homomorphism $f \mapsto f(x)$ from $L^\infty(\sigma(x), \mu)$ into A which agrees with Borel functional calculus. Moreover, we have that $\sigma(f(x))$ is the essential range of f.*

Note that $W^*(x)$ need not be separable in general. For example, $\ell^\infty([0,1]) \subset \mathcal{B}(\ell^2([0,1]))$ is generated by the multiplication operator corresponding to the function $t \mapsto t$.

Lemma 2.8.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there exists a self-adjoint operator $x \in A$ such that $A = \{x\}''$.*

Proof. Since A is separable we have that A is countably generated as a von Neumann algebra. Indeed, just take a countable family in A which is dense in the SOT. By functional calculus we can approximate any self-adjoint element by a linear combination of projections and thus A is generated by a countable collection of projections $\{p_k\}_{k=0}^\infty$.

Define a sequence of self adjoint elements $x_n = \sum_{k=0}^n 4^{-k} p_k$, and let $x = \sum_{k=0}^\infty 4^{-k} p_k$. We denote by $A_0 = \{x\}''$. Define a continuous function $f : [-1, 2] \to \mathbb{R}$ such that $f(t) = 1$ if $t \in [1 - \frac{1}{3}, 1 + \frac{1}{3}]$ and $f(t) = 0$ if $t \leq \frac{1}{3}$, then we have that $f(x_n) = p_0$ for every n and hence by continuity of continuous functional calculus we have $p_0 = f(x) \in A_0$. The same argument shows that $p_1 = f(4(x - p_0)) \in A_0$ and by induction it follows easily that $p_k \in A_0$ for all $k \geq 0$, thus $A_0 = A$. ■

Theorem 2.8.9. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there is a countable (possibly empty) set K such that either A is*

$*$-isomorphic to $\ell^\infty K$, or else A is $*$-isomorphic to $L^\infty([0,1],\lambda) \oplus \ell^\infty K$ where λ is Lebesgue measure.

Proof. Since A is separable we have from Lemma 2.8.8 that as a von Neumann algebra A is generated by a single self-adjoint element $x \in A$.

We define $K = \{a \in \sigma(x) \mid 1_{\{a\}}(x) \neq 0\}$. Since the projections corresponding to elements in K are pairwise orthogonal it follows that K is countable. Further, if we denote by $p_K = \sum_{a \in K} 1_{\{a\}}$ then we have that $Ap_K \cong \ell^\infty K$. Thus, all that remains is to show that if $(1 - p_K) \neq 0$ then $(1 - p_K)A = \{(1 - p_K)x\}'' \cong L^\infty([0,1],\lambda)$.

Set $x_0 = (1 - p_K)x \neq 0$. By our definition of K above we have that $\sigma(x_0)$ has no isolated points. Thus, we can inductively define a sequence of partitions $\{A_k^n\}_{k=1}^{2^n}$ of $\sigma(x_0)$ such that $A_k^n = A_{2k-1}^{n+1} \cup A_{2k}^{n+1}$, and A_k^n has non-empty interior, for all $n > 0$, $1 \leq k \leq 2^n$. If we then consider the elements $y_n = \sum_{k=1}^{\infty} \frac{k}{2^n} 1_{A_k}(x_0)$ then we have that $y_n \to y$ where $0 \leq y \leq 1$, $\{x_0\}'' = \{y\}''$ and every dyadic rational is contained in the spectrum of y (since the space of invertible operators is open in the norm topology), hence $\sigma(y) = [0,1]$.

By Theorem 2.8.6 it then follows that $\{x_0\}'' = \{y\}'' \cong L^\infty([0,1],\mu)$ for some Radon measure μ on $[0,1]$ which has full support and no atoms. If we define the function $\theta : [0,1] \to [0,1]$ by $\theta(t) = \mu([0,t])$ then θ gives a continuous bijection of $[0,1]$, and we have $\theta_*\mu = \lambda$, since both are Radon probability measures such that for intervals $[a,b]$ we have $\theta_*\mu([a,b]) = \mu([\theta^{-1}(a), \theta^{-1}(b)]) = \lambda([a,b])$. The map $\theta^* : L^\infty([0,1],\lambda) \to L^\infty([0,1],\mu)$ given by $\theta^*(f) = f \circ \theta^{-1}$ is then easily seen to be a $*$-isomorphism. ∎

Chapter 3

Types of von Neumann algebras

3.1 Projections

If M is a von Neumann algebra we denote by $\mathcal{P}(M)$ the space of projections. The following proposition we leave as an exercise.

Proposition 3.1.1. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $p \in \mathcal{P}(M)$ or $p \in \mathcal{P}(M')$ then pMp is a von Neumann subalgebra of $\mathcal{B}(p\mathcal{H})$.*

3.1.1 The projection lattice

If $K \subset \mathcal{H}$ is a subset, then we use the notation $[K]$ to denote the orthogonal projection onto the closure of $\mathrm{sp}\, K$.

Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose $\{p_\alpha\}_\alpha \subset \mathcal{P}(M)$ is a family of projections. The **infimum** of the family $\{p_\alpha\}_\alpha$ is $[\cap_\alpha p_\alpha \mathcal{H}]$ and is denoted by $\wedge_\alpha p_\alpha$. The **supremum** of the family is given by $\vee_\alpha p_\alpha = [\sum_\alpha p_\alpha \mathcal{H}] = 1 - \wedge_\alpha (1 - p_\alpha)$.

Given two projections $p, q \in \mathcal{P}(M)$ we say that q is **sub-equivalent** to p and write $p \preceq q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$. The projections $p, q \in \mathcal{P}(M)$ are **equivalent** and we write $p \sim q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. If $p \preceq q$ but $p \not\sim q$ then we write $p \prec q$.

Proposition 3.1.2. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then the relation $p \preceq q$ is a partial ordering on $\mathcal{P}(M)$, and the relation $p \sim q$ is an equivalence relation on $\mathcal{P}(M)$.*

Proof. To show that $\preceq$ is transitive, suppose that $p, q, r \in \mathcal{P}(M)$ such that $p = u^*u$, $uu^* \leq q = v^*v$, and $vv^* \leq r$. Since $uu^* \leq q$ we have

$$qu = q(uu^*)u = (uu^*)u = u.$$

Hence,

$$(vu)^*(vu) = u^*qu = u^*u = p$$

and

$$(vu)(vu)^* \leq vv^* \leq r.$$

The same argument shows that $\sim$ is transitive, and $\sim$ is clearly reflexive and symmetric. ∎

Example 3.1.3. Consider a set X. Then each subset $S \subset X$ determines a closed subspace $\ell^2 S \subset \ell^2 X$ and hence a projection $[\ell^2 S] \in \mathcal{B}(\ell^2 X)$. Any bijection $f : S_1 \to S_2$ determines an isometry $v_f : \ell^2 S_1 \to \ell^2 S_2$ by the formula $v_f(\xi)(s) = \xi(f^{-1}(s))$. Thus, the two projections $[\ell^2 S_1]$ and $[\ell^2 S_2]$ are equivalent in $\mathcal{B}(\ell^2 X)$. Conversely, if $[\ell^2 S_1]$ and $[\ell^2 S_2]$ are equivalent then in particular the spaces have the same dimension and hence there exists a bijection between S_1 and S_2.

We also see similarly that $[\ell^2 S_1] \preceq [\ell^2 S_2]$ in $\mathcal{B}(\ell^2 X)$ if and only if there exists an injective function from S_1 to S_2.

More generally, for any Hilbert space $\mathcal{H}$ and any two projections $p, q \in \mathcal{B}(\mathcal{H})$ we have $p \preceq q$ in $\mathcal{B}(\mathcal{H})$ if and only if $\dim(p\mathcal{H}) \leq \dim(q\mathcal{H})$.

With the above example in mind we see that the following is a generalization of the Cantor-Bernstein-Schröeder theorem in set theory, and the proof is much the same.

Proposition 3.1.4. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, if $p, q \in \mathcal{P}(M)$ such that $p \preceq q$ and $q \preceq p$, then $p \sim q$.*

Proof. Suppose $vv^* \leq p = u^*u$, and $uu^* \leq q = v^*v$. Set $p_0 = p - vv^*$, $q_0 = up_0u^*$, and inductively define a pair of sequences of orthogonal projections $\{p_n\}_n, \{q_n\}_n$, as follows:

$$p_n = vq_{n-1}v^*, \qquad q_n = up_nu^*.$$

We also define the projections

$$p_\infty = p - \sum_{n=0}^\infty p_n, \qquad q_\infty = q - \sum_{n=0}^\infty q_n.$$

By construction we have $(up_n)^*(up_n) = p_n$, and $(up_n)(up_n)^* = q_n$, for every $n \geq 0$. Also, if we consider $v_k = v^*(p - \sum_{n=0}^k p_n)$ then it is easy to check that for $k \geq 0$ we have $vv^* = p - \sum_{n=0}^k p_n$ and $v^*v = q - \sum_{n=0}^{k-1} q_n$. Taking limits as $k \to \infty$ we see that $(vp_\infty)(vp_\infty)^* = p_\infty$ and $(vp_\infty)^*(vp_\infty) = q_\infty$. Hence, considering $w = u(\sum_{n=0}^\infty p_n) + v^*p_\infty$ we have $w^*w = p$ and $ww^* = q$. ∎

Lemma 3.1.5. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, if $\{p_i\}_i$ and $\{q_i\}_i$ are two families of pairwise orthogonal projections in M such that $p_i \preceq q_i$, for each i, then we have $\sum_i p_i \preceq \sum_i q_i$.*

Proof. Suppose $u_i \in M$ such that $u_i^* u_i = p_i$ and $u_i u_i^* = r_i \leq q_i$. By orthogonality we have $u_i^* u_j = u_j u_i^* = 0$ for $i \neq j$, and hence $(\sum_i u_i)^*(\sum_i u_i) = \sum_i p_i$ while $(\sum_i u_i)(\sum_i u_i)^* = \sum_i r_i \leq \sum_i q_i$. ∎

Lemma 3.1.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $x \in M$. Then $[x\mathcal{H}], [x^*\mathcal{H}] \in M$ and $[x\mathcal{H}] \sim [x^*\mathcal{H}]$.*

Proof. If we consider the polar decomposition $x = v|x|$. Then we see easily that $vv^* = [x\mathcal{H}]$, while $v^*v = [x^*\mathcal{H}]$. Since $v \in M$ the result folows. ∎

Proposition 3.1.7 (Kaplansky's formula)**.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $p, q \in \mathcal{P}(M)$ then*

$$p \vee q - p \sim q - p \wedge q.$$

Proof. If we consider $x = (1-p)q$ then we have $\ker(x) = \ker(q) \oplus (q\mathcal{H} \cap p\mathcal{H})$, hence $[x^*\mathcal{H}] = 1 - (1 - q + q \wedge p) = q - q \wedge p$. By symmetry it then follows that

$$\begin{aligned} [x\mathcal{H}] &= (1-p) - (1-p) \wedge (1-q) \\ &= (1-p) - (1 - p \vee q) = p \vee q - p. \end{aligned}$$

The result then follows from Lemma 3.1.6 ∎

If $x \in M$, the **central support** of x is the infimum of all central projections $z \in \mathcal{Z}(M)$ such that $zx = xz = x$. we denote the central support of x by $z(x)$. Two projections p, and q are **centrally orthogonal** if $z(p)z(q) = 0$.

Lemma 3.1.8. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p \in \mathcal{P}(M)$ be a projection, then the central support of p is*

$$z = \vee_{x \in M}[xp\mathcal{H}] = [Mp\mathcal{H}].$$

Proof. By considering $x = 1$ we see that $p \leq z$, and z is central since the range of z is clearly invariant to all operators in M. Thus, $z(p) \leq z$. We also have that the range of $z(p)$ is invariant to all operators in M and since $p \leq z(p)$ it follows that any operator in M maps the range of p into the range of $z(p)$. Thus $[xp\mathcal{H}] \leq z(p)$ for all $x \in M$ and hence $z \leq z(p)$. ∎

Proposition 3.1.9. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p, q \in \mathcal{P}(M)$ be two projections, then the following are equivalent:*

(*i*) *p and q are centrally orthogonal.*

(*ii*) *$pMq = \{0\}$.*

(*iii*) *There does not exist nonzero projections $p_0 \leq p$, and $q_0 \leq q$ such that $p_0 \sim q_0$.*

Proof. The equivalence between (i) and (ii) follows easily from the previous lemma. Indeed, for all $x \in M$ we have $[pxq\mathcal{H}] \leq p \leq z(p)$, and by the previous lemma we have $[pxq\mathcal{H}] \leq z(q)$, thus if p and q are centrally orthogonal then we have $pxq = 0$ for all $x \in M$. Conversely, if $pMq = \{0\}$ then we have $pz(q) = 0$ and since $z(q)$ is central it follows that $z(p) \leq 1 - z(q)$.

To see that (ii) and (iii) are equivalence note that if $x \in M$ such that $[pxq\mathcal{H}] \neq 0$ then $[pxq\mathcal{H}] \leq p$ and $[pxq\mathcal{H}] \sim [qx^*p\mathcal{H}] \leq q$. Conversely, if $p_0 \leq p$ is a nonzero projection and $v \in M$ such that $v^*v = p$ and $vv^* \leq q$ then $v^* = pv^*q \in pMq \setminus \{0\}$. ■

Theorem 3.1.10 (The comparison theorem)**.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, if $p, q \in \mathcal{P}(M)$ then there exists a central projection $z \in \mathcal{P}(\mathcal{Z}(M))$ such that*

$$zp \preceq zq, \quad \text{and} \quad (1-z)q \preceq (1-z)p.$$

Proof. By Zorn's lemma there exists a maximal families $\{p_\alpha\}_{\alpha \in I}$ and $\{q_\alpha\}_{\alpha \in I}$ of pairwise orthogonal projections such that $p_0 = \sum_\alpha p_\alpha \leq p$, $q_0 = \sum_\alpha q_\alpha \leq q$, and $p_\alpha \sim q_\alpha$ for all $\alpha \in I$. If we let z_1 be the central support of $p - p_0$, and z_2 be the central support of $q - q_0$ then by Proposition 3.1.9 we have $z_1 z_2 = 0$, and hence $p - p_0 \leq z_1 \leq 1 - z_2$. Thus, $(p - p_0)z_2 = 0$ and since $p_0 \sim q_0$ implies $p_0 z_2 \sim q_0 z_2$ we have

$$pz_2 = p_0 z_2 \sim q_0 z_2 \leq q z_2,$$

while

$$q(1 - z_2) = q_0(1 - z_2) \sim p_0(1 - z_2) \leq p(1 - z_2).$$ ■

A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is a **factor** if it has trivial center, i.e., $\mathcal{Z}(M) = \mathbb{C}$.

Corollary 3.1.11. *If $M \subset \mathcal{B}(\mathcal{H})$ is a factor and $p, q \in \mathcal{P}(M)$ then exactly one of the following is true.*

$$p \prec q, \quad p \sim q, \quad q \prec p.$$

Proof. Since M is a factor the only central projections are 0, or 1. Hence, by the comparison theorem we have $p \preceq q$ or $q \preceq p$. If both occur then by Proposition 3.1.4 we have $p \sim q$. ■

3.2 Types of projections

Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A projection $p \in \mathcal{P}(M)$ is said to be

- **minimal** if $p \neq 0$, and the only subprojections are 0 and p, or equivalently if $\dim(pMp) = 1$.
- **abelian** if pMp is abelian.
- **finite** if $q \leq p$ and $q \sim p$ implies $q = p$.

- **semi-finite** if there are pairwise orthogonal finite projections $p_\alpha \in \mathcal{P}(M)$ such that $p = \sum_\alpha p_\alpha$.
- **purely infinite** if $p \neq 0$ and there is no non-zero finite projection $q \leq p$.
- **properly infinite** if $p \neq 0$ and zp is not-finite for any non-zero central projection $z \in \mathcal{P}(M)$.

We say that M is finite, semi-finite, purely infinite, or properly infinite depending on whether or not 1 has the corresponding property in M.

Note that we have the trivial implications:

$$\text{minimal} \implies \text{abelian} \implies \text{finite} \implies \text{semi}-\text{finite} \implies \text{not purely infinite},$$

and also

$$\text{purely infinite} \implies \text{properly infnite}.$$

Note also that M is finite if and only if the only isometries in M are unitary. Thus, $\mathcal{B}(\mathcal{H})$ is finite if and only if $\mathcal{H}$ is finite dimensional. When $\mathcal{H}$ is infinite dimensional we can fix an orthonormal basis $\{\xi_\alpha\}_\alpha$ and we have $1 = \sum_\alpha [\mathbb{C}\xi_\alpha]$, hence in this case $\mathcal{B}(\mathcal{H})$ is semi-finite. The same argument shows that for the von Neumann algebra $\mathcal{B}(\mathcal{H})$, every projection is semi-finite.

Lemma 3.2.1. *Let $\{p_\alpha\}_\alpha$ be a family of centrally orthogonal projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$. If each projection p_α is abelian (resp. finite) then $p = \sum_\alpha p_\alpha$ is also abelian (resp. finite).*

Proof. If each p_α is abelian then since they are centrally orthogonal for $\alpha \neq \beta$ and $x, y \in M$ we have $(p_\alpha x p y p_\beta) = 0$. Hence

$$(pxp)(pyp) = \sum_\alpha p_\alpha x p_\alpha y p_\alpha = (pyp)(pxp),$$

thus p is abelian.

If each p_α is finite and $u \in M$ such that $uu^* \leq u^*u = p$. Then for all α we have $z(p_\alpha)u^*uz(p_\alpha) = p$ and $uz(p_\alpha)u^* = z(p_\alpha)uu^* \leq p_\alpha$. Thus $uz(p_\alpha)u^* = p_\alpha$ for each α and hence

$$uu^* = uz(p)u^* = \sum_\alpha uz(p_\alpha)u^* = p. \qquad \blacksquare$$

Proposition 3.2.2. *Let p, q be non-zero projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ such that $p \preceq q$. If q is finite (resp. purely infinite), then p is also finite (resp. purely infinite).*

Proof. For the case when q is finite let us suppose that $p \sim q$ and let $v \in M$ be such that $v^*v = p$ and $vv^* = q$. Suppose $u \in M$ such that $u^*u = p$, and $uu^* \leq p$, then we have $(vuv^*)^*(vuv^*) = q$ and $(vuv^*)(vuv^*)^* \leq q$. Therefore $(vuv^*)(vuv^*)^* = q$ and hence $uu^* = p$.

Next suppose that $p \leq q$. If $u^*u = p$ and $uu^* \leq p$ then setting $w = u + (q - p)$ we have $w^*w = q$ and $ww^* \leq q$. Therefore, $uu^* + (q - p) = ww^* = q$ and hence

$uu^* = p$. In general, if $p \preceq q$ then there exists $q_0 \leq q$ such that $p \sim q_0 \leq q$ and the result follows.

Since projections are purely infinite when they have no non-zero finite subprojections, the purely infinite case follows from the finite case. ■

Proposition 3.2.3. *A projection $p \in \mathcal{P}(M)$ in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is semi-finite if and only if p is the supremum of finite projections. In particular, a supremum of semi-finite projections is again semi-finite.*

Proof. If p is semi-finite then is is a sum (and hence also a supremum) of a family of pairwise orthogonal finite projections. Conversely, if $p = \vee_\alpha p_\alpha$ where each p_α is finite. Then let $\{q_\beta\}_\beta$ be a maximal family of pairwise orthogonal finite subprojections of p. If $q_0 = p - \sum_\beta q_\beta \neq 0$ then there exists some p_α such that p_α and q_0 are not orthogonal, and hence not centrally orthogonal. By Proposition 3.1.9 there then exists a non-zero subprojection $\tilde{q}_0 \leq q_0$ such that $q_0 \preceq p_\alpha$ and hence is finite is finite by Proposition 3.2.2, contradicting maximality of the family $\{q_\beta\}_\beta$. ■

Corollary 3.2.4. *Let p be projection in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$, if p is semi-finite (resp. purely infinite) then the central support $z(p)$ is also semi-finite (resp. purely infinite).*

Proof. The central support is the supremum over all equivalent projections and hence from the previous corollary this proves the case when p is semi-finite. It follows from Propositions 3.1.9 and 3.2.2 that a non-zero projection is purely infinite if and only if it is centrally orthogonal to every semi-finite projection which then proves the corollary in this case. ■

Corollary 3.2.5. *Let p, q be non-zero projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ such that $p \preceq q$. If q is semi-finite then p is also semi-finite.*

Proof. By Corollary 3.2.4 it is enough to consider the case when $q \in \mathcal{Z}(M)$ in which case we have $p \leq q$. Let p_0 be the maximal semi-finite subprojection of p (i.e., p_0 is the supremum of all finite subprojections of p). Since q is semi-finite, it is the supremum of all its finite subprojections. Thus, since $z(p - p_0) \leq q = z(q)$, if $p - p_0$ were not zero then there would exist a non-zero finite subprojection of q which would be equivalent to a subprojection of $p - p_0$, contradicting our definition of p_0. Therefore we have that p is the supremum if its finite subprojections and hence is semi-finite. ■

Lemma 3.2.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a properly infinite von Neuman algebra, then there exists a projection $p \in \mathcal{P}(M)$ such that $p \sim 1 - p \sim 1$.*

Proof. By hypothesis there exists $u \in M$ such that $uu^* < u^*u = 1$. Set $p_0 = 1 - uu^*$, then $p_n = u^n p_0 (u^n)^*$ is a pairwise orthogonal family of equivalent projections. Let $\{q_\iota\}_\iota$ be a maximal family of pairwise orthogonal equivalent projections in M which extends the family $\{p_n\}_n$, and consider $q_0 = 1 - \sum_\iota q_\iota$.

By the comparison theorem there exists a central projection z such that $q_0 z \preceq q_{\iota_0} z$, and $q_{\iota_0}(1-z) \preceq q_0(1-z)$. If $z = 0$ then $q_{\iota_0} \preceq q_0$ contradicting the maximality of $\{q_\iota\}_\iota$, thus $z \neq 0$ and we have

$$z = q_0 z + \sum_\iota q_\iota z \preceq q_{\iota_0} z + \sum_{\iota \neq \iota_0} q_\iota z = \sum_\iota q_\iota z \leq z.$$

Thus, $z \sim \sum_\iota q_\iota z$ by the Cantor-Bernstein-Schröeder theorem for projections. By decomposing $\{q_\iota\}_\iota$ into two infinite sets we construct two projections p and $z - p$ such that $p \sim z - p \sim z$.

Consider $\{r_j\}_j$ a maximal family of centrally orthogonal projections such that $r_j \sim z(r_j) - r_j \sim z(r_j)$, then the argument above shows that $\sum_j z(r_j) = 1$ and hence setting $p = \sum_j r_j$ finishes the proof. ∎

Proposition 3.2.7. *Let p, q be finite projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$, then $p \vee q$ is also finite.*

Proof. By Kaplansky's formula we have $p \vee q - p \sim q - p \wedge q \leq q$, and thus we may replace q by $p \vee q - p$ and assume that p and q are orthogonal. We will also assume that $p + q = 1$ by considering the von Neumann algebra $(p+q)M(p+q)$.

Let z_0 be the supremum of all central finite projections. By Lemma 3.2.1 z_0 is a finite projection and thus either $z_0 = 1$ in which case the proof is finished, or else by considering $(1 - z_0)p$ and $(1 - z_0)q$ we may assume $z_0 = 0$, i.e., we may assume that M is properly infinite.

Then by Lemma 3.2.6 there exists a projection $r \in \mathcal{P}(M)$ such that $r \sim 1 - r \sim 1$. By the comparison theorem there then exists $z \in \mathcal{P}(\mathcal{Z}(M))$ such that

$$z(p \wedge r) \preceq z(q \wedge (1 - r)), \quad \text{and} \quad (1 - z)(q \wedge (1 - r)) \preceq (1 - z)(p \wedge r).$$

Then $zr \sim z(1 - r) \sim z$ and

$$z(p \wedge r) = zp \wedge zr \preceq z(1 - r) \wedge zq.$$

Using Kaplansky's formula and Lemma 3.1.5 we then have

$$zr = z(r - r \wedge p) + z(r \wedge p) \preceq z(r \vee p - p) + z(q \wedge (1 - r)) = zq, \tag{3.1}$$

which implies that $z \sim zr = 0$, since zq is finite.

Thus, $q \wedge (1 - r) \preceq p \wedge r$, and replacing the roles of p with q, and of r with $1 - r$ in (3.1) we then have

$$\begin{aligned} 1 - r &= ((1 - r) - (1 - r) \wedge q) + ((1 - r) \wedge q) \\ &\preceq ((1 - r) \vee q - q) + (p \wedge r) = p, \end{aligned}$$

which gives a contradiction since p is finite. ∎

Proposition 3.2.8. *Let p and q be finite equivalent projections in a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$, then $1 - p$ and $1 - q$ are also equivalent. Thus, there exists a unitary operator $u \in M$ such that $upu^* = q$.*

Proof. By Proposition 3.2.7 we have that $p \vee q$ is finite, hence by considering $(p \vee q)M(p \vee q)$ we may assume that M is finite. By the comparison theorem there exists a central projection $z \in M$, and projections p_1, and q_1 such that

$$(1-p)z \sim q_1 \leq (1-q)z, \quad \text{and} \quad (1-q)(1-z) \sim p_1 \leq (1-p)(1-z).$$

Then,

$$z = (1-p)z + pz \sim q_1 + qz \leq (1-q)z + qz = z,$$

and

$$(1-z) = (1-q)(1-z) + q(1-z) \sim p_1 + p(1-z) \leq (1-z).$$

Since, z and $(1-z)$ are finite this implies that $q_1 = (1-q)z$ and $p_1 = (1-p)(1-z)$, and so $1-q \sim 1-p$. ∎

A projection $p \in \mathcal{P}(M)$ is **countably decomposable** if every family of non-zero pairwise orthogonal subprojections is countable. A von Neumann algebra is countably decomposable if the identity projection is. Note that separable von Neumann algebras are always countably decomposable. Also, note that if p is countably decomposable and $q \leq p$ then so is q.

Proposition 3.2.9. *Let $p, q \in \mathcal{P}(M)$ be properly infinite projections and suppose that M is countably decomposable. If $z(p) \leq z(q)$ then $p \preceq q$.*

Proof. By the comparison theorem we may assume that $q \preceq p$, and hence we may assume $q \leq p$. By considering pMp we may also assume $p = 1$.

By Lemma 3.2.6 there exists a subprojection $q_0 \leq q$ such that $q_0 \sim q - q_0 \sim q$. Take $u \in M$ such that $u^*u = q$ and $uu^* = q - q_0$. Setting $q_n = u^n q_0 (u^n)^*$ we obtain a family of pairwise orthogonal equivalent projections. Let $\{r_n\}_n$ be a maximal family of pairwise orthogonal projections such that $r_n \preceq q$. Since M is countably decomposable we have that $\{r_n\}_n$ is countable and by maximality we have that $1 - \sum_n r_n$ and q are centrally orthogonal, thus $\sum_n r_n = 1$ since $z(q) = z(p) = 1$. Hence,

$$1 = \sum_n r_n \preceq \sum_{n=0}^{\infty} q_n \leq q,$$

and so $q \sim 1$ by the Cantor-Bernstein-Schröeder theorem for projections. ∎

Corollary 3.2.10. *If $M \subset \mathcal{B}(\mathcal{H})$ is a countably decomposable factor, than any two infinite projections in M are equivalent.*

Exercise 3.2.11. Suppose that $M \subset \mathcal{B}(\mathcal{H})$ is a factor which is either finite, or is countably decomposable and purely infinite. Show that M is algebraically simple. I.e., the only two-sided ideals are $\{0\}$, or M.

Exercise 3.2.12. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and consider a non-zero vector $\xi \in \mathcal{H}$. Let p be the supremum of all projections q such that $q\xi = 0$. Show that $1 - p$ is countably decomposable.

Exercise 3.2.13. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and set $I = \{x \in M \mid [x\mathcal{H}]$ is countably decomposable.$\}$. Show that I is a norm closed 2-sided ideal and conclude that all finite factors are countably decomposable.

Exercise 3.2.14. Suppose that $M \subset \mathcal{B}(\mathcal{H})$ is a semi-finite factor which is not finite. Show that the set of elements whose support projection is finite forms a two-sided ideal $\mathbb{K}_0(M)$, whose closure $\mathbb{K}(M)$ does not equal M.

3.3 Type decomposition

A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is of **type** I if every non-zero projection has a non-zero abelian subprojection. M is of **type** II if it is semi-finite and has no non-zero abelian projections, if M is also finite then M is of type II_1, if M is properly infinite then M is of type II_∞. M is of **type** III if it has no non-zero finite projections.

Theorem 3.3.1. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then there exists unique projections $P_I, P_{II_1}, P_{II_\infty}$, and P_{III} in $\mathcal{Z}(M)$ such that MP_I, MP_{II_1}, MP_{II_∞}, and MP_{III} are of type I, II_1, II_∞, and III respectively, and such that $P_I + P_{II_1} + P_{II_\infty} + P_{III} = 1$.*

Proof. Let P_I be the supremum of all abelian projections in M. Since the family of abelian projections is closed under conjugation by unitaries it follows that $uP_Iu^* = P_I$ for all unitaries in M and since every operator is a linear combination of 4 unitaries it then follows that P_I is central. If $q \leq P_I$, $q \neq 0$, then since P_I is central it follows that for some abelian projection $r \in M$ we have that $z(r)z(q) \neq 0$ and hence by Proposition 3.1.9 there exists a non-zero subprojection $r_0 \leq r$ such that $r_0 \preceq q$, thus q has an abelian subprojection. Therefore, MP_I is type I, and $1 - P_I$ has no abelian subprojections.

Let P_{II_1} be the supremum of all finite central projections $p \in M$ such that $p \leq 1 - P_I$. Then MP_{II_1} is finite and $M(1 - P_I - P_{II_1})$ has no non-zero finite central subprojections.

Let P_{II_∞} be the supremum of all finite projections $p \in M$ such that $p \leq 1 - P_I - P_{II_1}$. Then since the family of finite projections is closed under conjugation by unitaries we again see that P_{II_∞} is central. By Proposition 3.2.3 we have that P_{II_∞} is semi-finite and has no non-zero finite central subprojections, hence MP_{II_∞} is type II_∞. By definition of P_{II_1} and P_{II_∞} we then have that there are no non-zero finite subprojections of $P_{III} = 1 - P_I - P_{II_1} - P_{II_\infty}$ and hence MP_{III} is type III.

To see that this decomposition is unique suppose that $Q_I + Q_{II_1} + Q_{II_\infty} + Q_{III} = 1$ gives another such decomposition. Since MQ_{III} is type III we have that Q_{III} is purely infinite, and from Proposition 3.2.2 it follows that $Q_{III}P_I = Q_{III}P_{II_1} = Q_{III}P_{II_\infty} = 0$, hence $Q_{III} \leq P_{III}$ and the same reasoning shows that $P_{III} \leq Q_{III}$.

Similarly, since $Q_{II_1} + Q_{II_\infty}$ has no abelian subprojections, we have that $(Q_{II_1} + Q_{II_\infty})P_I = 0$, hence $Q_{II_1} + Q_{II_\infty} \leq P_{II_1} + P_{II_\infty}$ and by symmetry $P_{II_1} + P_{II_\infty} \leq Q_{II_1} + Q_{II_\infty}$. Therefore, $P_I = Q_I$ as well.

Since Q_{II_1} is finite and central, and P_{II_∞} is properly infinite we have that $Q_{II_1}P_{II_\infty} = 0$. Similarly we have $P_{II_1}Q_{II_\infty} = 0$ and hence $P_{II_1} = Q_{II_1}$ and $P_{II_\infty} = Q_{II_\infty}$. ■

Corollary 3.3.2. *A factor is either of type I, type II_1, type II_∞, or type III.*

Proposition 3.3.3. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $p \in \mathcal{P}(M)$ such that $z(p) = 1$, then*

1. *M is type I if and only if pMp is type I;*
2. *M is type II if and only if pMp is type II;*
3. *M is type III if and only if pMp is type III.*

Proof. If M is type I then clearly pMp is type I since every projection in pMp is also a projection in M. Conversely, if pMp is type I and $q \in \mathcal{P}(M)$ is a non-zero projection, then by Proposition 3.1.9 there exists $v \in M$ such that $v \neq 0$, $v^*v \leq q$, and $vv^* \leq p$. Since pMp is type I we then have that there is a minimal projection $p_0 \leq vv^*$, and hence $v^*p_0v \leq q$ is also a minimal projection. Hence M is type I.

Essentially the same argument also works in the type II and type III cases by considering finite subprojections rather than minimal subprojections. ■

Theorem 3.3.4. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a cyclic vector $\xi \in \mathcal{H}$. Then for each $\eta \in \mathcal{H}$ there $x, y \in M$ with $x \geq 0$, and $\zeta \in \overline{x\mathcal{H}}$, such that $x\zeta = \xi$ and $y\zeta = \eta$.*

Proof. Fix $\eta \in \mathcal{H}$, and assume $\|\xi\|, \|\eta\| \leq 1$. Since ξ is cyclic there exists a sequence $x_n \in M$ such that $\|\eta - \sum_{n=1}^k x_n\xi\| \leq 4^{-k}$ for each $k \in \mathbb{N}$. Then $h_k^2 = 1 + \sum_{n=1}^k 4^n x_n^* x_n$ defines an increasing sequence in M. By Proposition 1.4.6 h_k^{-1} is then a decreasing sequence of positive elements and so by Lemma 2.7.1 must converge in the strong operator topology to a limit x.

For $k \in \mathbb{N}$ we have

$$\|h_k\xi\|^2 = \langle h_k^2\xi, \xi\rangle = \|\xi\|^2 + \sum_{n=1}^k 4^n \|x_n\xi\|^2 \leq 1 + 2\sum_{n=1}^k 4^{-n} < 2.$$

Therefore $\{h_k\xi\}_k$ is a bounded sequence and so must have a weak cluster point $\zeta \in \mathcal{H}$.

To see that $x\zeta = \xi$, fix $\xi_0 \in \mathcal{H}$, and $\varepsilon > 0$, and consider $k \in \mathbb{N}$ such that $\|\zeta - h_k\xi\| < \frac{\varepsilon}{2}$ and $\|(x - h_k^{-1})\xi_0\| < \frac{\varepsilon}{4}$.

Then

$$\begin{aligned}
|\langle x\zeta - \xi, \xi_0\rangle| &< |\langle h_k\xi, x\xi_0\rangle - \langle \xi, \xi_0\rangle| + \frac{\varepsilon}{2} \\
&= |\langle h_k\xi, x\xi_0 - h_k^{-1}\xi_0\rangle| + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{4}\|h_k\xi\| + \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

so that $x\zeta = \xi$.

For $m > k$ we have

$$0 \leq h_m^{-1} 4^k x_k^* x_k h_m^{-1} \leq h_m^{-1}(1 + \sum_{n=1}^{m} 4^n x_n^* x_n) h_m^{-1} = 1,$$

and since $h_m^{-1} 4^k x_k^* x_k h_m^{-1}$ is strong operator convergent to $4^k x x_k^* x_k x$ we then have $\|x_k x\|^2 = \|x^* x_k^* x_k x\| \leq 4^{-k}$. Therefore, $\sum_{n=1}^{\infty} x_k x$ converges in norm to an operator y, and we have $y\zeta = \sum_{n=1}^{\infty} x_k x\zeta = \sum_{n=1}^{\infty} x_k \xi = \eta$.

Since $\ker(y) \subset \ker(x)$ we can replace ζ by $[x\mathcal{H}]\zeta$ if needed. ∎

Proposition 3.3.5. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\xi, \eta \in \mathcal{H}$ are two cyclic vectors, then $[M'\xi] \sim [M'\eta]$.*

Proof. Let $\xi, \eta \in \mathcal{H}$ be cyclic vectors. By the previous theorem there exists $x, y \in M$ with $x \geq 0$, and $\zeta \in \overline{x\mathcal{H}}$ such that $x\zeta = \xi$ and $y\zeta = \eta$. Set $p = [M'\zeta]$.

Since $\zeta \in \overline{x\mathcal{H}}$, and $p\zeta = \zeta$, we then have $\zeta \in \overline{px\mathcal{H}}$. Hence $p \leq [M'px\mathcal{H}] \leq [px\mathcal{H}] \leq [p\mathcal{H}] = p$ and so $p = [px\mathcal{H}] \sim [xp\mathcal{H}] = [xM'\zeta] = [M'\xi]$.

On the other hand, we have $[M'\eta] = [yM'\zeta] = [yp\mathcal{H}] \sim [py\mathcal{H}] \leq p \sim [M'\xi]$. The result then follows by symmetry. ∎

Proposition 3.3.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose $\xi, \eta \in \mathcal{H}$. Then $[M'\xi] \preceq [M'\eta]$ in M if and only if $[M\xi] \preceq [M\eta]$ in M'.*

Proof. We will first show that $[M'\xi] \sim [M'\eta]$ in M if and only if $[M\xi] \sim [M\eta]$ in M'. For this, suppose $v \in M$ such that $v^*v = [M'\xi]$ and $vv^* = [M'\eta]$. Then $[M'v\xi] = [vM'\xi] = [M'\eta]$, and $[Mv\xi] \leq [M\xi] = [Mv^*v\xi] \leq [Mv\xi]$. Thus, replacing ξ with $v\xi$ we may assume that $[M'\xi] = [M'\eta]$.

Then the central support z of $[M\eta]$ is $[M'M\eta] = [MM'\eta] = [M'M\xi]$, and so by considering Mz we may assume that all projections $[M\xi]$, and $[M\eta]$ in M', and $p_0 = [M'\xi] = [M'\eta]$ in M, have central support equal to 1.

In particular then $x \mapsto xp_0$ gives an isomorphism from M' onto $M'p_0$, and so $[M\xi]$, and $[M\eta]$ are equivalent in M' if and only if $[M\xi]p_0$, and $[M\eta]p_0$ are equivalent in $M'p_0$. But $p_0\xi$ and $p_0\eta$ are cyclic vectors for $M'p_0$ and so by the previous proposition we have the equivalence.

For the general case, if $[M'\xi] \sim q \leq [M'\eta]$, then $q = [M'q\eta]$ and hence from above we have $[M\xi] \sim [Mq\eta] \leq [M\eta]$. ∎

Lemma 3.3.7. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra and suppose that M has a cyclic and separating vector, then M' is finite.*

Proof. Suppose $\xi \in \mathcal{H}$ is a cyclic and separating vector, and that M' is not finite. If $q \in \mathcal{P}(M')$ is a maximal central finite projection, then $(1-q)\xi$ is a cyclic and separating vector for $M(1-q)$, and replacing M with $M(1-q)$ we will assume that M' is properly infinite.

If M were abelian then M would be maximal abelian since it has a cyclic and separating vector (see the remark before Theorem 2.8.5), contradicting the fact that M' is properly infinite. Thus, there exists a projection $p \in \mathcal{P}(M)$ such that $p < z(p) = 1$.

Let $q = [Mp\xi] \in M'$. Since M' has a separating vector it is countably decomposable (see Exercise 3.2.12), and hence by Proposition 3.2.9 we have $q \sim z(q) = [M'Mp\xi] = [MpM'\xi] = [Mp\mathcal{H}] = z(p) = 1$.

Thus $[Mp\xi] \sim 1 = [M\xi]$ in M' and so by Proposition 3.3.6 we have $p = [pM'\xi] = [M'p\xi] \sim [M'\xi] = 1$ in M, showing that M is not finite. ∎

Lemma 3.3.8. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $p \in \mathcal{P}(M)$, and $q \in \mathcal{P}(M')$. Then $(pMpq)' \cap \mathcal{B}(pq\mathcal{H}) = qM'qp$.*

Proof. It's sufficient to check in the cases when either $p = 1$ or $q = 1$. When $p = 1$ then clearly $qM'q \subset (Mq)' \cap \mathcal{B}(q\mathcal{H})$, and if $x \in (Mq)' \cap \mathcal{B}(q\mathcal{H})$ and $y \in M$ then

$$xy = x(qy + (1-q)y) = xqy = yqx = (yq + y(1-q))x = yx,$$

thus $(Mq)' \cap \mathcal{B}(q\mathcal{H}) \subset qM'q$.

Taking commutants and using von Neumann's double commutant theorem also gives $Mq = (qM'q)' \cap \mathcal{B}(q\mathcal{H})$. Replacing M with M' and q with p then gives the other case. ∎

Theorem 3.3.9. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then*

1. *M is type I if and only if M' is type I;*
2. *M is type II if and only if M' is type II;*
3. *M is type III if and only if M' is type III.*

Proof. We only need to show one implication for each type as the other then follows from von Neumann's double commutant theorem.

First, suppose that M is type II, and take $q \in M'$ a non-zero projection. Consider $\xi \in \mathcal{H}$ such that $q\xi \neq 0$, and take $p \leq [qM'q\xi] \in Mq$ a non-zero finite projection. Then $pq\xi$ is cyclic and separating for $pMpq$. As $pMpq$ is not abelian, neither is $qM'qp$, and hence q cannot be an abelian projection. Also, by Lemmas 3.3.7, and 3.3.8 we have that $(pMpq)' = qM'qp$ is finite, showing that q has a finite subprojection. Since q was arbitrary this shows that M' is type II.

Next, if M is type III, and $q \in M'$ were a non-zero finite projection, then from above we would have $Mq = (qM'q)'$ semi-finite giving a contradiction. Thus, M' has no non-zero finite projections and hence is type III.

Finally, if M is type I then from above it follows that M' cannot have a type II or type III von Neumann algebra as a direct summand. Thus, M' must be type I. ∎

3.4 Type I von Neumann algebras

For a cardinal number n, a von Neumann algebra M is **type** I_n, if 1 is a sum of n equivalent non-zero abelian projections. We use the terminology **type** I_∞ to describe a properly infinite type I von Neumann algebra.

Lemma 3.4.1. *If n and m are cardinal numbers such that a von Neumann algebra M is both type I_n and type I_m, then $n = m$.*

Proof. If n or m is finite this is clear. For infinite cardinals suppose $1 = \sum_{i \in I} p_i = \sum_{j \in J} q_j$ where $\{p_i\}$, and $\{q_j\}$ are each infinite pairwise orthogonal collections of abelian projections with central support 1.

First note that for each $\xi \in \mathcal{H}$, there are only countably many $j \in J$ such that $q_j \xi \neq 0$, hence for each fixed $i \in I$, there are only countably many $j \in J$ such that $[p_i q_j \mathcal{H}]\xi \neq 0$. If we denote by $z_{i,j}$ the central support of $[p_i q_j \mathcal{H}]$ then as $z_{i,j}[p_i q_j \mathcal{H}] = [p_i q_j \mathcal{H}]$, and as p_i is abelian we then have $[p_i q_j \mathcal{H}] \leq p_i z_{i,j} = [p_i M p_i q_j \mathcal{H}] \leq [p_i q_j p_i M p_i \mathcal{H}] \leq [p_i q_j \mathcal{H}]$. Thus, $z_{i,j} p_i = [p_i q_j \mathcal{H}]$, and so for each $i \in I$, and $\xi \in p_i \mathcal{H}$, there are only countably many $j \in J$ such that $z_{i,j}\xi \neq 0$. As $\{p_i\}$ is a pairwise orthogonal family we then have that for each $\xi \in \mathcal{H}$, and $i \in I$, there are only countably many $j \in J$ such that $z_{i,j}\xi \neq 0$.

Fix $\xi_0 \in \mathcal{H}$ a non-zero vector, and for each $i \in I$ set $J_i = \{j \in J \mid z_{i,j}\xi_0 \neq 0\}$. For each $j \in J$, we have $\vee_{i \in I} z_{i,j} = z(p_j) = 1$, and hence there exists some $i \in I$ such that $z_{i,j}\xi_0 \neq 0$. Hence we have $J = \cup_{i \in I} J_i$, and as each J_i is countable we then have $|J| \leq |I \times \mathbb{N}| = |I|$. By symmetry we also have $|I| \leq |J|$, and so $|I| = |J|$ by the Cantor-Bernstein-Schröeder theorem. ∎

Proposition 3.4.2. *Every type I von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is a direct sum of type I_n von Neumann algebras. Moreover, this decomposition is unique.*

Proof. If M is any type I von Neumann algebra and we fix q a non-zero abelian projection, then we may consider a maximal family $\{q_i\}_{i \in I}$ of pairwise orthogonal abelian projections such that $q_i \sim q$ for each $i \in I$. We then have that $\sum_{i \in I} q_i \leq z(q)$, and if $\sum_{i \in I} q_i z < z$ for some non-zero central subprojection $z \leq z(q)$, then there is a non-zero abelian subprojection $q_0 \leq z - \sum_{i \in I} q_i z$. Thus, if $\sum_{i \in I} q_i z < z$ for all non-zero central subprojections $z \leq z(q)$ then a maximality argument would produce an abelian subprojection $\tilde{q_0} \leq z(q) - \sum_{i \in I} q_i$ whose central support is $z(q)$. Buth then $q_0 \sim q$ contradicting maximality of the set $\{q_i\}_{i \in I}$. Hence, there exists some non-zero central subprojection $z \leq z(q)$ such that $\sum_{i \in I} q_i z = z$, and hence Mz is type $I_{|I|}$. Thus, every type I von Neumann algebra contains a type I_n direct summand for some cardinal number n.

For each cardinal number n, let $\{z_i\}_i$ be a maximal family of orthogonal central projections, such that Mz_i is type I_n for each i. If we set $p_n = \sum_i z_i$ then have that Mp_n is type I_n, and $M(1 - p_n)$ has no type I_n summand. By Lemma 3.4.1 we have that $\{p_n\}$ is a pairwise orthogonal family of projections and from the argument above, for a large enough cardinal number N, we have $1 = \sum_{n \leq N} p_n$. Uniqueness of this decomposition follows directly from Lemma 3.4.1. ∎

Theorem 3.4.3. *A type I factor M is $*$-isomorphic to $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$.*

Proof. By Zorn's Lemma there exists a maximal family of pairwise orthogonal minimal projection $\mathcal{F} \subset M$. If we denote by $P = \vee_{p \in \mathcal{F}} p$ then $1 - P$ can have no minimal subprojections and hence since M is a type I factor it follows that $P = 1$.

If $p, q \in \mathcal{F}$, then by minimality we cannot have $p \prec q$ or $q \prec q$, thus by Corollary 3.1.11 we must have that $p \sim q$. Hence, if we fix $p_0 \in \mathcal{F}$, then for any $q \in \mathcal{F}$ we can chose a partial isometry $v_q \in M$ such that $v_q^* v_q = p_0$, and $v_q v_q^* = q$. We may assume in addition that $v_{p_0} = p$.

For each $p, q \in \mathcal{F}$ we define the partial isometry $v_{p,q} = v_p v_q^*$. It is then easy to verify that $v_{p,p} = p$, $v_{p,q}^* = v_{q,p}$, and $v_{p,q} v_{q,r} = v_{p,r}$, for all $p, q, r \in \mathcal{F}$.

We may then define a map $\theta : \mathcal{B}(\ell^2 \mathcal{F}) \to M$ by $\theta(T) = \sum_{p,q \in \mathcal{F}} \langle T\delta_q, \delta_p \rangle v_{p,q}$, where the sum is taken in the SOT. The map θ is clearly unital, linear, adjoint preserving, and injective. From Parseval's identity we see that it is multiplicative as well. Indeed, for $T, S \in \mathcal{B}(\ell^2 \mathcal{F})$ we have

$$\begin{aligned}
\theta(T)\theta(S) &= \Big(\sum_{p,r \in \mathcal{F}} \langle T\delta_r, \delta_p \rangle v_{p,r}\Big)\Big(\sum_{r',q \in \mathcal{F}} \langle S\delta_q, \delta_{r'} \rangle v_{r',q}\Big) \\
&= \sum_{p,q,r \in \mathcal{F}} \langle \delta_r, T^* \delta_p \rangle \langle S\delta_q, \delta_r \rangle v_{p,q} \\
&= \sum_{p,q \in \mathcal{F}} \langle S\delta_q, T^* \delta_p \rangle v_{p,q} = \theta(TS).
\end{aligned}$$

The fact that θ is onto follows from the fact that for $x \in M$ we have $pxq \in \mathbb{C} v_{p,q}$, and $x = \sum_{p,q \in \mathcal{F}} pxq$. ∎

A similar classification for type I von Neumann algebras can be made, but we will delay this discussion until after we introduce tensor products of von Neumann algebras.

Chapter 4

States and traces

If A is a C^*-algebra then A^* is a Banach space which is also an A-bimodule given by $(a \cdot \psi \cdot b)(x) = \psi(bxa)$. Moreover, the bimodule structure is continuous since

$$\|a \cdot \psi \cdot b\| = \sup_{x \in (A)_1} |\psi(bxa)| \leq \sup_{x \in (A)_1} \|\psi\| \|bxa\| \leq \|\psi\| \|b\| \|a\|.$$

4.1 States

A linear functional $\varphi : A \to \mathbb{C}$ on a C^*-algebra A is **positive** if $\varphi(x) \geq 0$, whenever $x \in A_+$. Note that if $\varphi : A \to \mathbb{C}$ is positive then so is $a^* \cdot \varphi \cdot a$ for all $a \in A$. A positive linear functional is **faithful** if $\varphi(x) \neq 0$ for every non-zero $x \in A_+$, and a **state** if φ is positive, and $\|\varphi\| = 1$. The state space $S(A)$ is a convex closed subspace of the unit ball of A^*, and as such it is a compact Hausdorff space when endowed with the weak*-topology.

Note that if $\varphi \in S(A)$ then for all $x \in A$, $x = x^*$, then $\varphi(x) = \varphi(x_+ - x_-) \in \mathbb{R}$. Hence, if $y \in A$ then writing $y = y_1 + iy_2$ where y_j are self-adjoint for $j = 1, 2$, we have $\varphi(y^*) = \varphi(y_1) - i\varphi(y_2) = \overline{\varphi(y)}$. In general, we say a functional is **Hermitian** if $\varphi(y^*) = \overline{\varphi(y)}$, for all $y \in A$. Note that by defining $\varphi^*(y) = \overline{\varphi(y^*)}$ then we have that $\varphi + \varphi^*$, and $i(\varphi - \varphi^*)$ are each Hermitian.

Also note that a positive linear functional $\varphi : A \to \mathbb{C}$ is bounded. Indeed, if $\{x_n\}_n$ is any sequence of positive elements in $(A)_1$ then for any $(a_n)_n \in \ell^1\mathbb{N}$ we have $\sum_n a_n \varphi(x_n) = \varphi(\sum_n a_n x_n) < \infty$. This shows that $(\varphi(x_n))_n \in \ell^\infty\mathbb{N}$ and since the sequence was arbitrary we have that φ is bounded on the set of positive elements in $(A)_1$. Writing an element x in the usual way as $x = x_1 - x_2 + ix_3 - ix_4$ then shows that φ is bounded on the whole unit ball.

Lemma 4.1.1. *Let $\varphi : A \to \mathbb{C}$ be a positive linear functional on a C^*-algebra A, then for all $x, y \in A$ we have $|\varphi(y^* x)|^2 \leq \varphi(y^* y)\varphi(x^* x)$.*

Proof. Since φ is positive, the sesquilinear form defined by $\langle x, y \rangle = \varphi(y^* x)$ is non-negative definite. Thus, the result follows from the Cauchy-Schwarz inequality. ∎

Lemma 4.1.2. *Suppose A is a unital C^*-algebra. A linear functional $\varphi : A \to \mathbb{C}$ is positive if and only $\|\varphi\| = \varphi(1)$.*

Proof. First suppose φ is a positive linear functional, then for all $x \in A$ we have $\varphi(\|x + x^*\| \pm (x + x^*)) \geq 0$. Since φ is Hermitian we then have

$$|\varphi(x)| = |\varphi(\frac{x + x^*}{2})| \leq \|\frac{x + x^*}{2}\|\varphi(1) \leq \|x\|\varphi(1),$$

showing $\|\varphi\| \leq \varphi(1) \leq \|\varphi\|$.

Now suppose $\|\varphi\| = \varphi(1)$, and $x \in A$ is a positive element such that $\varphi(x) = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. For all $t \in \mathbb{R}$ we have

$$\begin{aligned} \alpha^2 + (\beta + t\|\varphi\|)^2 &= |\varphi(x + it)|^2 \\ &\leq \|x + it\|^2\|\varphi\|^2 = (\|x\|^2 + t^2)\|\varphi\|^2. \end{aligned}$$

Subtracting $t^2\|\varphi\|^2$ from both sides of this inequality shows $2\beta t\|\varphi\| \leq \|x\|^2\|\varphi\|$, thus $\beta = 0$.

Also, we have

$$\|x\|\|\varphi\| - \varphi(x) = \varphi(\|x\| - x) \leq \|\|x\| - x\|\|\varphi\| \leq \|x\|\|\varphi\|,$$

hence $\alpha > 0$. ∎

Corollary 4.1.3. *If $\varphi : A \to \mathbb{C}$ is a positive linear functional on a C^*-algebra A, then φ has a unique extension to a positive linear functional on the unitization $\tilde{A}$.*

Proposition 4.1.4. *Let A be a C^*-algebra and $x \in A$. For each $\lambda \in \sigma(x)$ there exists a state $\varphi \in S(A)$ such that $\varphi(x) = \lambda$.*

Proof. By considering the unitization, we may assume that A is unital. Consider the subspace $\mathbb{C}x + \mathbb{C}1 \subset A$, with the linear functional φ_0 on this space defined by $\varphi_0(\alpha x + \beta) = \alpha\lambda + \beta$, for $\alpha, \beta \in \mathbb{C}$. Since $\varphi_0(\alpha x + \beta) \in \sigma(\alpha x + \beta)$ we have that $\|\varphi_0\| = 1$.

By the Hahn-Banach theorem there exists an extension $\varphi : A \to \mathbb{C}$ such that $\|\varphi\| = 1 = \varphi(1)$. By Lemma 4.1.2 $\varphi \in S(A)$, and we have $\varphi(x) = \lambda$. ∎

Proposition 4.1.5. *Let A be a C^*-algebra, and $x \in A$.*

(i) $x = 0$ if and only if $\varphi(x) = 0$ for all $\varphi \in S(A)$.

(ii) x is self-adjoint if and only if $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(A)$.

(iii) x is positive if and only if $\varphi(x) \geq 0$ for all $\varphi \in S(A)$.

Proof. (i) If $\varphi(x) = 0$ for all $\varphi \in S(A)$ then writing $x = x_1 + ix_2$ where $x_j = x_j^*$, for $j = 1, 2$, we have $\varphi(x_j) = 0$ for all $\varphi \in S(A)$, $j = 1, 2$. Thus, $x_1 = x_2 = 0$ by Proposition 4.1.4

(ii) If $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(A)$ then $\varphi(x - x^*) = \varphi(x) - \overline{\varphi(x)} = 0$, for all $\varphi \in S(A)$. Hence $x - x^* = 0$.

(iii) If $\varphi(x) \geq 0$ for all $\varphi \in S(A)$ then $x = x^*$ and by Proposition 4.1.4 we have $\sigma(x) \subset [0, \infty)$. ∎

4.1.1 The Gelfand-Naimark-Segal construction

A **representation** of a C^*-algebra A is a $*$-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$. If $\mathcal{K} \subset \mathcal{H}$ is a closed subspace such that $\pi(x)\mathcal{K} \subset \mathcal{K}$ for all $x \in A$ then the restriction to this subspace determines a **sub-representation**. If the only sub-representations are the restrictions to $\{0\}$ or $\mathcal{H}$ then π is **irreducible** , which by the double commutant theorem is equivalent to the von Neumann algebra generated by $\pi(A)$ being $\mathcal{B}(\mathcal{H})$. Two representations $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\rho : A \to \mathcal{B}(\mathcal{K})$ are **equivalent** if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that $U\pi(x) = \rho(x)U$, for all $x \in A$.

If $\pi : A \to \mathcal{B}(\mathcal{H})$ is a representation, and $\xi \in \mathcal{H}$, $\|\xi\| = 1$ then we obtain a state on A by the formula $\varphi_\xi(x) = \langle \pi(x)\xi, \xi \rangle$. Indeed, if $x \in A$ then $\langle \pi(x^*x)\xi, \xi \rangle = \|\pi(x)\xi\|^2 \geq 0$. We now show that every state arises in this way.

Theorem 4.1.6 (The GNS construction)**.** *Let A be a unital C^*-algebra, and consider $\varphi \in S(A)$, then there exists a Hilbert space $L^2(A, \varphi)$, and a unique (up to equivalence) representation $\pi : A \to \mathcal{B}(L^2(A, \varphi))$, with a unit cyclic vector $1_\varphi \in L^2(A, \varphi)$ such that $\varphi(x) = \langle \pi(x)1_\varphi, 1_\varphi \rangle$, for all $x \in A$.*

Proof. Consider $A_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$. Since

$$(x + y)^*(x + y) \leq (x + y)^*(x + y) + (x - y)^*(x - y) = 2(x^*x + y^*y)$$

we see that N_φ is a closed linear subspace. We then see that N_φ is a left ideal from the inequality $(ax)^*(ax) \leq \|a\|x^*x$.

We consider $\mathcal{H}_0 = A/N_\varphi$ which we endow with the inner product $\langle [x], [y] \rangle = \varphi(y^*x)$, where $[x]$ denotes the equivalence class of x in A/N_φ, (this is well defined since N_φ is a left ideal). From Lemma 4.1.1 this inner product is positive definite, and hence we denote by $L^2(A, \varphi)$ the Hilbert space completion.

For $a \in A$ we consider the map $\pi_0(a) : \mathcal{H}_0 \to \mathcal{H}_0$ given by $\pi_0(a)[x] = [ax]$. Since N_φ is a left ideal this is well defined, and since $\|\pi_0(a)[x]\|^2 = \varphi((ax)^*(ax)) \leq \|a\|^2\varphi(x^*x)$ we have that this extends to a bounded operator $\pi(a) \in \mathcal{B}(L^2(A, \varphi))$ such that $\|\pi(a)\| \leq \|a\|$. The map $a \mapsto \pi(a)$ is clearly a homomorphism, and for $x, y \in A$ we have $\langle [x], \pi(a^*)[y] \rangle = \varphi(y^*a^*x) = \langle \pi(a)[x], [y] \rangle$, thus $\pi(a^*) = \pi(a)^*$. Also, if we consider $1_\varphi = [1] \in \mathcal{H}_0 \subset L^2(A, \varphi)$ then we have $\langle \pi(a)1_\varphi, 1_\varphi \rangle = \varphi(a)$.

If $\rho : A \to \mathcal{B}(\mathcal{K})$ and $\eta \in \mathcal{K}$ is a cyclic vector such that $\varphi(a) = \langle \rho(a)\eta, \eta \rangle$, then we can consider the map $U_0 : \mathcal{H}_0 \to \mathcal{K}$ given by $U_0([x]) = \rho(x)\eta$. We then have

$$\langle U_0([x]), U_0([y]) \rangle = \langle \rho(x)\eta, \rho(y)\eta = \langle \rho(y^*x)\eta, \eta = \varphi(y^*x) = \langle [x], [y] \rangle$$

which shows that U_0 is well defined and isometric. Also, for $a, x \in A$ we have

$$U_0(\pi(a)[x]) = U_0([ax]) = \rho(ax)\eta = \rho(a)U_0([x]).$$

Hence, U_0 extends to an isometry $U : L^2(A, \varphi) \to \mathcal{K}$ such that $U\pi(a) = \rho(a)U$ for all $a \in A$. Since η is cyclic, and $\rho(A)\eta \subset U(L^2(A, \varphi))$ we have that U is unitary. ∎

Corollary 4.1.7. *Let A be a C^*-algebra, then there exists a faithful representation.*

Proof. If we let π be the direct sum over all GNS representations corresponding to states, then this follows easily from Proposition 4.1.5. Note also that if A is separable, then so is $S(A)$ and by considering a countable dense subset of $S(A)$ we can construct a faithful representation onto a separable Hilbert space. ∎

If φ and ψ are two Hermitian linear functionals, we write $\varphi \leq \psi$ if $\varphi(a) \leq \psi(a)$ for all $a \in A_+$, alternatively, this is if and only if $\psi - \varphi$ is a positive linear functional. The following is a Radon-Nikodym type theorem for positive linear functionals.

Proposition 4.1.8. *Suppose φ and ψ are positive linear functionals on a C^*-algebra A such that ψ is a state. Then $\varphi \leq \psi$, if and only if there exists a unique $y \in \pi_\psi(A)'$ such that $0 \leq y \leq 1$ and $\varphi(a) = \langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle$ for all $a \in A$.*

Proof. First suppose that $y \in \pi_\psi(A)'$, with $0 \leq y \leq 1$. Then for all $a \in A$, $a \geq 0$ we have $\pi_\psi(a)y = \pi_\psi(a)^{1/2} y \pi_\psi(a)^{1/2} \leq \pi_\psi(a)$, hence $\langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle \leq \langle \pi_\psi(a) 1_\psi, 1_\psi \rangle = \psi(a)$.

Conversely, if $\varphi \leq \psi$, the Cauchy-Schwarz inequality implies

$$|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b) \leq \psi(a^*a)\psi(b^*b) = \|\pi_\psi(a)1_\psi\|^2 \|\pi_\psi(b)1_\psi\|^2.$$

Thus $\langle \pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle_\varphi = \varphi(b^*a)$ is a well defined non-negative definite sesquilinear form on $\pi_\psi(A)1_\psi$ which is bounded by 1, and hence extends to the closure $L^2(A, \psi)$.

Therefore there is an operator $y \in \mathcal{B}(L^2(A,\psi))$, $0 \leq y \leq 1$, such that $\varphi(b^*a) = \langle y\pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle$, for all $a, b \in A$.

If $a, b, c \in A$ then

$$\begin{aligned} \langle y\pi_\psi(a)\pi_\psi(b)1_\psi, \pi_\psi(c)1_\psi \rangle &= \langle y\pi_\psi(ab)1_\psi, \pi_\psi(c)1_\psi \rangle = \varphi(c^*ab) \\ &= \langle y\pi_\psi(b)1_\psi, \pi_\psi(a^*)\pi_\psi(c)1_\psi \rangle \\ &= \langle \pi_\psi(a)y\pi_\psi(b)1_\psi, \pi_\psi(c)1_\psi \rangle. \end{aligned}$$

Thus, $y\pi_\psi(a) = \pi_\psi(a)y$, for all $a \in A$.

To see that y is unique, suppose that $0 \leq z \leq 1$, $z \in \pi_\psi(A)'$ such that $\langle \pi_\psi(a) z 1_\psi, 1_\psi \rangle = \langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle$ for all $a \in A$. Then $\langle (z-y)1_\psi, \pi_\psi(a^*)1_\psi \rangle = 0$ for all $a \in A$ and hence $z - y = 0$ since 1_ψ is a cyclic vector for $\pi_\psi(A)$. ∎

4.1.2 Pure states

A state φ on a C^*-algebra A is said to be **pure** if it is an extreme point in $S(A)$.

Proposition 4.1.9. *A state φ on a C^*-algebra A is a pure state if and only if the corresponding GNS representation $\pi_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$ with corresponding cyclic vector 1_φ is irreducible.*

Proof. Suppose first that φ is pure. If $\mathcal{K} \subset L^2(A,\varphi)$ is a closed invariant subspace, then so is $\mathcal{K}^\perp$ and we may consider $\xi_1 = [\mathcal{K}](1_\varphi) \in \mathcal{K}$ and $\xi_2 = 1_\varphi - \xi_1 \in \mathcal{K}^\perp$. For $x \in A$ we have

$$\langle x\xi_1, \xi_1\rangle + \langle x\xi_2, \xi_2\rangle = \langle x1_\varphi, 1_\varphi\rangle = \varphi(x).$$

Thus, either $\xi_1 = 0$, or $\xi_2 = 0$, since φ is pure. Since 1_φ is cyclic, we have that ξ_1 is cyclic for $\mathcal{K}$ and ξ_2 is cyclic for $\mathcal{K}^\perp$ showing that either $\mathcal{K} = \{0\}$ or else $\mathcal{K}^\perp = \{0\}$.

Conversely, if $\varphi = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$ where $\varphi_j \in S(A)$ for $j = 1, 2$, then we may consider the map $U : L^2(A,\varphi) \to L^2(A,\varphi_1) \oplus L^2(A,\varphi_2)$ such that $U(x1_\varphi) = (x\frac{1}{\sqrt{2}}1_{\varphi_1}) \oplus (x\frac{1}{\sqrt{2}}1_{\varphi_2})$, for all $x \in A$. It is not hard to see that U is a well defined isometry and $U\pi_\varphi(x) = (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))U$ for all $x \in A$. If we denote by $p_1 \in \mathcal{B}(L^2(A,\varphi_1) \oplus L^2(A,\varphi_2)$ the orthogonal projection onto $L^2(A,\varphi_1)$ then the operator $U^*p_1U \in \mathcal{B}(L^2(A,\varphi))$ commutes with $\pi(A)$, and for all $x \in A$ we have

$$\langle U^*p_1U\pi_\varphi(x)1_\varphi, 1_\varphi\rangle = \frac{1}{2}\langle \pi_{\varphi_1}(x)1_{\varphi_1}, 1_{\varphi_1}\rangle = \frac{1}{2}\varphi_1(x).$$

Hence $U^*p_1U \in \pi_\varphi(A)' \setminus \mathbb{C}$ which shows that π_φ is not irreducible. ∎

Note that the previous proposition, together with Proposition 4.1.8 shows also that a state φ is pure if and only if for any positive linear functional ψ such that $\psi \leq \varphi$ there exists a constant $\alpha \geq 0$ such that $\psi = \alpha\varphi$.

Exercise 4.1.10. Show that every finite-dimensional C^*-algebra is isomorphic to a finite direct sum of finite-dimensional type I factors

4.2 Normal representations

Recall that if M is a von Neumann algebra, then a bounded linear functional $\varphi \in M^*$ is normal if it is continuous in the σ-WOT, (or equivalently, by Lemma 2.4.3, $\varphi(\cdot) = \mathrm{Tr}(\cdot A)$ for some trace class operator A).

Lemma 4.2.1. *Suppose φ and ψ are positive linear functionals on a von Neumann algebra M, and $p \in \mathcal{P}(M)$ such that $p \cdot \psi \cdot p$ is normal and $\varphi(p) < \psi(p)$, then there exists a non-zero projection $q \in \mathcal{P}(M)$, $q \leq p$ such that $\varphi(x) < \psi(x)$ for all $x \in qMq$, $x > 0$.*

Proof. Consider the set $\mathcal{C}$ of all operators $0 \leq x \leq p$ such that $\varphi(x) \geq \psi(x)$. If x_i is any increasing family in $\mathcal{C}$ then since $p\cdot\psi\cdot p$ is normal we have $\psi(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} \psi(x_i)$ and since for each i we have $\varphi(\lim_{j\to\infty} x_j) \geq \varphi(x_i)$ it follows that $\lim_{i\to\infty} x_i \in \mathcal{C}$. Thus, by Zorn's lemma there exists a maximal operator $x_0 \in \mathcal{C}$. Moreover, $x_0 \neq p$ since $\varphi(p) < \psi(p)$.

Since $x_0 \neq p$, there exists $\varepsilon > 0$ such that $q = 1_{[\varepsilon,1]}(p - x_0) \neq 0$. We then have $q \leq p$ and if $0 < y \leq \varepsilon q$ then $x_0 < x_0 + y \leq x_0 + \varepsilon q \leq p$, hence $\varphi(x_0 + y) < \psi(x_0 + y) \leq \varphi(x_0) + \psi(y)$, and so $\varphi(y) < \psi(y)$. ∎

Proposition 4.2.2. *Let φ be a positive linear functional on a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

(i) *φ is normal.*

(ii) *There is a positive trace class operator A such that $\varphi(x) = \mathrm{Tr}(xA)$, for all $x \in M$.*

(iii) *If $x_i \in M$ is any bounded increasing net, then we have $\varphi(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} \varphi(x_i)$.*

(iv) *If $\{p_i\}_i$ is any family of pairwise orthogonal projections in M, then $\varphi(\sum_i p_i) = \sum_i \varphi(p_i)$.*

Proof. (i) $\implies$ (ii) If φ is normal then there exist Hilbert-Schmidt operators B and C such that $\varphi(x) = \langle xB, C\rangle_2$ for all $x \in A$. If we set $\psi(x) = \frac{1}{4}\langle x(B + C), (B + C)\rangle_2$, then ψ is a positive linear functional and for $x \in M$, $x \geq 0$ we have

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\langle xB, C\rangle_2 + \frac{1}{2}\overline{\langle xB, C\rangle_2} \\ &= \frac{1}{4}\langle x(B + C), (B + C)\rangle_2 - \frac{1}{4}\langle x(B - C), (B - C)\rangle_2 \leq \psi(x).\end{aligned}$$

By Proposition 4.1.8 there exists $T \in (M\overline{\otimes}\mathbb{C})' \subset \mathcal{B}(\mathcal{H}\overline{\otimes}\overline{\mathcal{H}})$ such that $0 \leq T \leq 1$ and $\varphi(x) = \frac{1}{4}\langle xT^{1/2}(B + C), T^{1/2}(B + C)\rangle_2$, for all $x \in M$. The result then follows.

(ii) $\implies$ (iii) This follows easily since $x \mapsto \mathrm{Tr}(xA)$ is SOT continuous and since $x_i \to \lim_{i\to\infty} x_i$ in the SOT topology.

(iii) $\implies$ (iv) This is obvious.

(iv) $\implies$ (i) If $p \in \mathcal{P}(M)$ is a non-zero projection, then we can consider $\xi \in \mathcal{H}$ such that $\varphi(p) < \langle p\xi, \xi\rangle$. By Lemma 4.2.1 there then exists a non-zero projection $q \leq p$ such that $\varphi(x) < \langle x\xi, \xi\rangle$ for all $x \in qMq$. By the Cauchy-Schwarz inequality for any $x \in M$ we then have

$$|\varphi(xq)|^2 \leq \varphi(qx^*xq)\varphi(1) \leq \langle qx^*xq\xi, \xi\rangle\varphi(1) = \|xq\xi\|^2\varphi(1).$$

Thus $q \cdot \varphi$ is SOT continuous, and hence normal.

By Zorn's lemma we may consider a maximal family $\{p_i\}_{i\in I}$ of pairwise orthogonal projections such that $p_i \cdot \varphi$ is SOT continuous for all $i \in I$. From the previous paragraph we have that $\sum_i p_i = 1$. By hypothesis, for any $\varepsilon > 0$ there exists a finite subcollection $J \subset I$ such that if $p = \sum_{j\in J} p_j$ then $\varphi(p) > \varphi(1) - \varepsilon$, but then for $x \in M$ we have

$$|(\varphi - p \cdot \varphi)(x)|^2 \leq \varphi(xx^*)\varphi(1 - p) \leq \|x\|^2\varphi(1)(\varphi(1) - \varphi(p)),$$

hence $\|\varphi - p \cdot \varphi\|^2 < \varphi(1)\varepsilon$. Therefore the finite partial sums of $\sum_{i\in I} p_i \cdot \varphi$ converge to φ in norm, and since each $p_i \cdot \varphi$ is normal it then follows that φ is normal. ∎

Corollary 4.2.3. *If φ is a normal state on a von Neumann algebra M then the GNS-representation $(\pi_\varphi, L^2(M,\varphi), 1_\varphi)$ is a normal representation.*

Proof. From the implication (i) $\implies$ (ii) of the previous theorem we have that φ is of the form $x \mapsto \langle xT, T\rangle$ for some Hilbert-Schmidt operator T. By uniqueness of the GNS-construction it then follows that the GNS-representation $(\pi_\varphi, L^2(M,\varphi), 1_\varphi)$ is equivalent to a subrepresentation of the normal representation $x \mapsto x \otimes 1 \in \mathcal{B}(\mathcal{H}\overline{\otimes}\overline{\mathcal{H}})$. ∎

Corollary 4.2.4. *Every $*$-isomorphism between von Neumann algebras is normal.*

Proof. If $\theta : M \to N$ is a $*$-isomorphism, then for any bounded increasing net $x_i \in M$ we have $\theta(\lim_{i\to\infty} x_i) \geq \lim_{i\to\infty} \theta(x_i)$ and applying θ^{-1} gives the reverse inequality as well. Hence θ is normal just as in the previous corollary. ∎

4.3 Polar and Jordan decomposition

Lemma 4.3.1. *Let M be a von Neumann algebra and $I \subset M$ a left ideal which is closed in the WOT, then there exists a projection $p \in \mathcal{P}(M)$ such that $I = Mp$. If, in addition, I is a two sided ideal then p is central. If $V \subset M_*$ is a closed left invariant subspace (i.e., $x \cdot \varphi \in V$ for all $x \in M$, $\varphi \in V$), then there exists a projection $q \in \mathcal{P}(M)$ such that $V = M_* q$. If, in addition, V is also right invariant then q is central.*

Proof. By Theorem 1.4.9 any closed left ideal $I \subset M$ has a right approximate identity. Since I is closed in the WOT it then follows that I has a right identity p. Since p is positive and $p^2 = p$ we have that p is a projection, and $Mp = Ip = I$.

If I is a two sided ideal then p is also a left identity, hence for all $x \in M$ we have $xp = pxp = px$, and so $p \in \mathcal{Z}(M)$.

If $V \subset M_*$ is a closed left invariant subspace then $V^0 = \{x \in M \mid \varphi(x) = 0, \text{ for all } \varphi \in V\}$ is a right ideal which is closed in the WOT. Hence there exists $q \in \mathcal{P}(M)$ such that $V^0 = qM$. and then it is easy to check that $V = M_* q$. If V is also right invariant then V^0 will be a two sided ideal and hence q will be central. ∎

If $\varphi : M \to \mathbb{C}$ is a normal positive linear functional, then $\{x \in M \mid \varphi(x^*x) = 0\}$ is a left ideal which is closed in the WOT, thus by the previous lemma there exists a projection $p \in \mathcal{P}(M)$ such that $\varphi(x^*x) = 0$ if and only if $x \in Mp$. We denote by $s(\varphi) = 1 - p$ the **support projection** of φ. Note that if $q = s(\varphi)$ then $\varphi(xq) = \varphi(qx) = \varphi(x)$ for all $x \in M$, and moreover, φ will be faithful when restricted to qMq.

Theorem 4.3.2 (Polar decomposition)**.** *Suppose M is a von Neumann algebra and $\varphi \in M_*$, then there exists a unique partial isometry $v \in M$ and positive linear functional $\psi \in M_*$ such that $\varphi = v \cdot \psi$ and $v^*v = s(\psi)$.*

Proof. We will assume that $\|\varphi\| = 1$. Since $(M_*)^* = M$, if $\varphi \in M_*$ there exists $a \in M$, $\|a\| \leq 1$, such that $\varphi(a) = \|\varphi\|$. Consider $a^* = v|a^*|$, the polar decomposition of a^*. Then if $\psi = v^* \cdot \varphi$ we have $\psi(|a^*|) = \varphi(a) = \|\varphi\| = 1$. Since $0 \leq |a^*| \leq 1$, we have $\||a^*| + e^{i\theta}(1 - |a^*|)\| \leq 1$ for every $\theta \in \mathbb{R}$. If we fix $\theta \in \mathbb{R}$ such that $e^{i\theta}\psi(1 - |a^*|) \geq 0$ then we have

$$\psi(|a^*|) \leq \psi(|a^*|) + e^{i\theta}\psi(1 - |a^*|) = \psi(|a^*| + e^{i\theta}(1 - |a^*|)) \leq \|\psi\| = \psi(|a^*|).$$

Thus $\psi(1) = \psi(|a^*|) = \|\psi\|$ and hence ψ is a positive linear functional.

Set $p = v^*v$. By replacing a with $avs(\psi)v^*$ we may assume that $p \leq s(\psi)$, and for $x \in M$ such that $\|x\| \leq 1$, we have that $\psi(|a^*| + (1-p)x^*x(1-p)) \leq \|\psi\| = \varphi(|a^*|)$ which shows that $\psi((1-p)x^*x(1-p)) = 0$ and hence $p \geq s(\psi)$.

To see that $\varphi = v \cdot \psi$ it suffices to show that $\varphi(x(1-p)) = 0$ for all $x \in M$. Suppose that $\|x\| = 1$ and $\varphi(x(1-p)) = \beta \geq 0$. Then for $n \in \mathbb{N}$ we have

$$\begin{aligned} n + \beta = \varphi(na + x(1-p)) &\leq \|na + x(1-p)\| \\ &= \|(na + x(1-p))(na + x(1-p))^*\|^{1/2} \\ &= \|n^2|a^*|^2 + x(1-p)x^*\|^{1/2} \leq \sqrt{n^2 + 1}, \end{aligned}$$

which shows that $\beta = 0$.

To see that this decomposition is unique, suppose that $\varphi = v_0 \cdot \psi_0$ gives another decomposition, and set $p_0 = v_0^* v_0 = s(\psi_0)$. Then for $x \in M$ we have

$$\psi(x) = \varphi(xv^*) = \psi_0(xv^*v_0) = \psi_0(p_0 x v^* v_0).$$

Setting $x = 1 - p_0$ we then have $p = s(\psi) \leq p_0$, and by symmetry we have $p_0 \leq p$.

In particular we have $v_0^* v \in pMp$ and so we may write $v_0^* v = h + ik$ where h and k are self-adjoint elements in pMp. Then $\psi(h) + i\psi(k) = \psi(v_0^* v) = \psi_0(p) = \|\psi_0\| = \|\varphi\|$.

Hence, $\psi(h) = \|\varphi\|$ and $\psi(k) = 0$. We then have $p - h \geq 0$ and $\psi(p-h) = 0$, thus since ψ is faithful on pMp it follows that $h = p$, and we must then also have $k = 0$ since $\|v_0^* v\| \leq 1$. Hence, $v_0^* v = p$ and taking adjoints gives $v^* v_0 = p$.

Thus, $v = vp = vv^* v_0$ and so $vv^* \leq v_0 v_0^*$. Similarly, we have $v_0 v_0^* \leq vv^*$ from which it then follows that $v = v_0$. Therefore, $\psi = v^* \cdot \varphi = v_0^* \cdot \varphi = \psi_0$. ∎

If $\varphi \in M_*$ as in the previous theorem then we denote by $|\varphi| = \psi$ the **absolute value** of φ. We also denote by $s_r(\varphi) = v^*v$ the **right support projection** of φ, and $s_l(\varphi) = vv^*$ the **left support projection** of φ, so that $s_l(\varphi) \cdot \varphi \cdot s_r(\varphi) = \varphi$.

Theorem 4.3.3 (Jordan decomposition)**.** *Suppose M is a von Neumann algebra and φ is a normal Hermitian linear functional, then there exist unique normal positive linear functionals φ_+, φ_- such that $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$.*

Proof. As in the previous theorem, we will take $a \in M$, $\|a\| \leq 1$, such that $\varphi(a) = \|\varphi\|$. Note that since φ is Hermitian we may assume that $a^* = a$, and hence if we consider the polar decomposition $a = |a|v$ we have that $v^* = v$ and hence $v = p - q$ for orthogonal projections $p, q \in M$. For $\psi = |\varphi|$, since $\varphi = v \cdot \psi = (v \cdot \psi)^* = \psi \cdot v$ it follows that any spectral projection of v will commute with ψ, and hence $p \cdot \psi$ and $q \cdot \psi$ will both be positive.

Since $p \cdot \psi$ and $q \cdot \psi$ have orthogonal supports and since $p \cdot \varphi - q \cdot \varphi = \psi$ it follows that $\varphi_+ = p \cdot \varphi$ and $\varphi_- = -q \cdot \varphi$ are both positive. Thus, $\varphi = v^2 \cdot \varphi = \varphi_+ - \varphi_-$, and

$$\|\varphi\| = \psi(1) = \varphi_+(1) + \varphi_-(1) = \|\varphi_+\| + \|\varphi_-\|.$$

To see that this decomposition is unique, suppose that $\varphi = \varphi_1 - \varphi_2$ where φ_1, φ_2 are positive, and such that $\|\varphi\| = \|\varphi_1\| + \|\varphi_2\|$. Then $\|\varphi_+\| = \varphi(s(\varphi_+)) \leq \varphi_1(s(\varphi_+)) \leq \|\varphi_1\|$, and similarly $\|\varphi_-\| \leq \|\varphi_2\|$. However, $\|\varphi_+\| + \|\varphi_-\| = \|\varphi_1\| + \|\varphi_2\|$ and so we have

$$\|\varphi_+\| = \varphi_1(s(\varphi_+)) = \|\varphi_1\|;$$

$$\|\varphi_-\| = \varphi_2(s(\varphi_-)) = \|\varphi_2\|.$$

Thus, $s(\varphi_1)$ and $s(\varphi_2)$ are orthogonal and hence $\varphi = (s(\varphi_1) - s(\varphi_2))(\varphi_1 + \varphi_2)$.

By the uniqueness for polar decomposition we then have $s(\varphi_1) - s(\varphi_2) = s(\varphi_+) - s(\varphi_-)$ from which it follows that $s(\varphi_1) = s(\varphi_+)$ and $s(\varphi_2) = s(\varphi_-)$. Therefore, $\varphi_1 = s(\varphi_1) \cdot \varphi = \varphi_+$, and $\varphi_2 = s(\varphi_2) \cdot \varphi = \varphi_-$. ■

Corollary 4.3.4. *Let M be a von Neumann algebra, then M_* is spanned by normal positive linear functionals.*

By combining the previous corollary with Proposition 4.2.2 we obtain the following corollaries.

Corollary 4.3.5. *Let $\varphi : M \to \mathbb{C}$ be a continuous linear functional on a von Neumann algebra M, then φ is normal if and only if for any family $\{p_i\}_i$ of pairwise orthogonal projections we have $\varphi(\sum_i p_i) = \sum_i \varphi(p_i)$.*

Corollary 4.3.6. *Let $\varphi : M \to \mathbb{C}$ be a continuous linear functional on a von Neumann algebra M, then φ is normal if and only if φ is normal when restricted to any abelian von Neumann subalgebra.*

4.4 Unique preduals

Lemma 4.4.1. *Let A be a C^*-algebra, and suppose that X is a Banach space such that $X^* \cong A$. Then $(A)_1 \cap A_+$ is $\sigma(A, X)$-compact.*

Proof. Note that by Alaoglu's theorem we have that $(A)_1$ is $\sigma(A, X)$-compact. Suppose $\{x_\alpha\}$ is a net of positive elements in $(A)_1$ which converge to $a + ib$ where a and b are self-adjoint.

If we fix $t \in \mathbb{R}$, then since $x_\alpha + it \in (A)_{\sqrt{1+t^2}}$ converges in the $\sigma(A, X)$-topology to $a + i(b + t)$ it follows from the $\sigma(A, X)$-compactness of $(A)_{\sqrt{1+t^2}}$

that $\|b+t\|^2 \leq \|a+i(b+t)\|^2 \leq 1+t^2$ for all $t \in \mathbb{R}$. In particular, if $\lambda \in \sigma(b)$, then we have $|\lambda+t|^2 \leq \|b+t\|^2 \leq 1+t^2$, for all $t \in \mathbb{R}$, which implies $\lambda = 0$. Hence, $b = 0$.

We then have $\|a\| \leq 1$, and since $\|1-x_\alpha\| \leq 1$ and $1-x_\alpha$ converges to $1-a$ it follows that $\|1-a\| \leq 1$, hence $a \geq 0$. Thus, $(A)_1 \cap A_+$ is $\sigma(A,X)$-compact. ∎

Lemma 4.4.2. *Let A be a C^*-algebra, and suppose that X is a Banach space such that $X^* \cong A$. Then for $x \in A_+$, $x = 0$ if and only if $\varphi(x) = 0$ for every positive linear functional $\varphi \in X$.*

Proof. From Lemma 4.4.1 we have that $(A)_1 \cap A_+$ is $\sigma(A,X)$ compact, hence if $a \in A$, $a < 0$, by the Hahn-Banach separation theorem there exists $\varphi \in X$ such that $\varphi(a) < 0$, and $\varphi(b) \geq 0$ for all $b \in (A)_1 \cap A_+$. Then $\varphi \in X$ is a positive linear functional such that $\varphi(a) \neq 0$. ∎

Lemma 4.4.3. *Let A be a C^*-algebra, and suppose that X is a Banach space such that $X^* \cong A$. Then every bounded increasing net $\{a_\alpha\}_\alpha \subset A_+$ $\sigma(A,X)$-converges to a least upper bound. Moreover, if a_α converges to a then $x^*a_\alpha x$ converges to x^*ax for each $x \in A$.*

Proof. Suppose that $\{a_\alpha\}$ is a bounded increasing net of positive operators. By Lemma 4.4.1 $(A)_1 \cap A_+$ is $\sigma(A,X)$-compact, and so there exists a $\sigma(A,X)$-cluster point $a \geq 0$. Moreover, since for fixed α_0 we have that $a - a_{\alpha_0}$ is a cluster point of $\{a_\alpha - a_{\alpha_0}\}$ it follows that $a_{\alpha_0} \leq a$. Similarly, if b is an upper bound for $\{a_\alpha\}$ then $b-a$ is a cluster point of $\{b-a_\alpha\}$ and hence it follows that $a \leq b$. Therefore, a_α converges in the $\sigma(M,X)$-topology to a unique least upper bound.

Note that by taking an approximate identity for A, this in particular shows that A is unital.

If $x \in A$ is invertible then x^*ax is clearly the least upper bound for $\{x^*a_\alpha x\}$. In general, if b is the least upper bound of $\{x^*a_\alpha x\}$, then taking $\lambda \in \mathbb{R} \setminus \sigma(x)$, for any $\varphi \in X$ a positive linear functional we have

$$\begin{aligned}\lambda^2\varphi(a_\alpha) - \lambda\varphi(x^*a_\alpha) - \lambda\varphi(a_\alpha x) + \varphi(x^*a_\alpha x) &= \varphi((x-\lambda)^*a_\alpha(x-\lambda)) \\ &\to \varphi((x-\lambda)^*a(x-\lambda)).\end{aligned}$$

On the other hand, for $\beta > \alpha$ we have

$$\begin{aligned}|\varphi(x^*(a_\beta - a_\alpha)| &= |\varphi(x^*(a_\beta - a_\alpha)^{1/2}(a_\beta - a_\alpha)^{1/2})| \\ &\leq \varphi(x^*(a_\beta - a_\alpha)x)^{1/2}\varphi(a_\beta - a_\alpha)^{1/2}.\end{aligned}$$

And similarly,

$$|\varphi((a_\beta - a_\alpha)x) \leq \varphi(a_\beta - a_\alpha)^{1/2}\varphi(x^*(a_\beta - a_\alpha)x)^{1/2}.$$

Hence,

$$\begin{gathered}\lambda^2\varphi(a_\alpha) - \lambda\varphi(x^*a_\alpha) - \lambda\varphi(a_\alpha x) + \varphi(x^*a_\alpha x) \\ \to \lambda^2(a) - \lambda\varphi(x^*a) - \lambda\varphi(ax) + \varphi(b),\end{gathered}$$

And so $x^*ax = b$ by Lemma 4.4.2. ∎

Theorem 4.4.4 (Sakai)**.** *Let M be a von Neumann algebra, then M_* is the unique predual of M in the sense that if X is a Banach space, and $\theta : X^* \to M$ is an isomorphism, then $\theta^* : M^* \to X^{**}$ restricted to M_* defines an isomorphism from M_* onto X.*

Proof. By the Hahn-Banach Theorem, it is enough to show that under the identification $M \cong X^*$, we have $X \subset M_*$.

If $\varphi \in X \subset X^{**} = M^*$, then from Lemma 4.4.3 it follows that for any bounded increasing net $\{x_\alpha\}$ we have $\lim_\alpha \varphi(x_\alpha) = \varphi(\lim_\alpha x_\alpha)$, therefore $\varphi \in M_*$ by Corollary 4.3.5. ■

Proposition 4.4.5 (Kadison, Pedersen)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a C^*-algebra such that $1 \in A$, and the SOT-limit of any monotone increasing net in A is contained in A. Then A is a von Neumann algebra.*

Proof. As the span of projections in A'' is norm dense it is enough to show that A contains every projection in A''.

If $a \in A$, then taking $f_n : [0, \infty) \to [0, \infty)$ defined by $f_n(t) = nt$ for $0 \leq t \leq \frac{1}{n}$, and $f_n(t) = 1$ for $t > \frac{1}{n}$, we have that $f_n(aa^*)$ is increasing to $[a\mathcal{H}]$, and thus $[a\mathcal{H}] \in A$ for all $a \in A$.

If $p, q \in A$ are projections then $[(p + q)\mathcal{H}] = p \vee q$, thus A is closed under taking finite supremums of projections, and as an arbitrary supremum is an increasing net of finite supremums it follows that A is closed under taking arbitrary supremums (or infimums) of projections.

If $p \in \mathcal{P}(A'')$ then $p = \vee_{\xi \in \mathcal{H}} [A' p\xi]$ and hence it is enough to show that projections of the form $[A'\xi]$ are contained in A for each $\xi \in \mathcal{H}$. To show this it is enough to show that for $\eta \perp \overline{A'\xi}$ there exists a positive operator $y \in A$ such that $y\xi = \xi$, and $y\eta = 0$. Indeed, we would then have $[y\mathcal{H}] \geq [A'\xi]$ and $[y\mathcal{H}]\eta = 0$. Taking the infimum of such projections would then show that $[A'\xi] \in A$.

So suppose $\xi, \eta \in \mathcal{H}$ such that $\eta \perp \overline{A'\xi}$. By Kaplansky's density theorem there exists a sequence $a_n \in A_{s.a.} \cap (A)_1$ such that $\|\xi - a_n\xi\| \leq 1/n$, and $\|a_n\eta\| \leq 1/(n2^n)$. By considering a_n^2 we may assume that $a_n \geq 0$.

For $n \leq m$ we define

$$y_{n,m} = (1 + \sum_{n \leq k \leq m} k a_k)^{-1} \sum_{n \leq k \leq m} k a_k.$$

Then $y_{n,m} \in A_+ \cap (A)_1$ and $y_{n,m} \leq \sum_{n \leq k \leq m} k a_k$. Thus, for $i \leq n$ we have

$$\langle y_{n,m}\eta, \eta \rangle \leq \sum_{n \leq k \leq m} 2^{-k} < 2^{-n+1}.$$

As $\sum_{n \leq k \leq m} k a_k \geq m a_m$ we have $y_{n,m} \geq (1 + m a_m)^{-1} m a_m$ (since $t/(1 + t) = 1 - 1/(1 + t)$ is an operator monotone function), hence

$$1 - y_{n,m} \leq (1 + m a_m)^{-1} \leq (1 + m)^{-1}(1 + m(1 - a_m)).$$

Thus, for $n \leq m$ we have

$$\langle \xi - y_{n,m}\xi, \xi \rangle \leq 2/(1+m).$$

For fixed n the sequence $y_{n,m}$ is increasing and hence converges to an element $y_n \in A_+ \cap (A)_1$. Since $y_{n+1,m} \leq y_{n,m}$ for all m it follows that $y_{n+1} \leq y_n$, and hence y_n is decreasing and so converges to an element $y \in A_+ \cap (A)_1$. For each $n \in \mathbb{N}$ we have

$$\langle y_n \eta, \eta \rangle \leq 2^{-n+1} \text{ and } \langle \xi - y_n \xi, \xi \rangle \leq 0,$$

therefore $y\eta = 0$, and $y\xi = \xi$, completing the proof. ■

Theorem 4.4.6 (Sakai)**.** *Let A be a C^*-algebra such that there is a Banach space X, and an isomorphism $X^* \cong A$. Then A is isomorphic to a von Neumann algebra.*

Proof. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be the direct sum of all GNS-representations corresponding to states in X. Then by Lemma 4.4.2 π is faithful and so all that remains is to show that $\pi(A)$ is a von Neumann algebra. Let $\{a_\alpha\}$ be a bounded increasing net in A_+, and let $x \in \mathcal{B}(\mathcal{H})$ be the SOT limit of $\pi(a_\alpha)$. From Lemma 4.4.3 we know that there also exists $a \in A$ such that a is the $\sigma(A, X)$-limit of $\{a_\alpha\}$.

For each state $\varphi \in X$ we have $\langle (x - \pi(a)) 1_\varphi, 1_\varphi \rangle = \lim_{\alpha \to \infty} \varphi(a_\alpha) - \varphi(a) = 0$, and similarly if $b \in A$, then $\langle (x - \pi(a))\pi(b)1_\varphi, \pi(b)1_\varphi \rangle = 0$. The polarization identity then gives $\langle (x - \pi(a))\pi(b)1_\varphi, \pi(c)1_\varphi \rangle = 0$ for all $b, c \in A$. As the net $\{a_\alpha\}$ is uniformly bounded, and as φ was arbitrary it then follows that $x = \pi(a)$. Proposition 4.4.5 then shows that $\pi(A)$ is a von Neumann algebra. ■

4.5 Standard representations

Theorem 4.5.1. *Let M be a von Neumann algebra, suppose that for $i \in \{1, 2\}$, $\pi_i : M \to \mathcal{B}(\mathcal{H}_i)$ is a normal faithful representation, and set $\mathcal{K} = \overline{\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2}$. Then the representations $\pi_i \otimes \mathrm{id} : M \to \mathcal{B}(\mathcal{H}_i \overline{\otimes} \mathcal{K})$ are unitarily equivalent.*

Proof. Let $\{\xi_\alpha\}_{\alpha \in I}$ be a maximal family of unit vectors in $\mathcal{H}_i \overline{\otimes} \mathcal{K}$ such that if P_α denotes the projection onto the closure of the subspace $(\pi_1 \otimes \mathrm{id})(M)\xi_\alpha$, then $\{P_\alpha\}_\alpha$ is a pairwise orthogonal family. Note that by maximality we have $\sum_\alpha P_\alpha = 1$.

By Proposition 4.2.2, for any normal state $\varphi \in M_*$ there exists a unit vector $\xi \in \mathcal{H}_2 \overline{\otimes} \overline{\mathcal{H}_2}$ such that $\varphi(x) = \langle (\pi_2 \otimes \mathrm{id})(x)\xi, \xi \rangle$ for all $x \in N$. It then follows that there exists a family $\{\eta_\alpha\}_{\alpha \in I}$ of unit vectors in $\mathcal{H}_2 \overline{\otimes} \mathcal{K}$ such that the projections Q_α onto the closure of the subspaces $(\pi_2 \otimes \mathrm{id})(N)\xi_\alpha$ are pairwise orthogonal, and such that for each $\alpha \in I$, and $x \in N$ we have

$$\langle (\pi_1 \otimes \mathrm{id})(x)\xi_\alpha, \xi_\alpha \rangle = \langle (\pi_2 \otimes \mathrm{id})(x)\eta_\alpha, \eta_\alpha \rangle.$$

By uniqueness of the GNS-construction there then exists a family of partial isometries $\{V_\alpha\}_{\alpha \in I} \subset \mathcal{B}(\mathcal{H}_1 \overline{\otimes} \mathcal{K}, \mathcal{H}_2 \overline{\otimes} \mathcal{K})$ such that $V_\alpha^* V_\alpha = P_\alpha$, and $V_\alpha V_\alpha^* = Q_\alpha$.

Setting $V = \sum_\alpha V_\alpha$ we then have that V is an isometry such that $V(\pi_1 \otimes \mathrm{id})(x) = (\pi_2 \otimes \mathrm{id})(x)V$ for all $x \in N$.

By symmetry, there also exists an isometry $W : \mathcal{H}_2 \overline{\otimes} \mathcal{K} \to \mathcal{H}_1 \overline{\otimes} \mathcal{K}$ such that $W(\pi_2 \otimes \mathrm{id})(x) = (\pi_1 \otimes \mathrm{id})(x)W$ for all $x \in N$. We then have $VV^*, VW \in (\pi_2 \otimes \mathrm{id})(N)'$, and $(VW)(VW)^* \leq VV^* \leq 1 = (VW)^*(VW)$, and so $VV^* \sim 1$ in $(\pi_2 \otimes \mathrm{id})(N)'$. Hence there exists an isometry $V_0 \in (\pi_2 \otimes \mathrm{id})(N)'$ such that $V_0 V_0^* = VV^*$.

Setting $U = V_0^* V$ we then have that U is a unitary operator such that $U(\pi_1 \otimes \mathrm{id})(x) = (\pi_2 \otimes \mathrm{id})(x)U$ for all $x \in N$. ■

Proposition 4.5.2. *Let M be a von Neumann algebra. Then M is countably decomposable if and only if M has a normal faithful state.*

Proof. If M has a normal faithful state φ, and $\{p_\alpha\}_{\alpha \in I}$ is a family of pairwise orthogonal projections such that $\sum_\alpha p_\alpha = 1$, then as $\varphi(\sum_\alpha p_\alpha) = \sum_\alpha \varphi(p_\alpha)$ it follows that $\varphi(p_\alpha) > 0$ for only countably many $\alpha \in I$. Faithfulness then implies that $p_\alpha \neq 0$ for only countably many $\alpha \in I$, and hence M is countably decomposable.

Conversely, suppose M is countably decomposable and by Zorn's lemma let $\{p_n\}$ be a maximal family of pairwise orthogonal projections, such that there exists a faithful normal state φ_n on $p_n M p_n$. Since M is countably decomposable we have that $\{p_n\}$ is countable and hence we will assume that the projections by indexed the natural numbers (the case when it is finite follows similarly). We must have $\sum_n p_n = 1$ since otherwise taking any normal state φ on $(1 - \sum_n p_n)M(1 - \sum_n p_n)$, we would have that φ is faithful on $s(\varphi)Ms(\varphi)$, contradicting the maximality of $\{p_n\}$.

If we then define $\psi(x) = \sum_{n \in \mathbb{N}} 2^{-n} \varphi_n(p_n x p_n)$, then it follows easily from Proposition 4.2.2 that ψ defines a normal state. Moreover, if $x \in M$, such that $\psi(x^* x) = 0$, then $\varphi_n(p_n x^* x p_n) = 0$ for each $n \in \mathbb{N}$ and hence $p_n x^* x p_n = 0$, and so $x p_n = 0$, for each $n \in \mathbb{N}$. Thus, ψ is faithful. ■

If M is a countably decomposable von Neumann algebra then a normal faithful representation $\pi : M \to \mathcal{B}(\mathcal{H})$ is **standard**[1] if there exists a cyclic and separating vector.

Example 4.5.3. Let M be a countably decomposable von Neumann algebra and suppose that $\varphi \in M_*$ is a normal faithful state, then the GNS representation $M \subset \mathcal{B}(L^2(M, \varphi))$ is a standard representation.

Theorem 4.5.4. *Let M be a countably decomposable von Neumann algebra. Then all standard representations are unitarily equivalent.*

Proof. Suppose for $i \in \{1, 2\}$ we have a standard representation $\pi_i : M \to \mathcal{B}(\mathcal{H}_i)$. By Theorem 4.5.1 we may assume that there is a normal representation $\pi : M \to \mathcal{B}(\mathcal{K})$ and projections $p_i \in \pi(M)'$, $i = 1, 2$, such that $\pi_i = p_i \pi$. If $\xi_i \in p_i \mathcal{K}$ are cyclic and separating vectors for $p_i \pi(M)$, then in particular we

[1] Standard representations can be defined in general, but for simplicity we will only consider the case when M is countably decomposable.

have that ξ_i are separating for $\pi(M)$, and hence $[\pi(M)'\xi_1] = 1 = [\pi(M)'\xi_2]$, thus by Proposition 3.3.6 we have $p_1 = [\pi(M)\xi_1] \sim [\pi(M)\xi_2] = p_2$ in $\pi(M)'$, and hence π_1 and π_2 are unitarily equivalent. ■

It will be convenient to give the notation $M \subset \mathcal{B}(L^2M)$ to the standard representation. This is only defined up to unitary conjugacy but for each normal faithful state φ we obtain a concrete realization as $M \subset \mathcal{B}(L^2(M, \varphi))$.

Corollary 4.5.5. *Let $M \subset \mathcal{B}(L^2M)$, and $N \subset \mathcal{B}(L^2N)$ be two countably decomposable von Neumann algebras. If $\theta : M \to N$ is an isomorphism, then there exists a unitary $U : L^2M \to L^2N$, such that $\theta(x) = UxU^*$ for all $x \in M$.*

4.6 The universal enveloping von Neumann algebra

If A is a C^*-algebra then we can consider the direct sum of all GNS-representations $\pi = \bigoplus_{\varphi \in S(A)} \pi_\varphi$, we call this representation the **universal $*$-representation**, and the von Neumann algebra $\pi(A)''$ generated by this representation is the **universal enveloping von Neumann algebra** of A. We will denote the universal enveloping von Neumann algebra by $\tilde{A}$.

Theorem 4.6.1. *Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation of a C^*-algebra A. Then there exists a unique linear map $\tilde{\pi} : A^{**} \to \pi(A)''$ with $\tilde{\pi} \circ i = \pi$ where i is the canonical embedding of A into A^{**}, such that $\tilde{\pi}$ takes the unit ball of A^{**} onto the unit ball of $\pi(A)''$ and is continuous with respect to the weak* and σ-weak topologies. Moreover, in the case when π is the universal $*$-representation, $\tilde{\pi}$ will be isometric, and a homeomorphism with respect to the weak* and σ-weak topologies.*

Proof. Set $M = \pi(A)''$. Then π induces a linear map from M^* to A^* and we will denote by π_* the restriction of this map to $M_* \subset M^*$. Taking the dual again we obtain the map $\tilde{\pi}$ from A^{**} into $(M_*)^* \cong M$.

This map is continuous by construction, and clearly satisfies $\tilde{\pi} \circ i = \pi$. Since homomorphisms of C^*-algebras are contractions, it follows that $\tilde{\pi}$ applied to the unit ball of A^{**} is compact and contains the image of unit ball of A under the map π as a dense subset. By Kaplansky's density theorem it then follows that $\tilde{\pi}$ applied to the unit ball of A^{**} is equal to the unit ball of M.

If π is the universal $*$-representation then $\pi_*(M_*)$ contains all positive linear functionals, and hence by Theorem 4.6.2 is equal to A^*. Thus, it follows that $\tilde{\pi}$ is injective and hence gives an isometry, and since A^{**} and M are locally compact with respect to the weak* and σ-weak topologies, it follows that $\tilde{\pi}$ is a homeomorphism. ■

The previous theorem allows us to extend the Jordan decomposition of linear functionals to arbitrary C^*-algebras.

Corollary 4.6.2. *Suppose that A is a C^*-algebra, and $\varphi \in A^*$ is Hermitian, then there exists unique positive linear functionals φ_+ and φ_- on A such that $\varphi = \varphi_+ - \varphi_-$.*

Corollary 4.6.3. *Let A be a C^*-algebra, then A^* is spanned by positive linear functionals.*

4.7 Traces on finite von Neumann algebras

Lemma 4.7.1. *Let M be a finite von Neumann algebra, and $p \in \mathcal{P}(M)$ a non-zero projection. If $\{p_i\}_{i \in I}$ is a family of pairwise orthogonal projection in M such that $p_i \sim p$ for each $i \in I$, then I is finite.*

Proof. If I were infinite then there would exist a proper subset $J \subset I$ with the same cardinality and we would then have that $\sum_{i \in I} p_i \sim \sum_{j \in J} p_j < \sum_{i \in I} p_i$, showing that $\sum_{i \in I} p_i$ is not finite, contradicting Proposition 3.2.2. ■

Lemma 4.7.2. *Let M be a type II_1 von Neumann algebra. Then there exists a projection $p_{1/2} \in \mathcal{P}(M)$ such that $p_{1/2} \sim 1 - p_{1/2}$.*

Proof. Let $\{p_i, q_i\}$ be a maximal family of pairwise orthogonal projections, such that $p_i \sim q_i$ for each i. If $p_{1/2} = \sum_i p_i$, and $q = \sum_i q_i$ then $p_{1/2} \sim q$, and if $p_{1/2} + q \neq 1$, then taking $p_0, q_0 \leq 1 - (p_{1/2} + q)$ which are orthogonal, but not centrally orthogonal, (this would be possible since $(1-(p_{1/2}+q))M(1-(p_{1/2}+q))$ is not abelian), there would then exist equivalent subprojections of p_0, and q_0 contradicting maximality. Thus we have $q = 1 - p_{1/2}$. ■

If M is a type II_1 von Neumann algebra, and if we set $p_1 = 1$, $p_0 = 0$, and $p_{1/2} \in \mathcal{P}(M)$ is as in the previous lemma, so that $p_{1/2} = v^*v$, and $p_1 - p_{1/2} = vv^*$ for some partial isometry $v \in M$, then $p_{1/2}Mp_{1/2}$ is also type II_1 and so we may iterate the previous lemma to produce $p_{1/4} \leq p_{1/2}$ such that $p_{1/4} \sim p_{1/2} - p_{1/4}$. If we set $p_{3/4} = p_{1/2} + vp_{1/4}v^*$, then we have $p_{1/4} \sim p_{(k+1)/4} - p_{k/4}$ for all $0 \leq k \leq 3$.

Proceeding by induction we may construct for each dyadic rational $r \in [0,1]$, a projection $p_r \in \mathcal{P}(M)$ such that $p_r \leq p_s$ if $r \leq s$, and if $0 \leq r \leq s \leq 1$, and $0 \leq r' \leq s \leq 1$, such that $s - r = s' - r'$, then we have $p_s - p_r \sim p_{s'} - p_{r'}$.

Lemma 4.7.3. *Let M by a type II_1 von Neumann algebra, and let $\{p_r\}_r$ be as above. If $p \in \mathcal{P}(M)$, $p \neq 0$, then there exists a central projection $z \in \mathcal{P}(\mathcal{Z}(M))$ such that $pz \neq 0$, and $p_r z \preceq pz$ for some positive dyadic rational r.*

Proof. By considering $Mz(p)$ we may assume that $z(p) = 1$. If the above does not hold, then by the comparison theorem we would have $p \preceq p_r$ for every positive dyadic rational r. Thus, p would be equivalent to a subprojection of $p_{2^{-k}} - p_{2^{-(k+1)}}$ for every $k \in \mathbb{N}$, which would contradict Lemma 4.7.1. ■

If M is a von Neumann algebra, then a projection $p \in \mathcal{P}(M)$ is **monic** if there exists a finite collection of pairwise orthogonal projections $\{p_1, p_2, \ldots, p_n\}$

such that $p_i \sim p$ for each $1 \leq i \leq n$, and $\sum_{i=1}^n p_i \in \mathcal{Z}(M)$. Note that in the type II_1 case, any of the projections $p_{1/2^k}$ as defined above, are monic.

Proposition 4.7.4. *If M is a finite von Neumann algebra, then every projection is the sum of pairwise orthogonal monic projections.*

Proof. By a maximality argument it is sufficient to show that every non-zero projection has a non-zero monic subprojection. Also, by restricting with central projections it suffices to consider separately the cases when M is type II, or type I_n, with $n < \infty$.

The type II case follows from Lemma 4.7.3, and the type I_n case follows by considering any non-zero abelian projection. ∎

If M is a von Neumann algebra, then a **center-valued state** is a linear map $\varphi : M \to \mathcal{Z}(M)$ such that $\varphi(x^*x) \geq 0$ for all $x \in M$, $\varphi_{|\mathcal{Z}(M)} = \text{id}$, and $\varphi(zx) = z\varphi(x)$ for all $x \in M$, $z \in \mathcal{Z}(M)$. We say that φ is faithful if $\varphi(x^*x) \neq 0$ whenever $x \neq 0$.

Lemma 4.7.5. *Let M be a von Neumann algebra, and $\varphi : M \to \mathcal{Z}(M)$ a center-valued state, then φ is bounded and $\|\varphi\| = 1$.*

Proof. This is exactly the same as the proof of Lemma 4.1.2. First note that φ is Hermitian since if y is self-adjoint we have $\varphi(y) = \varphi(y_+) - \varphi(y_-)$ is also self-adjoint, and in general, if $y = y_1 + iy_2$ where $y_i^* = y_i$ then we have $\varphi(y^*) = \varphi(y_1) - i\varphi(y_2) = \varphi(y)^*$.

Next, note that for all $y \in M$ we have $\varphi(\|y + y^*\| \pm (y + y^*)) \geq 0$, and so $|\varphi(y + y^*)| \leq \|y + y^*\|$, hence $\|\varphi(y)\| = \|\varphi(\frac{y+y^*}{2})\| \leq \|\frac{y+y^*}{2}\| \leq \|y\|$, showing $\|\varphi\| \leq 1 \leq \|\varphi\|$. ∎

Lemma 4.7.6. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then M has a normal center-valued state.*

Proof. The von Neumann algebra $\mathcal{Z}(M)'$ is type I, and hence has an abelian projection q with central support equal to 1. We then have $qMq \subset q\mathcal{Z}(M)'q = \mathcal{Z}(M)q$, and $\theta(z) = zq$ defines a normal isomorphism from $\mathcal{Z}(M)$ onto $\mathcal{Z}(M)q$. If we set $\varphi(x) = \theta^{-1}(qxq)$, for $x \in M$, then φ is a normal center-valued state. ∎

Lemma 4.7.7. *Let M be a von Neumann algebra, and $\tau : M \to \mathcal{Z}(M)$ a center-valued state. The following are equivalent:*

(i) *$\tau(xy) = \tau(yx)$, for all $x, y \in M$.*

(ii) *$\tau(xx^*) = \tau(x^*x)$, for all $x \in M$.*

(iii) *$\tau(p) = \tau(q)$, for all equivalent projections $p, q \in \mathcal{P}(M)$.*

Proof. The implication (i) $\implies$ (ii) is obvious, as is (ii) $\implies$ (iii). Suppose (iii) holds, then for all $p \in \mathcal{P}(M)$, and $u \in \mathcal{U}(M)$ we have $\tau(upu^*) = \tau(p)$. Since τ is bounded it then follows from functional calculus that $\tau(uxu^*) = \tau(x)$ for all $x = x^* \in M$, and $u \in \mathcal{U}(M)$. Considering the real and imaginary parts this then holds for all $x \in M$, and replacing x with xu it then follows that $\tau(ux) = \tau(xu)$ for all $x \in M$, $u \in \mathcal{U}(M)$. As every operator is a span of four unitaries this then shows (i). ∎

We say that τ is a **center-valued trace** if it satisfies the equivalent conditions of the previous lemma.

Lemma 4.7.8. *Let M be a finite von Neumann algebra. If $\varphi : M \to \mathcal{Z}(M)$ is a normal center-valued state, then for each $\varepsilon > 0$ there exists $p \in \mathcal{P}(M)$, such that $\varphi(p) \neq 0$, and for all $x \in pMp$ we have*

$$\varphi(xx^*) \leq (1+\varepsilon)\varphi(x^*x).$$

Proof. Let $q_0 = 1 - \sum_i q_i$ where $\{q_i\}$ is a maximal family of pairwise orthogonal projections with $\varphi(q_i) = 0$. By normality we have $\varphi(q_0) = 1$, and φ is faithful when restricted to $q_0 M q_0$.

We let $\{e_i, f_i\}$ be a maximal family of projections such that $\{e_i\}$, and $\{f_i\}$ are each pairwise orthogonal, $e_i \sim f_i$ for each i, and $\varphi(e_i) > \varphi(f_i)$ for each i. If we set $e = q_0 - \sum_i e_i$, and $f = q_0 - \sum_i f_i$ then unless φ was already a trace we have $\varphi(f) > \varphi(e) \geq 0$, Hence, $f \neq 0$, and by Proposition 3.2.8 we have $e \sim f$, hence $e \neq 0$.

If we let μ be the smallest number such that $\varphi(\tilde{e}) \leq \mu\varphi(\tilde{f})$ whenever $\tilde{e} \leq e$, $\tilde{f} \leq f$, and $\tilde{e} \sim \tilde{f}$ then $\mu \neq 0$ since $\varphi(e) \neq 0$, and there exists $\tilde{e} \leq e$, $\tilde{f} \leq f$, such that $\tilde{e} \sim \tilde{f}$ and $(1+\varepsilon)\varphi(\tilde{e}) \not\leq \mu\varphi(\tilde{f})$, and thus cutting down by a suitable central projection we may assume $(1+\varepsilon)\varphi(\tilde{e}) > \mu\varphi(\tilde{f})$.

If we now take $\{\hat{e}_i, \hat{f}_i\}$ a maximal family such that $\{\hat{e}_i\}$, and $\{\hat{f}_i\}$ are each pairwise orthogonal, $\hat{e}_i \leq \tilde{e}$, $\hat{f}_i \leq \tilde{f}_i$, $\hat{e}_i \sim \hat{f}_i$, and $(1+\varepsilon)\varphi(\hat{e}_i) \leq \mu\varphi(\hat{f}_i)$, then $p = \tilde{e} - \sum_i \hat{e}_i$ is non-zero, and equivalent to $q = \tilde{f} - \sum_i \hat{f}_i$. Moreover, if $p_1, p_2 \leq p$, such that $p_1 \sim p_2$, then there exists $r \leq q$ such that $r \sim p_1$, and hence

$$\varphi(p_1) \leq \mu\varphi(r) \leq (1+\varepsilon)\varphi(p_2).$$

Since φ is bounded and every positive operator can be approximated uniformly by the span of its spectral projections it then follows that

$$\varphi(ux^*xu) \leq (1+\varepsilon)\varphi(x^*x),$$

for each $x \in pMp$, and $u \in \mathcal{U}(pMp)$. If x^* has polar decomposition $x^* = v|x^*|$, then since pMp is finite we can extend v to a unitary $u^* \in \mathcal{U}(pMp)$ and from above this shows $\varphi(xx^*) \leq (1+\varepsilon)\varphi(x^*x)$. ∎

Lemma 4.7.9. *Let M be a finite von Neumann algebra, and $\varepsilon > 0$. Then there is a normal center-valued state φ such that for all $x \in M$ we have*

$$\varphi(xx^*) \leq (1+\varepsilon)\varphi(x^*x).$$

Proof. We need only show existence of such a state on Mz for some non-zero central projection z as a maximality argument will then finish the proof.

By Lemma 4.7.6 there exists a normal center-valued state ψ, and so by Lemma 4.7.8 there exists some non-zero projection $p \in \mathcal{P}(M)$ such that $\psi(xx^*) \leq (1+\varepsilon)\psi(x^*x)$ for all $x \in pMp$, and by Proposition 4.7.4 we may assume that p is monic.

Let $\{p_1, p_2, \ldots, p_n\}$ be a finite family of pairwise orthogonal projections such that $p_i \sim p$, and $z_0 = \sum_i p_i \in \mathcal{Z}(M)$. Take $v_i \in M$ such that $v_i^* v_i = p_i$ and $v_i v_i^* = p$. For $x \in Mz_0$ we then set $\varphi_0(x) = \sum_{i=1}^n \psi(v_i x v_i^*)$. If $x \in Mz_0$ we have

$$\begin{aligned}
0 \leq \varphi_0(xx^*) = \varphi_0(xz_0x^*) &= \sum_{j=1}^n \varphi_0(xp_jx^*) \\
&= \sum_{i,j=1}^n \psi(v_i x v_j^* v_j x^* v_i^*) \leq (1+\varepsilon) \sum_{i,j=1}^n \psi(v_j x^* v_i^* v_i x v_j^*) \\
&= (1+\varepsilon) \sum_{i=1}^n \varphi_0(x^* p_i x) = (1+\varepsilon)\varphi_0(x^*x).
\end{aligned}$$

For $\tilde{z} \in \mathcal{Z}(M)z_0$, and $x \in Mz_0$ we have $\varphi_0(\tilde{z}x) = \sum_{i=1}^n \psi(v_i \tilde{z} x v_i^*) = \tilde{z}\varphi_0(x)$.

In general, it may not be the case that $\varphi_0(z_0) = z_0$. However, we do have $\varphi_0(z_0) > 0$, and hence, taking a spectral projection z of the form $z = 1_{[\varepsilon,\infty)}(\varphi_0(z_0))$, then we have that $0 \neq z = y\varphi_0(z_0)$ for some $y \geq 0$, $y \in \mathcal{Z}(M)$. If we set $\varphi(x) = y\varphi_0(x)$, then we still have $0 \leq \varphi(xx^*) \leq (1+\varepsilon)\varphi(xx^*)$, for $x \in Mz$, and φ is then a center-valued state on Mz. ∎

Theorem 4.7.10. *A von Neumann algebra M is finite if and only if there exists a (normal) center-valued trace. Moreover, any such trace is faithful.*

Proof. First note that if τ is a center-valued trace on M and if p is any monic projection, say $\sum_{i=1}^n p_i = z \in \mathcal{Z}(M)$ where $p_i \sim p$, then $z = \tau(\sum_{i=1}^n p_i) = n\tau(p)$, and hence $\tau(p) > 0$. Proposition 4.7.4 then shows that $\tau(p) > 0$ for any non-zero projection, and hence it follows that $\tau(x) > 0$ for any non-zero positive operator $x \in M$, so that τ is faithful.

If $v \in M$ is an isometry we then have $1 = \tau(v^*v) = \tau(vv^*)$, and hence $\tau(1 - vv^*) = 0$ which shows that $vv^* = 1$ and hence M is finite.

Conversely, if M is finite, then from Lemma 4.7.9, if $\{a_n\}$ is a strictly decreasing sequence of real number that converge to 1, there exists a sequence of normal center-valued states τ_n such that $\tau_n(xx^*) \leq a_n\tau_n(x^*x)$ for each $n \in \mathbb{N}$, and $x \in M$. We claim that if $1 \leq m < n$, then the function $a_m^2\tau_m - \tau_n$ is a positive linear map. From this Lemma 4.7.5 would then imply that $\|a_m^2\tau_m - \tau_n\| \leq a_m^2 - 1$.

To see that $a_m^2\tau_m - \tau_n$ is positive it is enough to consider projections, and since this map is normal, by Proposition 4.7.4 it is then enough to consider monic projections. So let $p \sim p_1 \sim \cdots \sim p_k$ be non-zero projections such that $z = \sum_{i=1}^k p_i$ is a projection in $\mathcal{Z}(M)$. We then have $\tau_n(p) \leq a_n\tau_n(p_i)$, and

$\tau_m(p) \leq a_m \tau_m(p_i)$ for each $1 \leq i \leq k$. Hence,

$$\begin{aligned} k\tau_n(p) &\leq a_n \sum_{i=1}^{k} \tau_n(p_i) = a_n \tau_n(z) = a_n z \\ &= a_n \tau_m(z) = a_n \sum_{i=1}^{k} \tau_m(p_i) \\ &\leq k a_n a_m \tau_m(p) \leq k a_m^2 \tau_m(p). \end{aligned}$$

Thus, we have shown that $\|a_m^2 \tau_m - \tau_n\| \leq a_m^2 - 1$, and hence it follows that there is a bounded linear map τ, such that $\|\tau - \tau_m\| \to 0$.

We then have $0 \leq \tau(xx^*) \leq \tau(x^*x)$ and so we must have equality for all $x \in M$. Considering the polar decomposition of x it then follows that $\tau(u^*yu) = \tau(y)$ for all $u \in \mathcal{U}(M)$, and $y \geq 0$, invertible. Taking linear combinations it then follows that $\tau(u^*yu) = \tau(y)$ for all $u \in \mathcal{U}(M)$, and $y \in M$, or equivalently that $\tau(yu) = \tau(uy)$ for all $u \in \mathcal{U}(M)$, $y \in M$. Since every element is a linear combination of unitaries we then have $\tau(xy) = \tau(yx)$ for all $x, y \in M$.

Clearly, $\tau_{|\mathcal{Z}(M)} = \mathrm{id}$ and $\tau(zx) = z\tau(x)$ for all $z \in \mathcal{Z}(M)$, and $x \in M$. Thus, the only thing remaining to check is that τ is normal. If $\varphi \in M_*$, then $\|\varphi \circ \tau - \varphi \circ \tau_m\| \leq \|\varphi\| \|\tau - \tau_m\|$, and hence $\varphi \circ \tau$ since M_* is closed. Thus, τ is normal. ∎

Proposition 4.7.11. *Let M be a finite von Neumann algebra with center-valued trace τ. If $p, q \in \mathcal{P}(M)$, then $p \preceq q$ if and only if $\tau(p) \leq \tau(q)$*

Proof. If $p = v^*v$, and $vv^* \leq q$ then $\tau(p) = \tau(vv^*) \leq \tau(q)$. Conversely, if $\tau(p) \leq \tau(q)$, then by the comparison theorem there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ such that $pz \preceq qz$ and $(1-z)q \preceq (1-z)p$. If $v^*v = (1-z)q$ and $vv^* \leq (1-z)p$ then we have $\tau(vv^*) \leq \tau((1-z)p) = (1-z)\tau(p) \leq \tau((1-z)q) = \tau(vv^*)$, and since τ is faithful we have $vv^* = (1-z)p$, hence $(1-z)q \sim (1-z)p$ and so $p \preceq q$. ∎

Proposition 4.7.12. *Let M be a finite von Neumann algebra with normal faithful trace τ, then conjugation $x \mapsto x^*$ extends to an anti-linear isometry $J : L^2(M,\tau) \to L^2(M,\tau)$, and we have $M' \cap \mathcal{B}(L^2(M,\tau)) = JMJ$.*

Proof. For $x \in M$ we have $\|x^*\|_2^2 = \tau(xx^*) = \tau(x^*x) = \|x\|_2^2$, thus this extends to an anti-linear isometry $J : L^2(M,\tau) \to L^2(M,\tau)$. Note that we have $\langle J\xi, J\eta \rangle = \langle \eta, \xi \rangle$ for all $\xi, \eta \in L^2(M,\tau)$.

For a vector $\xi \in L^2(M,\tau)$ we define the unbounded operators $L_\xi^0 : M1_\tau \to L^2(M,\tau)$ and $R_\xi^0 : M1_\tau \to L^2(M,\tau)$ by

$$L_\xi^0 x1_\tau = (Jx^*J)\xi; \qquad R_\xi^0 x1_\tau = x\xi.$$

Note that if $x_i \in M$, and $\eta \in L^2(M,\tau)$ such that $\|x_i\|_2 \to 0$, and $L_\xi^0 x_i 1_\tau \to \eta$, then for all $z \in M$ we have

$$\begin{aligned} |\langle \eta, z\rangle| = \lim_{i\to\infty} |\langle (Jx_i^* J)\xi, z\rangle| &= \lim_{i\to\infty} |\langle \xi, (Jx_i J)z\rangle| \\ &= \lim_{i\to\infty} |\langle \xi, z x_i^*\rangle| \\ &\leq \lim_{i\to\infty} \|\xi\|_2 \|z\| \|x_i\|_2 = 0. \end{aligned}$$

Thus, the operator L_ξ^0 is closable and completing the graph gives rise to a closed operator L_ξ. Similarly, R_ξ^0 is closable and we have the corresponding operator R_ξ. Note that $L_{x1_\tau} = x$ for all $x \in M$.

If $x, y \in M$, and $\xi \in L^2(M, \tau)$ then we have

$$\langle R_{J\xi} x1_\tau, y1_\tau\rangle = \langle xJ\xi, y1_\tau\rangle = \langle y^* x1_\tau, \xi\rangle = \langle x1_\tau, R_\xi y1_\tau\rangle,$$

hence $R_{J\xi} = R_\xi^*$.

We also have

$$(JL_\xi J)x1_\tau = xJ\xi = R_{J\xi} x1_\tau = R_\xi^* x1_\tau,$$

hence $JL_\xi J = R_\xi^*$. If we set $\tilde{M} = \{L_\xi \mid L_\xi \in \mathcal{B}(L^2(M,\tau))\}$ and $\tilde{N} := \{R_\xi \mid R_\xi \in \mathcal{B}(L^2(M,\tau))\}$, then since $J\tilde{M}J = \tilde{N}$ to finish the proposition it is enough to show $M = \tilde{M}$ and $M' = \tilde{N}$.

Note that we clearly have $M \subset \tilde{M}$, and $\tilde{N} \subset M'$. If $x \in M'$, then consider $\xi = x1_\tau$. For $y \in M$ we then have

$$R_\xi y1_\tau = y\xi = yx1_\tau = xy1_\tau,$$

Thus $R_\xi = x$, showing $M' \subset \tilde{N}$. If $L_\xi \in \tilde{M}$, then $L_\xi \in \tilde{N}' = M'' = M$ showing that $\tilde{M} \subset M$. ■

4.8 Dixmier's property

Lemma 4.8.1. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $x = x^* \in M$, then there exists a unitary $u \in \mathcal{U}(M)$, and $y = y^* \in \mathcal{Z}(M)$ such that $\|\frac{1}{2}(x + u^* x u) - y\| \leq \frac{3}{4}\|x\|$.*

Proof. We may assume that $\|x\| = 1$. Let $p = 1_{[0,\infty)}(x)$, and $q = 1 - p$. By the comparison theorem there exists $z \in \mathcal{P}(\mathcal{Z}(M))$, $q_1, q_2, p_1, p_2 \in \mathcal{P}(M)$ such that

$$zq \sim p_1 \leq p_1 + p_2 = zp, \quad \text{and} \quad (1-z)p \sim q_1 \leq q_1 + q_2 = (1-z)q.$$

Suppose $v, w \in M$ such that $v^* v = zq$, $vv^* = p_1$, $w^* w = (1-z)p$, and $ww^* = q_1$. Set $u = v + v^* + w + w^* + q_2 + p_2$. Then $u \in \mathcal{U}(M)$, and we have

$$\begin{aligned} &u^* p_1 u = zq, & &u^* zq u = p_1, & &u^* p_2 u = p_2; \\ &u^* q_1 u = (1-z)p, & &u^*(1-z)pu = q_1, & &u^* q_2 u = q_2. \end{aligned}$$

We then have $-zq \leq zx \leq zp = p_1 + p_2$, and conjugating by u gives $-p_1 \leq zu^*xu \leq zq + p_2$. Hence,

$$-\frac{1}{2}(zq + p_1) \leq \frac{1}{2}(zx + zu^*xu) \leq \frac{1}{2}(p_1 + zq) + p_2.$$

As $zq + p_1 + p_2 = z$ we then have

$$-\frac{1}{2}z \leq \frac{1}{2}(zx + zu^*xu) \leq z,$$

hence,

$$-\frac{3}{4}z \leq \frac{1}{2}(zx + zu^*xu) - \frac{1}{4}z \leq \frac{3}{4}z.$$

A similar argument shows

$$-\frac{3}{4}(1-z) \leq \frac{1}{2}((1-z)x + (1-z)u^*xu) + \frac{1}{4}(1-z) \leq \frac{3}{4}(1-z).$$

Thus,

$$\|\frac{1}{2}(x + u^*xu) - \frac{1}{4}(2z-1)\| \leq \frac{3}{4}.$$

■

Theorem 4.8.2 (Dixmier's property)**.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. For all $x \in M$ denote by $\overline{K}(x)$ the norm closed convex hull of the unitary orbit of x. Then we have $\mathcal{Z}(M) \cap \overline{K}(x) \neq \emptyset$.*

Proof. We denote by $\mathcal{K}$ the set of all maps α from M to M of the form $\alpha(y) = \sum_{i=1}^n \alpha_i u_i^* y u_i$, where $u_1, \ldots, u_n \in \mathcal{U}(M)$, $\alpha_1, \ldots, \alpha_n \geq 0$, and $\sum_{i=1}^n \alpha_i = 1$.

Suppose $x = a_0 + ib_0$ where a_0 and b_0 are self-adjoint. By iterating Lemma 4.8.1, there exists a sequence $\alpha_k \in \mathcal{K}$, and $y_k = y_k^* \in \mathcal{Z}(M)$ such that if we set $\tilde{y}_k = \sum_{i=1}^k y_i$, and $a_k = \alpha_k(a_{k-1})$ then

$$\|a_k - \tilde{y}_k\| = \|\alpha_k(a_{k-1} - \tilde{y}_{k-1}) - y_k\| \leq \left(\frac{3}{4}\right)^k \|a_0\|.$$

Hence, for any $\varepsilon > 0$ there exists $\alpha \in \mathcal{K}$, and $y \in \mathcal{Z}(M)$ such that $\|\alpha(a_0) - y\| < \varepsilon$. Similarly, there then exists $\beta \in \mathcal{K}$, and $z \in \mathcal{Z}(M)$ such that $\|\beta(\alpha(b_0)) - z\| < \varepsilon$, and note that we still have $\|\beta(\alpha(a_0)) - y\| = \|\beta(\alpha(a_0) - y)\| < \varepsilon$. Thus, we have $\|\beta \circ \alpha(x) - (y + iz)\| < 2\varepsilon$.

We can therefore take a sequence $\alpha_k \in \mathcal{K}$, and $z_k \in \mathcal{Z}(M)$ such that if we define $x_0 = x$, and $x_k = \alpha_k(x_{k-1})$ then

$$\|x_k - z_k\| < 1/2^k.$$

In particular, we have $\|x_{k+1} - x_k\| \leq \|\alpha_{k+1}(x_k - z_k) + (z_k - x_k)\| \leq 1/2^{k-1}$, and so the sequences $\{x_k\}$ and $\{z_k\}$ converge in norm to an element $z \in \mathcal{Z}(M) \cap \overline{K}(x)$. ■

Corollary 4.8.3. *A von Neumann algebra M is finite if and only if $\mathcal{Z}(M) \cap \overline{K}(x)$ consists of a single point for each $x \in M$. Moreover, if M is finite then it has a unique center-valued trace τ, and we have $\mathcal{Z}(M) \cap \overline{K}(x) = \{\tau(x)\}$ for all $x \in M$.*

Proof. If M is finite, then for any center-valued trace τ we have that τ is constant on $\overline{K}(x)$, and so $\emptyset \neq \overline{K}(x) \cap \mathcal{Z}(M) \subset \{\tau(x)\}$. Since the trace τ was arbitrary, and since M has a trace by Theorem 4.7.10, it follows that the trace must be unique.

Conversely, if $\overline{K}(x) \cap \mathcal{Z}(M)$ consists of a single element $\tau(x)$ for each $x \in M$, then τ defines a center-valued state, and we have $\tau(u^*xu) = \tau(x)$ for each $u \in \mathcal{U}(M)$ and $x \in M$, hence τ is a trace and so M is finite by Theorem 4.7.10. ■

4.8.1 The fundamental group of a II_1 factor

Let M be a II_1 factor, and let τ be the unique trace on M. Then for all $n \in \mathbb{N}$ we have that $\mathbb{M}_n(M)$ is again a II_1 factor with unique trace given by $\tau_n([x_{i,j}]) = \frac{1}{n}\sum_{i=1}^n \tau(x_{i,i})$. If $0 < t \leq n$ then we know from Proposition 4.7.11 that for any two projections $p, q \in \mathcal{P}(\mathbb{M}_n(M))$ with trace $\tau_n(p) = \tau_n(q) = t/n$ there is a unitary $u \in \mathcal{U}(\mathbb{M}_n(M))$ such that $upMpu^* = qMq$. Thus, up to isomorphism the factor pMp only depends on t. Note that this is also independent of n since any two matrix algebras over M can be embedded into a larger common matrix algebra. The **amplification** of M with parameter t is the factor M^t which is define as the II_1 factor pMp (which is unique up to isomorphism class).

Note that $M^1 \cong M$, and also $(M^t)^s \cong M^{ts}$ for all $t, s > 0$. The **fundamental group**[2] of M is $\mathcal{F}(M) = \{t > 0 \mid M^t \cong M\}$ which is easily seen for form a subgroup of the multiplicative group $\mathbb{R}_{>0}$. Note that if $\theta : M \to N$ is a $*$-isomorphism then $\mathcal{F}(M) = \mathcal{F}(N)$ and hence the fundamental group is an isomorphism class invariant of M.[3] Note that for all $t > 0$ we have $\mathcal{F}(M^t) = \mathcal{F}(M)$.

If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and we consider the conjugate Hilbert space $\overline{\mathcal{H}}$. Recall that to each operator $x \in M$ we may associate the operator $\overline{x} \in \mathcal{B}(\overline{\mathcal{H}})$ which is defined by $\overline{x}\overline{\xi} = \overline{x\xi}$. The **opposite von Neumann algebra** $M^{\mathrm{o}} = \{\overline{x} \mid x \in M\} \subset \mathcal{B}(\overline{\mathcal{H}})$ is clearly a von Neumann algebra, and the map $x \mapsto \overline{x}$ defines an anti-linear isomorphism between M and M^{o}. If one prefers to work with linear maps then consider $x^{\mathrm{o}} = \overline{x^*}$, the map $x \mapsto x^{\mathrm{o}}$ is then a normal linear isometry from M to M^{o}, however, this is not an isomorphism but rather an **anti-isomorphism**, i.e., $(xy)^{\mathrm{o}} = y^{\mathrm{o}}x^{\mathrm{o}}$ for all $x, y \in M$. It is clear that $\mathcal{F}(M^{\mathrm{o}}) = \mathcal{F}(M)$, for all II_1 factors M.

[2]This has no relation to the better known notion in topology.

[3]Better terminology might be the fundamental subgroup, since it is an invariant of M not just as an abstract group but rather as a subgroup of $\mathbb{R}_{>0}$.

4.9 Traces on semi-finite von Neumann algebras

4.9.1 Weights

By the Riesz representation theorem, if K is a compact Hausdorff space then states on $C(K)$ are in 1-1 correspondence with Radon probability measures on K. If we consider positive Radon measures which are not necessarily finite then this leads to the notion of a weight. Specifically, a **weight** on a C^*-algebra A is a map $\varphi : A_+ \to [0, \infty]$ satisfying the following conditions for all $x, y \in A_+$, $\lambda > 0$:

$$\varphi(x + y) = \varphi(x) + \varphi(y), \quad \varphi(\lambda x) = \lambda\varphi(x), \quad \varphi(0) = 0.$$

The weight is **faithful** if $\varphi(x) \neq 0$ for every non-zero $x \in A_+$, and φ is **tracial** if $\varphi(xx^*) = \varphi(x^*x)$, for all $x \in A$.

If A is a von Neumann algebra then a weight φ is **semi-finite** if

$$\mathfrak{p}_\varphi = \{x \in A_+ \mid \varphi(x) < \infty\}$$

generates A as a von Neumann algebra, and φ is **normal** if $\varphi(\sup x_i) = \sup \varphi(x_i)$, for every bounded increasing net $\{x_i\}$ in A_+.

Example 4.9.1. The trace Tr is a normal faithful semi-finite tracial weight on $\mathcal{B}(\mathcal{H})$.

Lemma 4.9.2. *Let φ be a weight on a C^*-algebra A, then*

a) $\mathfrak{n}_\varphi = \{x \in A \mid x^*x \in \mathfrak{p}_\varphi\}$ *is a left ideal of A.*

b) $\mathfrak{m}_\varphi = \{\sum_{i=1}^n y_i^* x_i \mid x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{n}_\varphi\}$ *is a $*$-subalgebra of $\mathfrak{n}_\varphi$ which is equal to the span of $\mathfrak{p}_\varphi$.*

Proof. That $\mathfrak{n}_\varphi$ is a linear subspace follows from the inequality

$$(x + y)^*(x + y) \leq (x + y)^*(x + y) + (x - y)^*(x - y) = 2(x^*x + y^*y).$$

That $\mathfrak{n}_\varphi$ is a left ideal then follows from the inequality $(ax)^*(ax) \leq \|a\|x^*x$. Since $\mathfrak{n}_\varphi$ is a linear subspace we have that $\mathfrak{m}_\varphi$ is a $*$-subalgebra. It is easy to see that $\mathfrak{m}_\varphi$ contains $\mathfrak{p}_\varphi$, and the fact that it is equal to the span of $\mathfrak{p}_\varphi$ follows from the polarization identity

$$4y^*x = \sum_{k=0}^{3} i^k (x + i^k y)^*(x + i^k y).$$

■

We refer to $\mathfrak{m}_\varphi$ as the definition domain of φ. If $\varphi(1) < \infty$, then we can linearly extend φ to all of A, and after rescaling obtain a state.

If we fix a weight φ on a C^*-algebra A, then the set

$$N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$$

is a left ideal of A contained in $\mathfrak{n}_\varphi$. We may then consider the quotient space $\mathfrak{n}_\varphi/N_\varphi$ with the quotient map $\eta_\varphi : \mathfrak{n}_\varphi \to \mathfrak{n}_\varphi/N_\varphi$. This then gives a positive definite sesqui-linear form on $\mathfrak{n}_\varphi/N_\varphi$ by the formula

$$\langle \eta_\varphi(x), \eta_\varphi(y) \rangle = \varphi(y^* x),$$

for $x, y \in \mathfrak{n}_\varphi$. We denote by $L^2(A, \varphi)$ the Hilbert space completion of this form. Since $\mathfrak{n}_\varphi$, and N_φ are both left ideals we obtain a A module structure on $\mathfrak{n}_\varphi/N_\varphi$ by left multiplication, which from the inequality $(ax)^*(ax) \leq \|a\|^2 x^* x$ extends to a representation $\pi_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$.

The following proposition follows as in the GNS-construction and so we leave it to the reader.

Proposition 4.9.3. *Let φ be a weight on a C^*-algebra A, then $\pi_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$ is a continuous $*$-representation, which is faithful if φ is faithful. Moreover, if A is a von Neumann algebra, and φ is normal, then so is the representation π_φ.*

We call the triple $(\pi_\varphi, L^2(A, \varphi), \eta_\varphi)$ the **semi-cyclic representation** of A. Note that if A is unital, and φ is a state on A, then the map $\eta_\varphi : A \to \mathcal{H}_\varphi$ is completely determined by the value $\eta_\varphi(1)$. And so in this case we can think of this triple as a representation, together with a cyclic vector.

Theorem 4.9.4. *Let M be a semi-finite factor, then there exists a unique, up to scalar multiplication, normal semi-finite tracial weight* $\mathrm{Tr} : M_+ \to [0, \infty]$. *Moreover,* Tr *is faithful, and $p \in \mathcal{P}(M)$ is finite if and only if* $\mathrm{Tr}(p) < \infty$.

Proof. We have already constructed a tracial state on finite factors, thus for existence we need only consider the case when M is properly infinite. Let $p_0 \in M$ be a non-zero finite projection, then there exists an infinite family $\{p_n\}_{n \in I}$ of pairwise orthogonal projections, such that $\sum_n p_n = 1$, and $p_0 \sim p_n$ for each $n \in I$. Take $v_n \in M$ such that $v_n^* v_n = p_0$, and $v_n^* v_n = p_n$, and let τ_0 be the unique tracial state on $p_0 M p_0$. We define $\mathrm{Tr} : M_+ \to [0, \infty]$ by $\mathrm{Tr}(x) = \sum_{n \in I} \tau_0(v_n^* x v_n)$. Then Tr is a weight on M, which is normal since $x \mapsto \tau_0(v_n^* x v_n)$ is normal for each $n \in I$.

Note that we have $v_n^* M v_n \subset \mathfrak{m}_{\mathrm{Tr}}$ for all $n \in I$, and thus $\mathfrak{m}_{\mathrm{Tr}}$ is weakly dense in M, so that Tr is semi-finite. For each $x \in M$ we apply Fubini's theorem to obtain

$$\mathrm{Tr}(xx^*) = \sum_{n,m \in I} \tau_0(v_n^* x v_m v_m^* x^* v_n) = \sum_{n,m \in I} \tau_0(v_m x^* v_n v_n^* x v_m^*) = \mathrm{Tr}(xx^*).$$

If $p \in \mathcal{P}(M)$ is finite then so is $p_0 \vee p$, and there is a non-zero subprojection $q \leq p_0$ which is monic in $(p_0 \vee p) M (p_0 \vee p)$. We then have $0 < \mathrm{Tr}(q)$, and $\mathrm{Tr}(p_0 \vee p) < \infty$. Thus, Tr is faithful and is finite on finite projections. Conversely, if $p \in \mathcal{P}(M)$ is infinite then there is a subprojection $q \leq p$ such that $p \sim q \sim p - q$, hence $\mathrm{Tr}(p) = \mathrm{Tr}(q) = \mathrm{Tr}(p - q)$, and so $\mathrm{Tr}(p) = 2\,\mathrm{Tr}(p)$, and since Tr is faithful we must have $\mathrm{Tr}(p) = \infty$.

It only remains to prove uniqueness. So suppose M is a semi-finite factor and ω is a normal semi-finite tracial weight on M. Since ω is semi-finite, for each $x \in M_+$, $x \neq 0$, there exists $y \in \mathfrak{n}_\omega$ such that $x^{1/2}y \neq 0$. Hence, $x^{1/2}yy^*x^{1/2} \in \mathfrak{m}_\omega$ and so $\omega(x^{1/2}yy^*x^{1/2}) < \infty$. Taking a suitable non-zero spectral projection q of $x^{1/2}yy^*x^{1/2}$ there then exists some number $c > 0$ such that $cq \leq x^{1/2}yy^*x^{1/2} \leq \|y\|^2 x$.

From above it follows that q is a finite projection (since $\omega(q) < \infty$), and so $\omega(q) \neq 0$ since otherwise we would have $\omega(p) = 0$ for all finite projections, contradicting semi-finiteness. Hence, $\omega(x) \neq 0$ showing that ω is faithful, and it then follows from the argument above that $\omega(p) < \infty$ if and only if p is finite. In particular, if $p \in \mathcal{P}(M)$ is any non-zero finite projection then $\omega_{|pMp}$ defines a tracial positive linear functional and hence must be a scalar multiple of the trace on pMp. Since M is semi-finite, every projection is an increasing limit of finite projections we then obtain uniqueness of Tr up to a scalar multiple. ■

Proposition 4.9.5. *Let M be a countably decomposable semi-finite factor. If $p, q \in \mathcal{P}(M)$, then $p \preceq q$ if and only if* $\mathrm{Tr}(p) \leq \mathrm{Tr}(q)$.

Proof. If p and q are finite then $\mathrm{Tr}_{(p \vee q)M(p \vee q)}$ is a scalar multiple of the trace on $(p \vee q)M(p \vee q)$, hence the conclusion follows from Proposition 4.7.11. By Theorem 4.9.4 we have $\mathrm{Tr}(p) = \infty$ if and only if p is not finite, and so the result then follows from Corollary 3.2.10. ■

Chapter 5

Examples of von Neumann algebras

5.1 Group von Neumann algebras

5.1.1 Group representations

Let Γ be a discrete group. A (unitary) **representation** of Γ is a homomorphism $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$. The **trivial** representation of Γ on $\mathcal{H}$ is given by $\pi(g) = 1$, for all $g \in \Gamma$. The **left-regular** (resp. **right-regular**) representation of Γ is $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ (resp. $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$) given by $(\lambda_g \xi)(x) = \xi(g^{-1}x)$ (resp. $(\rho_g \xi)(x) = \xi(xg)$). If $\Lambda < \Gamma$ is a subgroup, then the representation $\pi : \Gamma \to \ell^2(\Gamma/\Lambda)$ given by $(\pi_g \xi)(x) = \xi(g^{-1}x)$ is a **quasi-regular** representation.

Two representations $\pi_i : \Gamma \to \mathcal{U}(\mathcal{H}_i)$, $i = 1, 2$, are **equivalent** if there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(g) = \pi_2(g)U$, for all $g \in \Gamma$. Note that the left and right-regular representations are seen to be equivalent by considering the unitary $U : \ell^2\Gamma \to \ell^2\Gamma$ given by $(U\xi)(x) = \xi(x^{-1})$.

Given a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ we define the **adjoint representation** $\overline{\pi} : \Gamma \to \mathcal{U}(\overline{\mathcal{H}})$ by setting $\overline{\pi}_g \overline{\xi} = \overline{\pi_g \xi}$. We then have the natural identification $\pi = \overline{\overline{\pi}}$.

Given a family of unitary representations $\pi_\iota : \Gamma \to \mathcal{U}(\mathcal{H}_\iota)$, with $\iota \in I$, the **direct-sum representation** is $\bigoplus_{\iota \in I} \pi_\iota : \Gamma \to \mathcal{U}(\bigoplus_{\iota \in I} \mathcal{H}_\iota)$ defined by

$$(\bigoplus_{\iota \in I} \pi_\iota)(g) = \bigoplus_{\iota \in I} (\pi_\iota(g)).$$

If I is finite, then the **tensor product representation** is given by the map $\bigotimes_{\iota \in I} \pi_\iota : G \to \mathcal{U}(\overline{\bigotimes}_{\iota \in I} \mathcal{H}_\iota)$ defined by

$$(\bigotimes_{\iota \in I} \pi_\iota)(g) = \bigotimes_{\iota \in I} (\pi_\iota(g)).$$

If we use the identification $\mathcal{H}\overline{\otimes}\overline{\mathcal{K}} \cong \mathrm{HS}(\mathcal{H},\mathcal{K})$, then for representations $\pi_1 : \Gamma \to \mathcal{U}(\mathcal{H})$, and $\pi_2 : \Gamma \to \mathcal{U}(\mathcal{K})$, the representation $\pi_1 \otimes \overline{\pi}_2$ is realized on $\mathrm{HS}(\mathcal{H},\mathcal{K})$ as $(\pi_1 \otimes \overline{\pi}_2)(g)(T) = \pi_1(g)T\pi_2(g^{-1})$.

Lemma 5.1.1 (Fell's Absorption Principle)**.** *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a discrete group Γ, and let $1_{\mathcal{H}}$ denote the trivial representation of Γ on $\mathcal{H}$. Then the representations $\lambda \otimes \pi$ and $\lambda \otimes 1_{\mathcal{H}}$ are equivalent.*

Proof. Consider the unitary $U \in \mathcal{U}(\ell^2\Gamma \otimes \mathcal{H})$ determined by $U(\delta_g \otimes \xi) = \delta_g \otimes \pi(g)\xi$, for all $g \in \Gamma$, $\xi \in \mathcal{H}$. Then for all $h, g \in \Gamma$, and $\xi \in \mathcal{H}$ we have

$$\begin{aligned} U^*(\lambda \otimes \pi)(h)U(\delta_g \otimes \xi) &= U^*(\lambda \otimes \pi)(h)(\delta_g \otimes \pi_g\xi) \\ &= U^*(\delta_{hg} \otimes \pi_h\pi_g\xi) \\ &= \delta_{hg} \otimes \pi_{(hg)^{-1}}\pi_h\pi_g\xi = (\lambda \otimes 1_{\mathcal{H}})(h)(\delta_g \otimes \xi). \end{aligned}$$

■

If $\xi, \eta \in \ell^2\Gamma$, the **convolution** of ξ with η is the function $\xi * \eta : \Gamma \to \mathbb{C}$ given by

$$(\xi * \eta)(x) = \sum_{g\in\Gamma} \xi(g)\eta(g^{-1}x) = \sum_{g\in\Gamma} \xi(xg^{-1})\eta(g).$$

Note that by the Cauchy-Schwarz inequality we have that $\xi * \eta \in \ell^\infty\Gamma$, and $\|\xi * \eta\|_\infty \leq \|\xi\|_2\|\eta\|_2$. If $\xi, \eta \in \ell^1\Gamma$ then we also have the estimate $\|\xi * \eta\|_1 \leq \|\xi\|_1\|\eta\|_1$. Also note that for $g \in \Gamma$ we have $\delta_g * \xi = \lambda_g\xi$, and $\xi * \delta_g = \rho_{g^{-1}}\xi$.

If $\xi \in \ell^2\Gamma$ we let $\overline{\xi}$ be the function defined by $\overline{\xi}(x) = \overline{\xi(x^{-1})}$. Also, it is easy to see that if $\xi, \eta, \zeta \in \ell^2\Gamma$, then $(\xi*\eta)*\zeta \in \ell^2\Gamma$ if and only if $\xi*(\eta*\zeta) \in \ell^2\Gamma$, and if both are in $\ell^2\Gamma$ then we have $(\xi*\eta)*\zeta = \xi*(\eta*\zeta)$. In particular $\ell^1\Gamma$ with the norm $\|\cdot\|_1$ forms a unital involutive Banach algebra which is the **convolution algebra** of Γ.

Given $\xi \in \ell^2\Gamma$ we set $D_\xi = \{\eta \in \ell^2\Gamma \mid \xi * \eta \in \ell^2\Gamma\}$. We then define the convolution operator $L_\xi : D_\xi \to \ell^2\Gamma$ by $L_\xi\eta = \xi * \eta$. We also set $D'_\xi = \{\eta \in \ell^2\Gamma \mid \eta * \xi \in \ell^2\Gamma\}$, and $R_\xi : D'_\xi \to \ell^2\Gamma$, by $R_\xi\eta = \eta * \xi$.

Lemma 5.1.2. *For each $\xi \in \ell^2\Gamma$, the operators L_ξ, and R_ξ have closed graph in $\ell^2\Gamma \oplus \ell^2\Gamma$.*

Proof. Let $\{\eta_n\} \subset \ell^2\Gamma$ be a sequence such that $\eta_n \to \eta \in \ell^2\Gamma$, and $L_\xi\eta_n \to \zeta \in \ell^2\Gamma$. Then for $x \in \Gamma$ we have $|\zeta(x) - (\xi*\eta)(x)| = \lim_{n\to\infty} |(\xi*\eta_n)(x) - (\xi*\eta)(x)| \leq \lim_{n\to\infty} \|\xi\|_2\|\eta_n - \eta\|_2 = 0$. Hence, $\xi * \eta = \zeta \in \ell^2\Gamma$, and so $\eta \in D_\xi$ and $L_\xi\eta = \zeta$. The proof for R_ξ is identical. ■

A **left-convolver** (resp. **right-convolver**) is a vector $\xi \in \ell^2\Gamma$ such that $\xi * \ell^2\Gamma \subset \ell^2\Gamma$ (resp. $\ell^2\Gamma * \xi \subset \ell^2\Gamma$). If ξ is a left-convolver then by the closed graph theorem we have that $L_\xi \in \mathcal{B}(\ell^2\Gamma)$, and similarly $R_\xi \in \mathcal{B}(\ell^2\Gamma)$ for ξ a right-convolver. Note that the space of left (resp. right) convolvers is a linear space which contains δ_g for each $g \in \Gamma$.

We set

$$L\Gamma = \{L_\xi \mid \xi \in \ell^2\Gamma \text{ is a left-convolver.}\} \subset \mathcal{B}(\ell^2\Gamma);$$
$$R\Gamma = \{R_\xi \mid \xi \in \ell^2\Gamma \text{ is a right-convolver.}\} \subset \mathcal{B}(\ell^2\Gamma).$$

If ξ is a left-convolver then it is easy to see that $\overline{\xi}$ is also a left-convolver and we have $L_{\overline{\xi}} = L_\xi^*$. Similarly, we have $R_{\overline{\xi}} = R_\xi^*$ for right-convolvers. Also, since convolution is associative we have $L_{\xi*\eta} = L_\xi L_\eta$, and $R_{\xi*\eta} = R_\eta R_\xi$. Hence, $L\Gamma$ and $R\Gamma$ are unital $*$-subalgebras of $\mathcal{B}(\ell^2\Gamma)$. We next show that actually $L\Gamma$ and $R\Gamma$ are von Neumann algebras.

Theorem 5.1.3. *Let Γ be a discrete group, then $L\Gamma$ and $R\Gamma$ are von Neumann algebras. Moreover, we have $L\Gamma = R\Gamma' = \rho(\Gamma)'$, and $R\Gamma = L\Gamma' = \lambda(\Gamma)'$.*

Proof. By von Neumann's double commutant theorem it is enough to show that $L\Gamma = R\Gamma' = \rho(\Gamma)'$. Note that we trivially have the inclusions $L\Gamma \subset R\Gamma' \subset \rho(\Gamma)'$ and so we need only show $\rho(\Gamma)' \subset L\Gamma$.

Suppose $T \in \rho(\Gamma)'$ and define $\xi = T\delta_e$. Then for all $g \in \Gamma$ we have

$$\xi * \delta_g = \rho_{g^{-1}} T\delta_e = T\rho_{g^{-1}}\delta_e = T\delta_g.$$

By linearity we then have $\xi*\eta = T\eta$ for all η in the dense subspace $\mathrm{sp}\{\delta_g \mid g \in \Gamma\}$. Hence it follows that ξ is a left-convolver and $T = L_\xi \in L\Gamma$. ∎

The von Neumann aglebra $L\Gamma$ is the (left) **group von Neumann algebra** of Γ, and $R\Gamma$ is the right group von Neumann algebra of Γ. Note that since the left and right-regular representations are equivalent it follows that $L\Gamma \cong R\Gamma$.

Proposition 5.1.4. *Let Γ be a discrete group, then $\tau(x) = \langle x\delta_e, \delta_e\rangle$ defines a normal faithful trace on $L\Gamma$. In particular, $L\Gamma$ is a finite von Neumann algebra.*

Proof. If $\tau(x^*x) = 0$, where $x = L_\xi$, then $\|\xi\|^2 = \|L_\xi\delta_e\|^2 = \tau(x^*x) = 0$, hence $x = 0$, and so τ is faithful. As a vector state, τ is clearly normal, thus to check that it is a trace it is enough to check the tracial property on a weakly dense subalgebra, and by linearity it is then enough to show $\tau(\lambda_{ghg^{-1}}) = \tau(\lambda_h)$ for all $g, h \in \Gamma$. By a direct calculation we see $\tau(\lambda_{ghg^{-1}}) = \delta_e(ghg^{-1}) = \delta_e(h) = \tau(\lambda_h)$. ∎

Example 5.1.5. If Γ is abelian then we may consider the dual group $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{T})$ which is a compact group when endowed with the topology of pointwise convergence. We consider this group endowed with a Haar measure μ normalized so that $\mu(\hat{\Gamma}) = 1$. The Fourier transform $\mathcal{F} : \ell^2\Gamma \to L^2\hat{\Gamma}$ is defined as $(\mathcal{F}\xi)(\chi) = \sum_{g\in\Gamma} \xi(g)\langle \chi, g\rangle$. The Fourier transform implements a unitary between $\ell^2\Gamma$ and $\ell^2\hat{\Gamma}$.

If $\xi \in \ell^2\Gamma$ is a (left) convolver, then we have $L_\xi = \mathcal{F}^{-1}M_{\mathcal{F}(\xi)}\mathcal{F}$, and hence we obtain a canonical isomorphism $L\Gamma \cong L^\infty\hat{\Gamma}$. Moreover, we have $\tau(L_\xi) = \int \mathcal{F}(\xi)\, d\mu$, for each $L_\xi \in L\Gamma$.

Recall that when (X,μ) was a probability space, we could view $L^\infty(X,\mu)$ both as a von Neumann subalgebra of $\mathcal{B}(L^2(X,\mu))$, and as a subspace of $L^2(X,\mu)$. When we wanted to make a distinction between the two embeddings we would write M_f to explicitly denote the multiplication operator by f. Similarly, we may view $L\Gamma$ both as a von Neumann subalgebra of $\mathcal{B}(\ell^2\Gamma)$, and as a subspace of $\ell^2\Gamma$ under the identification $L_\xi \mapsto \xi$. We will therefore not always be specific as to which identification we are taking and leave it to the reader to determine from the context.

In particular, if $x = \sum_{g\in\Gamma} \alpha_g \delta_g \in \ell^2\Gamma$ is a left-convolver, then we will often also write x or $\sum_{g\in\Gamma} \alpha_g u_g$ to denote the operator $L_x \in L\Gamma$. (We switch δ_g to u_g to emphasize that u_g is a unitary operator.) By analogy with the abelian case we call the set $\{\alpha_g\}_{g\in\Gamma}$ the **Fourier coefficients** of x. This convention is quite standard, however we should issue a warning at this point that the sum $\sum_{g\in\Gamma} \alpha_g u_g$ does not in general converge to L_x in any operator space topology (e.g., norm, weak, or strong). This is already the case for $L\mathbb{Z}$ in fact. Thus, writing $x = \sum_{g\in\Gamma} \alpha_g u_g$ should be considered as an abbreviation for writing $L_x = L_{\sum_{g\in\Gamma} \alpha_g \delta_g}$, and nothing more.

A discrete group Γ is said to be **i.c.c.**[1] if every non-trivial conjugacy class of Γ is infinite.[2]

Theorem 5.1.6. *Let Γ be a discrete group. Then $L\Gamma$ is a factor if and only if Γ is i.c.c.*

Proof. First suppose that $h \in \Gamma \setminus \{e\}$, such that $h^\Gamma = \{ghg^{-1} \mid g \in G\}$ is finite. Let $x = \sum_{k\in h^\Gamma} u_k$. Then $x \neq \mathbb{C}$, and for all $g \in G$ we have $u_g x u_g^* = \sum_{k\in h^\Gamma} u_{gkg^{-1}} = x$, hence $x \in \{u_g\}'_{g\in\Gamma} \cap L\Gamma = \mathcal{Z}(L\Gamma)$.

Conversely, suppose that Γ is i.c.c. and $x = \sum_{g\in\Gamma} \alpha_g u_g \in \mathcal{Z}(L\Gamma) \setminus \mathbb{C}$, then for all $h \in \Gamma$ we have $x = u_h x u_h^* = \sum_{g\in\Gamma} \alpha_g u_{hgh^{-1}} = \sum_{g\in\Gamma} \alpha_{h^{-1}gh} u_g$. Thus the Fourier coefficients for x are constant on conjugacy classes, and since $x \in L\Gamma \subset \ell^2\Gamma$ we have $\alpha_g = 0$ for all $g \neq e$, hence $x = \tau(x) \in \mathbb{C}$. ∎

Examples of i.c.c. groups which can be verified directly include the symmetric group S_∞ of all finite permutations of $\mathbb{N}$, free groups $\mathbb{F}_n$ of rank $n \geq 2$, free products $\Gamma_1 * \Gamma_2$ when $|\Gamma_1|, |\Gamma_2| > 1$ and $|\Gamma_1| + |\Gamma_2| \geq 5$, projective special linear groups $PSL_n(\mathbb{Z})$, $n \geq 2$, groups without non-trivial finite index subgroups, and many others.

5.1.2 Group C^*-algebras

If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a representation of a discrete group Γ then note that we may extend π linearly to a representation of the convolution algebra $\ell^1\Gamma$, we will use the same notation π for this representation. The (full) **group C^*-algebra** $C^*\Gamma$, is defined as the C^*-algebra completion of $\ell^1\Gamma$ with respect to the norm

[1] Infinite conjugacy classes

[2] Note that the trivial group is i.c.c., but this is the only finite i.c.c. group.

$\|f\| = \sup_{\pi:\Gamma\to\mathcal{U}(\mathcal{H})} \|\pi(f)\|$. The **reduced group C^*-algebra** $C^*_r\Gamma$ is the C^*-algebra completion of $L^1\Gamma$ with respect to the left-regular representation. Note that we have a natural inclusions $C^*_r\Gamma \subset L\Gamma \subset \mathcal{B}(\ell^2\Gamma)$.

Note that by definition we have that any representation of Γ extends uniquely to a representation of $C^*\Gamma$, and conversely every representation of $C^*\Gamma$ arrises in this way. Moreover, two representations of Γ are equivalent if and only if the representations are equivalent when extended to $C^*\Gamma$. Thus, $C^*\Gamma$ is a C^*-algebra which encodes the representation theory of Γ. Given a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, a vector $\xi \in \mathcal{H}$ is **cyclic** if it is cyclic for $\pi(C^*\Gamma)$, i.e., $\overline{\mathrm{sp}}\{\pi(g)\xi \mid g \in \Gamma\} = \mathcal{H}$.

A function $\varphi : \Gamma \to \mathbb{C}$ is of **positive type** if for all $g_1, \ldots, g_n \in \Gamma$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \varphi(g_j^{-1} g_i) \geq 0.$$

Note, that by considering $g_1 = e$, and $\alpha_1 = 1$ we have $\varphi(e) \geq 0$. Also, by considering $g_1 = g$, $g_2 = e$, and $|\alpha_1| = \alpha_2 = 1$ we see that $\overline{\alpha_1}\varphi(g^{-1}) + \alpha_1\varphi(g) \leq 2\varphi(e)$, and from this it follows easily that $\varphi(g^{-1}) = \overline{\varphi(g)}$, and $\varphi \in \ell^\infty\Gamma$ with $\|\varphi\|_\infty = \varphi(e)$. Thus, a simple calculation shows that positive type is equivalent to the conditions $\varphi \in \ell^\infty\Gamma$, and

$$\sum_{g\in\Gamma} (f * \overline{f})(g)\varphi(g) \geq 0,$$

for all $f \in \ell^1\Gamma$.

The same proof as in the GNS-construction allows us to construct a representation $\pi_\varphi : \Gamma \to \mathcal{H}$, and a cyclic vector $\xi \in \mathcal{H}$ such that $\varphi(g) = \langle \pi(g)\xi, \xi\rangle$ for each $g \in \Gamma$. In particular, we may then extend φ to a positive linear functional on $C^*\Gamma$. Conversely, if φ is a positive linear functional on $C^*\Gamma$, then restricted to $\ell^1\Gamma$ this again gives a positive linear functional and hence restricted to Γ gives a function of positive type. We have thus proved the following theorem.

Theorem 5.1.7. *Let Γ be a discrete group, and let $\varphi : \Gamma \to \mathbb{C}$, then the following conditions are equivalent:*

(i) *φ is of positive type.*

(ii) *φ extends to a positive linear functional on $C^*\Gamma$.*

(iii) *There exists a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and a cyclic vector $\xi \in \mathcal{H}$ such that $\varphi(g) = \langle \pi(g)\xi, \xi\rangle$ for each $g \in \Gamma$.*

5.1.3 Other von Neumann algebras generated by groups

Given a unitary representation of a discrete group $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ one can always consider the von Neumann algebra it generates $\pi(\Gamma)''$. Properties of the representation can sometimes be reflected in the von Neumann algebra $\pi(\Gamma)''$, here we will discuss a couple of these properties.

A representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **reducible** if $\mathcal{H}$ contains a non-trivial closed Γ-invariant subspace. Otherwise, the representation is **irreducible**. Note that if $\mathcal{K} \subset \mathcal{H}$ is a closed Γ-invariant subspace then $\mathcal{K}^\perp$ is also Γ-invariant. Indeed, if $\xi \in \mathcal{K}, \eta \in \mathcal{K}^\perp$, and $g \in \Gamma$ then $\langle \pi(g)\eta, \xi\rangle = \langle \eta, \pi(g^{-1})\xi\rangle = 0$. Thus, π would then decompose as $\pi_{|\mathcal{K}} \oplus \pi_{|\mathcal{K}^\perp}$.

Lemma 5.1.8 (Schur's Lemma). *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ be two irreducible unitary representations of a discrete group Γ, if $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Γ-invariant then either $T = 0$, or else T is a scalar multiple of a unitary. In particular, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ has a non-zero Γ-invariant operator if and only if π and ρ are isomorphic.*

Proof. Let π and ρ be as above and suppose $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Γ-invariant. Thus, $T^*T \in \mathcal{B}(\mathcal{H})$ is Γ-invariant and hence any spectral projection of T^*T gives a Γ-invariant subspace. Since π is irreducible it then follows that $T^*T \in \mathbb{C}$. If $T^*T \neq 0$ then by multiplying T by a scalar we may assume that T is an isometry. Hence, $TT^* \in \mathcal{B}(\mathcal{K})$ is a non-zero Γ-invariant projection, and since ρ is irreducible it follows that $TT^* = TT^* = 1$. ■

Corollary 5.1.9. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a representation of a discrete group Γ. Then π is irreducible if and only if $\pi(\Gamma)'' = \mathcal{B}(\mathcal{H})$.*

A function $\varphi : \Gamma \to \mathbb{C}$ is a **character**[3] if it is of positive type, is constant on conjugacy classes, and is normalized so that $\varphi(e) = 1$. Characters arise from representations into finite von Neumann algebras. Indeed, if M is a finite von Neumann algebra with a normal faithful trace τ, and if $\pi : \Gamma \to \mathcal{U}(M) \subset \mathcal{U}(L^2(M, \tau))$ is a representation then $\varphi(g) = \tau(\pi(g)) = \langle \pi(g)1_\tau, 1_\tau\rangle$ defines a character on Γ. Conversely, if $\varphi : \Gamma \to \mathbb{C}$ is a character then the cyclic vector ξ in the corresponding GNS-representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ satisfies $\langle \pi(gh)\xi, \xi\rangle = \varphi(gh) = \varphi(hg) = \langle \pi(hg)\xi, \xi\rangle$ for all $g, h \in \Gamma$. Since the linear functional $T \mapsto \langle T\xi, \xi\rangle$ is normal we may then extend this to a normal faithful trace $\tau : \pi(\Gamma)'' \to \mathbb{C}$ by the formula $\tau(x) = \langle x\xi, \xi\rangle$. In particular this shows that $\pi(\Gamma)''$ is finite since it has a normal faithful trace.

If (M_i, τ_i) are finite von Neumann algebras with normal faithful traces τ_i, $i \in \{1, 2\}$, and $\pi_i : \Gamma \to \mathcal{U}(M_i)$ then we will consider π_1 and π_2 to be equivalent if there is a trace preserving automorphism $\alpha : M_1 \to M_2$, such that $\alpha(\pi_1(g)) = \pi_2(g)$ for all $g \in \Gamma$. Clearly, this is equivalent to requiring that there exist a unitary $U : L^2(M_1, \tau_1) \to L^2(M_2, \tau_2)$ such that $U1_{\tau_1} = 1_{\tau_2}$, and $U\pi_1(g) = \pi_2(g)U$ for all $g \in G$.

Note that the space of characters is a convex set, which is closed in the topology of pointwise convergence.

Theorem 5.1.10 (Thoma). *Let Γ be a discrete group. There is a one to one correspondence between:*

[3]Some authors use the term character to refer to a homomorphism into the circle $\mathbb{T}$, we will allow a more general definition and refer to homomorphisms into $\mathbb{T}$ as unitary characters.

1. *Equivalence classes of embeddings $\pi : \Gamma \to \mathcal{U}(M)$ where M is a finite von Neumann algebra with a given normal faithful trace τ, and such that $\pi(\Gamma)'' = M$, and*

2. *Characters $\varphi : \Gamma \to \mathbb{C}$,*

which is given by $\varphi(g) = \tau(\pi(g))$. Moreover, M is a factor if and only if φ is an extreme point in the space of characters.

Proof. The one to one correspondence follows from the discussion preceding the theorem, thus we only need to show the correspondence between factors and extreme points. If $p \in \mathcal{P}(\mathcal{Z}(M))$, is a non-trivial projection then we obtain characters φ_1, and φ_2 by the formulas $\varphi_1(g) = \frac{1}{\tau(p)}\tau(\pi(g)p)$, and $\varphi_2(g) = \frac{1}{\tau(1-p)}\tau(\pi(g)(1-p))$, and we have $\varphi = \tau(p)\varphi_1 + \tau(1-p)\varphi_2$. Since $p \in M = \pi(\Gamma)''$, there exists a sequence $x_n \in \mathbb{C}\Gamma$ such that $\frac{1}{\tau(p)}\tau(\pi(x_n)p) \to 1$, and $\frac{1}{\tau(1-p)}\tau(\pi(x_n)(1-p)) \to 0$, it then follows that $\varphi_1 \neq \varphi_2$ and hence φ is not an extreme point.

Conversely, if $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2)$ with $\varphi_1 \neq \varphi_2$ then if we consider the corresponding representations $\pi_i : \Gamma \to \mathcal{U}(N_i)$, we obtain a trace preserving embedding $\alpha : N \to N_1 \oplus N_2$, which satisfies $\alpha(\pi(g)) = \pi_1(g) \oplus \pi_2(g)$. If we denote by p the projection $1 \oplus 0$ then it need not be the case that $p \in \alpha(N)$, however by considering p we may then define a new trace τ' on N by $\tau'(x) = \frac{1}{4}\tau_1(\alpha(x)p) + \frac{3}{4}\tau_2(\alpha(x)(1-p))$. Since $\varphi_1 \neq \varphi_2$ we must have that $\tau'(\pi(g)) \neq \tau(\pi(g))$ for some $g \in \Gamma$. Thus, N does not have unique trace and so is not a factor by Corollary 4.8.3. ∎

5.2 The group-measure space construction

Let Γ be a discrete group and (X, μ) a σ-finite measure space. An action $\Gamma \curvearrowright (X, \mu)$ is **quasi-invariant** (or non-singular) if for each $g \in \Gamma$, and each measurable set $E \subset X$ we have that gE is also measurable, and $\mu(gE) = 0$ if and only if $\mu(E) = 0$. If in addition we have $\mu(gE) = \mu(E)$ for each $g \in \Gamma$ and each measurable set $E \subset X$, then we say that the action is **measure-preserving**.

If $\Gamma \curvearrowright (X, \mu)$ is quasi-invariant then we have an induced action $\Gamma \curvearrowright^{\sigma} \mathcal{M}(X, \mu)$ on the space of measurable functions $\mathcal{M}(X, \mu)$, given by $\sigma_g(a) = a \circ g^{-1}$ for all $g \in \Gamma$, $a \in \mathcal{M}(X, \mu)$. Note that if $a \in L^\infty(X, \mu)$, then $\|\sigma_g(a)\|_\infty = \|a\|_\infty$, and thus this action restricts to an action also on $L^\infty(X, \mu)$.

For $g \in \Gamma$ the **push-forward measure** $g\mu$ is given by $g\mu(E) = \mu(g^{-1}E)$ for $E \subset X$ measurable. Since the action is quasi-invariant we have $g\mu \prec \mu$ and hence we may consider the Radon-Nikodym derivative $\frac{dg\mu}{d\mu} \in L^1(X, \mu)_+$, which satisfies

$$\int \sigma_{g^{-1}}(a)\, d\mu = \int a\, dg\mu = \int a \frac{dg\mu}{d\mu}\, d\mu,$$

for all $a \in L^\infty(X, \mu)$. Note for $g, h \in \Gamma$, we have a cocycle relation

$$\frac{dgh}{d\mu} = \frac{dg\mu}{d\mu}\frac{dgh}{dg\mu} = \frac{dg\mu}{d\mu}\sigma_g\left(\frac{dh\mu}{d\mu}\right).$$

The **Koopman representation** is the representation $\pi : \Gamma \to \mathcal{U}(L^2(X, \mu))$ given by $\pi_g \xi = (\frac{dg\mu}{d\mu})^{1/2}\sigma_g(\xi)$. Note that this is indeed a unitary representation since

$$\begin{aligned}\langle \pi_g \xi, \pi_g \eta\rangle &= \int \sigma_g(\xi)\overline{\sigma_g(\eta)}\left(\frac{dg\mu}{d\mu}\right) d\mu \\ &= \int \sigma_g(\xi\overline{\eta})\, dg\mu = \int \xi\overline{\eta}\, d\mu = \langle \xi, \eta\rangle,\end{aligned}$$

and,

$$\begin{aligned}\pi_{gh}\xi &= \left(\frac{dgh\mu}{d\mu}\right)^{1/2}\sigma_{gh}(\xi) \\ &= \left(\frac{dg\mu}{d\mu}\right)^{1/2}\left(\sigma_g\left(\frac{dh\mu}{d\mu}\right)\right)^{1/2}\sigma_g(\sigma_h(\xi)) = \pi_g\pi_h\xi.\end{aligned}$$

If $a \in L^\infty(X, \mu)$, $\xi \in L^2(X, \mu)$, and $g \in \Gamma$ then we have

$$\begin{aligned}\pi_g M_a \pi_{g^{-1}}\xi &= \pi_g(a\left(\frac{dg^{-1}\mu}{d\mu}\right)^{1/2}\sigma_{g^{-1}}(\xi)) \\ &= \sigma_g(a)\left(\frac{dg\mu}{d\mu}\sigma_g\left(\frac{dg^{-1}\mu}{d\mu}\right)\right)^{1/2}\xi = M_{\sigma_g(a)}\xi.\end{aligned}$$

Hence $\pi_g M_a \pi_{g^{-1}} = M_{\sigma_g(a)}$, and in particular the action of Γ on the abelian von Neumann algebra $L^\infty(X, \mu)$ is normal.

If we consider the Hilbert space $\mathcal{H} = L^2(X, \mu)\overline{\otimes}\ell^2\Gamma$ then we have a normal representation of $L^\infty(X, \mu)$ on $\mathcal{H}$ given by $a \mapsto M_a \otimes 1 \in \mathcal{B}(\mathcal{H})$. We also may consider the diagonal action of Γ on $\mathcal{H}$ given by $u_g = \pi_g \otimes \lambda_g \in \mathcal{U}(\mathcal{H})$.

The **group-measure space construction** associated to the action $\Gamma \curvearrowright (X, \mu)$ is the von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma$, generated by all the operators $M_a \otimes 1$, and u_g. We will consider $L^\infty(X, \mu)$ as a von Neumann subalgebra of $L^\infty(X, \mu) \rtimes \Gamma$, and note that we have $u_g a u_{g^{-1}} = \sigma_g(a)$ under this identification. Note also that by Fell's absorption principle we have $\pi \otimes \lambda \sim 1 \otimes \lambda$, and hence it follows that the map $\lambda_g \mapsto u_g$ extends to $L\Gamma$, giving an inclusion $L\Gamma \subset L^\infty(X, \mu) \rtimes \Gamma$.

We will also consider $L^2(X, \mu)$ as a subspace of $L^2(X, \mu)\overline{\otimes}\ell^2\Gamma$ given by the isometry $U\xi = \xi \otimes \delta_e$. We then let $e : L^2(X, \mu)\overline{\otimes}\ell^2\Gamma \to L^2(X, \mu)$ be the orthogonal projection, and we denote by $E : L^\infty(X, \mu) \rtimes \Gamma \to \mathcal{B}(L^2(X, \mu))$ the map $E(x) = exe$.

Lemma 5.2.1. *Suppose $\Gamma \curvearrowright (X, \mu)$ is a quasi-invariant action, and $L^\infty(X, \mu) \rtimes \Gamma$ is the associated group-measure space construction. If E is defined as above* *then the range of E is contained in $L^\infty(X,\mu)$, and for $x \in L^\infty(X,\mu) \rtimes \Gamma$, $E(x^*x) = 0$ if and only if $x = 0$. Moreover, for $g \in \Gamma$, and $x \in L^\infty(X,\mu) \rtimes \Gamma$, we have $\sigma_g(E(x)) = E(u_g x u_g^*)$.*

Proof. If $x = \sum_{g\in\Gamma} a_g u_g$, where $a_g \in L^\infty(X,\mu)$ with only finitely many non-zero terms then we may compute directly $E(x) = a_e \in L^\infty(X,\mu)$, and in particular $E(u_h x u_h^*) = \sigma_h(a_e) = \sigma_h(E(x))$ for all $h \in \Gamma$. Since E is normal and the algebra generated by $L^\infty(X,\mu)$ and Γ is weak operator topology dense it then follows that the range of E is contained in $L^\infty(X,\mu)$, and $E(u_h x u_h^*) = \sigma_h(E(x))$, for all $x \in L^\infty(X,\mu) \rtimes \Gamma$, and $h \in \Gamma$.

If we consider $h \in \Gamma$ then $(1 \otimes \rho_h^*)e(1 \otimes \rho_h)$ is the orthogonal projection from $L^2(X,\mu)\overline{\otimes}\ell^2\Gamma$ onto $L^2(X,\mu) \otimes \delta_h$, and hence $1 = \sum_{h\in\Gamma}(1 \otimes \rho_h^*)e(1 \otimes \rho_h)$. If $E(x^*x) = 0$, then $xe = 0$, and hence

$$x(1 \otimes \rho_h^*)e(1 \otimes \rho_h) = (1 \otimes \rho_h^*)xe(1 \otimes \rho_h) = 0$$

for every $h \in \Gamma$ (note that $1 \otimes \rho_h \in (L^\infty(X,\mu) \rtimes \Gamma)'$), thus

$$x = x(\sum_{h\in\Gamma}(1 \otimes \rho_h^*)e(1 \otimes \rho_h)) = 0. \qquad \blacksquare$$

If $x \in L^\infty(X,\mu) \rtimes \Gamma$, then as we did for the group von Neumann algebra, we may define the Fourier coefficients $a_g \in L^\infty(X,\mu)$ by $a_g = E(xu_g^*)$. From the previous lemma we have that the Fourier coefficients completely determine the operator x and so we will write $x = \sum_{g\in\Gamma} a_g u_g$. Note that just as in the case for the group von Neumann algebra this summation does not in general converge in an operator space topology. However, it gives us a useful way to view operators in $L^\infty(X,\mu) \rtimes \Gamma$, and this behaves well with respect to multiplication so that we may calculate the Fourier coefficients of a product as

$$(\sum_{g\in\Gamma} a_g u_g)(\sum_{h\in\Gamma} b_h u_h) = \sum_{g\in\Gamma}(\sum_{h\in\Gamma} a_{gh}\sigma_{(gh)^{-1}}(b_h))u_g.$$

Where for each $g \in \Gamma$, $\sum_{h\in\Gamma} a_{gh}\sigma_{(gh)^{-1}}(b_h)$ converges in $L^2(X,\mu)$ to a function in $L^\infty(X,\mu)$.

A quasi-invariant action of a discrete group Γ on (X,μ) is (essentially) **free** if for all $E \subset X$ with $\mu(E) > 0$, and $g \in \Gamma \setminus \{e\}$ there exists $a \in L^\infty(X,\mu)$ such that $(a - \sigma_g(a))1_E \neq 0$. If X is a compact Hausdorff space and μ is a σ-finite Randon measure then it is not hard to see that an action is free if and only if for any $g \in \Gamma \setminus \{e\}$ we have $\mu(\{x \in X \mid gx = x\}) = 0$, or equivalently, the stabilizer subgroup Γ_x is trivial for almost every $x \in X$.

An action is **ergodic** if whenever $E \subset X$ is a measurable subset such that $gE = E$ for all $g \in \Gamma$, then we have $\mu(E) = 0$, or $\mu(X \setminus E) = 0$.

Theorem 5.2.2. *Let $\Gamma \curvearrowright (X,\mu)$ be a quasi-invariant action of a discrete group Γ on a σ-finite measure space (X,μ).*

(*i*) *The action $\Gamma \curvearrowright (X,\mu)$ is free if and only if $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes \Gamma$ is a maximal abelian subalgebra.*

(ii) *If $L^\infty(X, \mu) \rtimes \Gamma$ is a factor then $\Gamma \curvearrowright (X, \mu)$ is ergodic.*

(iii) *If the action $\Gamma \curvearrowright (X, \mu)$ is free and ergodic then $L^\infty(X, \mu) \rtimes \Gamma$ is a factor.*

Proof. For $g \in \Gamma \setminus \{e\}$ let p_g be the supremum of all projections p in $L^\infty(X, \mu)$ such that $(a - \sigma_g(a))p = 0$ for all $a \in L^\infty(X, \mu)$. If $p_g \neq 0$ for some $g \in \Gamma \setminus \{e\}$ then for all $a \in L^\infty(X, \mu)$ we have $p_g u_g a = \sigma_g(a) p_g u_g = a p_g u_g$, and hence $p_g u_g$ gives a non-trivial element in $L^\infty(X, \mu)'$, showing that $L^\infty(X, \mu)$ is not maximal abelian in $L^\infty(X, \mu) \rtimes \Gamma$. Conversely, if $x \in L^\infty(X, \mu)' \cap (L^\infty(X, \mu) \rtimes \Gamma)$, but $x \notin L^\infty(X, \mu)$, then considering the Fourier decomposition $x = \sum_{g \in \Gamma} a_g u_g$ we must have that $a_g \neq 0$ for some $g \in \Gamma \setminus \{e\}$. Since $ax = xa$ for each $a \in L^\infty(X, \mu)$ we may use the uniqueness for the Fourier decomposition to conclude that $(a - \sigma_g(a))a_g = 0$ for each $a \in L^\infty(X, \mu)$. Hence, the action is not free.

Next, suppose the action is not ergodic, then there exists $E \subset X$ such that $gE = E$ for all $g \in \Gamma$, and $1_E \notin \mathbb{C}$. Then $u_g 1_E u_g^* = \sigma_g(1_E) = 1_E$ for all $g \in \Gamma$ and hence 1_E commutes with $L^\infty(X, \mu)$ and $L\Gamma$. Since these two subalgebras generate $L^\infty(X, \mu) \rtimes \Gamma$ it follows that 1_E is a non-trivial element in the center.

Finally, suppose that the action is free and ergodic and fix p a projection in the center of $L^\infty(X, \mu) \rtimes \Gamma$. Since the action is free we have from the first part that $z \in L^\infty(X, \mu)$. Thus $z = 1_E$ for some measurable subset $E \subset X$. Since 1_E commutes with u_g for each $g \in \Gamma$ we see that $gE = E$ a.e. for each $g \in \Gamma$. By ergodicity we then have either $\mu(E) = 0$ in which case $z = 0$, or $\mu(X \setminus E) = 0$ in which case $z = 1$. ∎

We next turn to the question of the type of $L^\infty(X, \mu) \rtimes \Gamma$. For this we need a lemma which is reminiscent of Dixmier's property, the difference being that we consider only conjugating by unitaries in a subalgebra, and we consider the weak operator topology rather than the norm topology.

Lemma 5.2.3. *Let M be a von Neumann algebra and $A \subset M$ an abelian von Neumann subalgebra. For each $x \in M$ let $\mathcal{K}_x$ be the weak operator topology convex closure of $\{uxu^* \mid u \in \mathcal{U}(A)\}$, then $\mathcal{K}_x \cap (A' \cap M) \neq \emptyset$.*

Proof. Consider the space $\mathcal{F}$ of all finite dimensional subalgebras of A, directed by inclusion. Note that since A is abelian, if $A_1, A_2 \in \mathcal{F}$, then $(A_1 \cup A_2)'' \in \mathcal{F}$. Also, note that $\cup_{A_0 \in \mathcal{F}} A_0$ is weak operator topology dense in A by the spectral theorem.

Since each $B \in \mathcal{F}$ is finite dimensional $\mathcal{U}(B)$ is a compact group, and if we consider the Haar measure λ_B on $\mathcal{U}(B)$ then we have that $\int uxu^* \, d\lambda_B \in \mathcal{K}_x \cap B'$. Thus, if we denote by $\mathcal{K}_B = \mathcal{K}_x \cap B'$ then $\{\mathcal{K}_B\}_{B \in \mathcal{F}}$ has the finite intersection property, and by weak operator topology compactness we then have $\mathcal{K}_x \cap (A' \cap M) = \cap_{B \in \mathcal{F}} \mathcal{K}_B \neq \emptyset$. ∎

A von Neumann algebra M is **completely atomic** if 1 is an orthogonal sum of minimal projections in M, if M has no minimal projections then M is **diffuse**. If (X, μ) is a σ-finite measure space then (X, μ) is completely atomic (resp. diffuse) if $L^\infty(X, \mu)$ is.

Theorem 5.2.4. *Let $\Gamma \curvearrowright (X, \mu)$ be a quasi-invariant free ergodic action of a discrete group Γ on a σ-finite measure space. Then $L^\infty(X, \mu) \rtimes \Gamma$ is*

(*i*) *type I if and only if (X, μ) is completely atomic;*

(*ii*) *type II_1 if and only if (X, μ) is diffuse and there exists a Γ-invariant probability measure $\nu \sim \mu$;*

(*iii*) *type II_∞ if and only if (X, μ) is diffuse and there exists an infinite Γ-invariant σ-finite measure $\nu \sim \mu$;*

(*iv*) *type III if and only if there is no Γ-invariant σ-finite measure $\nu \sim \mu$.*

Proof. We prove part (*iv*) first. Suppose first that $\nu \sim \mu$ is a σ-finite Γ-invariant measure. Then we obtain a normal weight on $L^\infty(X, \mu) \rtimes \Gamma$ by the formula $\mathrm{Tr}(x) = \int E(x)\, d\nu$. Note that if $F \subset X$ such that $\nu(F) < \infty$ then $\mathrm{Tr}(1_F x) < \infty$ for all $x \geq 0$, thus Tr is semi-finite. Also, if $\mathrm{Tr}(x^* x) = 0$ then $E(x^* x) = 0$ and hence $x = 0$ by Lemma 5.2.1, thus Tr is faithful. If $x = \sum_{g \in \Gamma} a_g u_g \in L^\infty(X, \mu) \rtimes \Gamma$, then we can compute directly $E(x^* x) = \sum_{h \in \Gamma} \sigma_{h^{-1}}(a_h^* a_h)$,and $E(x x^*) = \sum_{h \in \Gamma} a_h^* a_h$. Since ν is measure preserving we then see that $\mathrm{Tr}(x^* x) = \mathrm{Tr}(x x^*)$ and hence Tr is a semi-finite normal faithful trace which shows that $L^\infty(X, \mu) \rtimes \Gamma$ is semi-finite by Theorem 4.9.4.

Conversely, if $\mathrm{Tr} : (L^\infty(X, \mu) \rtimes \Gamma)_+ \to [0, \infty]$ is a semi-finite normal faithful trace then by restriction we define a normal faithful weight ω on $L^\infty(X, \mu)$. We claim that ω is again semi-finite. Indeed, if $x_n \in L^\infty(X, \mu) \rtimes \Gamma$ is a net of positive operators increasing to 1 such that $\mathrm{Tr}(x_n) < \infty$, then by Lemma 5.2.3 there exists an increasing net of operators $y_n \in L^\infty(X, \mu)' \cap (L^\infty(X, \mu) \rtimes \Gamma) = L^\infty(X, \mu)$ such that each y_n is in the weak operator topology convex closure of $\{u x_n u^* \mid u \in \mathcal{U}(L^\infty(X, \mu))\}$. Since Tr is normal we then have $\mathrm{Tr}(y_n) = \mathrm{Tr}(x_n) < \infty$, and y_n is increasing to 1, and hence ω is semi-finite. By the Riesz representation theorem there then exists a σ-finite measure $\nu \sim \mu$ such that $\omega(a) = \int a\, d\nu$ for all $a \in L^\infty(X, \mu)_+$. Since Tr is a trace we have that ν is Γ-invariant.

Having established part (*iv*) we now consider the other parts. Note that in the correspondence describe above we have that ν is finite if and only if $\mathrm{Tr}(1) < \infty$, thus the only thing left to show is that $L^\infty(X, \mu) \rtimes \Gamma$ is completely atomic if and only if $L^\infty(X, \mu)$ is completely atomic. Since we are in the semi-finite case we let Tr be as above.

Suppose that $L^\infty(X, \mu) \rtimes \Gamma$ is completely atomic then from above we have that Tr restricted to $L^\infty(X, \mu)$ is semi-finite and hence $L^\infty(X, \mu)$ has finite projections. Since every finite projection is a finite sum of minimal projections it follows that $L^\infty(X, \mu)$ has a minimal projection p_0. Since the action is free we have that $\sigma_g(p_0)$ is orthogonal to p_0 for all $g \in \Gamma$, and since the action is ergodic we then have $1 = \sum_{g \in \Gamma} \sigma_g(p_0)$, showing that (X, μ) is atomic.

Conversely, if $p_0 \in L^\infty(X, \mu)$ is a minimal projection then we claim that p_0 is also a minimal projection in $L^\infty(X, \mu) \rtimes \Gamma$. Indeed, if $0 \neq q \leq p_0$, then for all $a \in L^\infty(X, \mu)$ we have $aq = (ap_0)q + (a(1 - p_0))q = q(ap_0) = qa$, since $ap_0 \in \mathbb{C}p_0$. By freeness $L^\infty(X, \mu)$ is maximal abelian and hence $q \in L^\infty(X, \mu)$.

Thus, $q = p_0$ showing that p_0 is a minimal projection in $L^\infty(X, \mu) \rtimes \Gamma$, which must then by type I. ■

Example 5.2.5. Consider the action $\mathbb{Z} \curvearrowright (\mathbb{T}, \lambda)$ by an irrational rotation. Then this action is clearly measure preserving and free. If $E \subset \mathbb{T}$ were invariant with $\lambda(E) > 0$, then we could consider the measure ν on $\mathbb{T}$ given by $\nu(F) = \frac{1}{\lambda(E)}\lambda(E \cap F)$. since E is invariant we have that ν is an invariant Randon measure on $\mathbb{T}$. Moreover, since the rotation is irrational we see that ν is invariant under a dense subgroup of $\mathbb{T}$ and hence it is invariant for all of $\mathbb{T}$. Uniqueness of the Haar measure implies $\nu = \lambda$ and hence $\lambda(E) = 1$. Thus, $\mathbb{Z} \curvearrowright (\mathbb{T}, \lambda)$ is ergodic and $L^\infty(\mathbb{T}, \lambda) \rtimes \mathbb{Z}$ is a II_1 factor.

Example 5.2.6. Consider the action $\mathbb{Q} \curvearrowright (\mathbb{R}, \lambda)$ by addition. Then this is measure preserving and free, and again uniqueness of the Haar measure up to scaling implies that this action is ergodic. Thus, $L^\infty(\mathbb{R}, \lambda) \rtimes \mathbb{Q}$ is a II_∞ factor.

Example 5.2.7. Consider the action $\mathbb{Q} \rtimes \mathbb{Q}^* \curvearrowright (\mathbb{R}, \lambda)$ where $\mathbb{Q}$ acts by addition, and $\mathbb{Q}^*$ acts by multiplication. This is (essentially) free, and is ergodic since it is ergodic when restricted to $\mathbb{Q}$. Moreover, if $\nu \sim \lambda$ were a σ-finite invariant measure then ν would be invariant under the action of $\mathbb{Q}$ and hence be a multiple of Haar measure. But then it would not be preserved by $\mathbb{Q}^*$. Thus $\mathbb{Q} \rtimes \mathbb{Q}^* \curvearrowright (\mathbb{R}, \lambda)$ has no σ-finite invariant measure and so $L^\infty(\mathbb{R}, \lambda) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}^*)$ is a type III factor.

Chapter 6

Completely positive maps

An **operator system** E is a closed self adjoint subspace of a unital C^*-algebra A such that $1 \in E$. We denote by $\mathbb{M}_n(E)$ the space of $n \times n$ matrices over E. If A is a C^*-algebra, then $\mathbb{M}_n(A) \cong A \otimes \mathbb{M}_n(\mathbb{C})$ has a unique norm for which it is again a C^*-algebra, where the adjoint given by $[a_{i,j}]^* = [a_{j,i}^*]$. This can be seen easily for C^*-subalgebras of $\mathcal{B}(\mathcal{H})$, and by Corollary 4.1.7 every C^*-algebra is isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. In particular, if E is an operator system then $\mathbb{M}_n(E)$ is again an operator system when viewed as a subspace of the C^*-algebra $\mathbb{M}_n(A)$.

If $\phi : E \to F$ is a linear map between operator systems, then we denote by $\phi^{(n)} : \mathbb{M}_n(E) \to \mathbb{M}_n(F)$ the map defined by $\phi^{(n)}([a_{i,j}]) = [\phi(a_{i,j})]$. We say that ϕ is **positive** if $\phi(a) \geq 0$, whenever $a \geq 0$. If $\phi^{(n)}$ is positive then we say that ϕ is n**-positive** and if ϕ is n-positive for every $n \in \mathbb{N}$ then we say that ϕ is **completely positive**. If A and B are unital and $\phi : A \to B$ such that $\phi(1) = 1$ then we say that ϕ is unital.

Note that just as in the case of states, if $\phi : E \to F$ is positive then $\phi(x^*) = \phi(x)^*$, for all $x \in E$. Also note that positive maps are continuous. Indeed, if $\phi : E \to F$ is positive and $\{x_n\}_n$ is a sequence which converges to 0 in E, such that $\lim_{n\to\infty} \phi(x_n) = y$, then since $\omega \circ \phi$ is positive (and hence continuous) for any state $\omega \in S(B)$ we have $\omega(y) = 0$, Proposition 4.1.5 then gives that $y = 0$. The closed graph theorem then shows that ϕ is bounded.

We also remark that the proof in Lemma 4.1.2, also shows that a linear functional $\varphi \in E^*$ is positive if and only if $\varphi(1) = \|\varphi\|$. In particular, it follows form the Hahn-Banach theorem that any positive linear functional on E extends to a positive linear functional on A which has the same norm.

Lemma 6.0.8. *If A and B are unital C^*-algebras and $\phi : A \to B$ is a unital contraction then ϕ is positive.*

Proof. We first show that ϕ is Hermitian. Suppose $x = x^* \in A$ such that $\phi(x) = a + ib$ where $a, b \in B$ are self-adjoint. Assume $\|x\| \leq 1$. If $\lambda \in \sigma(b)$ then for all $tin\mathbb{R}$ we have

$$(\lambda + t)^2 \leq \|b + t\|^2 \leq \|\phi(x + it)\|^2 \leq \|x + it\|^2 \leq 1 + t^2.$$

Hence $\lambda^2 + t\lambda \leq 1$, and as this is true for all t we must then have $\lambda = 0$, and hence $b = 0$.

By Lemma 4.1.2 we have that $\omega \circ \phi$ is a state, for any state ω. Hence if $x \geq 0$ then for any state ω we have $\omega(a) = \omega \circ \phi(x) \geq 0$. By Proposition 4.1.4 we must then have $a \geq 0$, and hence ϕ is positive. ∎

The following proposition follows easily from Proposition 4.2.2.

Proposition 6.0.9. *Let M and N be von Neumann algebras, and $\phi : M \to N$ a positive map, then the following conditions are equivalent.*

(*i*) *ϕ is normal.*

(*ii*) *For any bounded increasing net $\{x_i\}_i$ we have $\phi(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} \phi(x_i)$ where the limits are taken in the strong operator topologies.*

(*iii*) *For any family $\{p_i\}_i$ of pairwise orthogonal projections we have $\phi(\sum_i p_i) = \sum_i \phi(p_i)$.*

6.1 Dilation theorems

6.1.1 Stinespring's Dilation Theorem

If $\pi : A \to \mathcal{B}(\mathcal{K})$ is a representation of a C^*-algebra A and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the operator $\phi : A \to \mathcal{B}(\mathcal{H})$ given by $\phi(x) = V^*\pi(x)V$ is completely positive. Indeed, if we consider the operator $V^{(n)} \in \mathcal{B}(\mathcal{H}^{\oplus n}, \mathcal{K}^{\oplus n})$ given by $V^{(n)}((\xi_i)_i) = (V\xi_i)_i$ then for all $x \in \mathbb{M}_n(A)$ we have

$$\begin{aligned}\phi^{(n)}(x^*x) &= {V^{(n)}}^* \pi^{(n)}(x^*x)V^{(n)} \\ &= (\pi^{(n)}(x)V^{(n)})^*(\pi^{(n)}(x)V^{(n)}) \geq 0.\end{aligned}$$

Generalizing the GNS construction Stinespring showed that every completely positive map from A to $\mathcal{B}(\mathcal{H})$ arises in this way.

Theorem 6.1.1. *Let A be a unital C^*-algebra, and suppose $\phi : A \to \mathcal{B}(\mathcal{H})$, then ϕ is completely positive if and only if there exists a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ and a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\phi(x) = V^*\pi(x)V$. We also have $\|\phi\| = \|V\|^2$, and if ϕ is unital then V is an isometry. Moreover, if A is a von Neumann algebra and ϕ is a normal completely positive map, then π is a normal representation.*

Proof. Consider the sesquilinear form on $A \otimes \mathcal{H}$ given by $\langle a \otimes \xi, b \otimes \eta \rangle_\phi = \langle \phi(b^*a)\xi, \eta \rangle$, for $a, b \in A$, $\xi, \eta \in \mathcal{H}$. If $(a_i)_i \in A^{\oplus n}$, and $(\xi_i)_i \in \mathcal{H}^{\oplus n}$, then we have

$$\begin{aligned}\langle \sum_i a_i \otimes \xi_i, \sum_j a_j \otimes \xi_j \rangle_\phi &= \sum_{i,j} \langle \phi(a_j^* a_i)\xi_i, \xi_j \rangle \\ &= \langle \phi((a_i)_i^* (a_i)_i)(\xi_i)_i, (\xi_i)_i \rangle \geq 0.\end{aligned}$$

Thus, this form is non-negative definite and we can consider N_ϕ the kernel of this form so that $\langle \cdot, \cdot \rangle_\phi$ is positive definite on $\mathcal{K}_0 = (A \otimes \mathcal{H})/N_\phi$. Hence, we can take the Hilbert space completion $\mathcal{K} = \overline{\mathcal{K}_0}$.

As in the case of the GNS construction, we define a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ by first setting $\pi_0(x)(a \otimes \xi) = (xa) \otimes \xi$ for $a \otimes \xi \in A \otimes \mathcal{H}$. Note that since ϕ is positive we have $\phi(a^* x^* x a) \leq \|x\|^2 \phi(a^* a)$, applying this to $\phi^{(n)}$ we see that $\|\pi_0(x) \sum_i a_i \otimes \xi_i\|_\phi^2 \leq \|x\|^2 \| \sum_i a_i \otimes \xi_i\|_\phi^2$. Thus, $\pi_0(x)$ descends to a well defined bounded map on $\mathcal{K}_0$ and then extends to a bounded operator $\pi(x) \in \mathcal{B}(\mathcal{K})$.

If we define $V_0 : \mathcal{H} \to \mathcal{K}_0$ by $V_0(\xi) = 1 \otimes \xi$, then we see that V_0 is bounded by $\|\phi(1)\|$ and hence extends to a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. For any $x \in A$, $\xi, \eta \in \mathcal{H}$ we then check that

$$\begin{aligned} \langle V^* \pi(x) V \xi, \eta \rangle &= \langle \pi(x)(1 \otimes \xi), 1 \otimes \eta \rangle_\phi \\ &= \langle x \otimes \xi, 1 \otimes \eta \rangle_\phi = \langle \phi(x) \xi, \eta \rangle. \end{aligned}$$

Thus, $\phi(x) = V^* \pi(x) V$ as claimed. ∎

Corollary 6.1.2 (Kadison's inequality)**.** *If A and B are unital C^*-algebras, and $\phi : A \to B$ is unital compleley positive then for all $x \in A$ we have $\phi(x)^* \phi(x) \leq \phi(x^* x)$*

Proof. By Corollary 4.1.7 we may assume that $B \subset \mathcal{B}(\mathcal{H})$. If we consider the Stinespring dilation $\phi(x) = V^* \pi(x) V$, then since ϕ is unital we have that V is an isometry. Hence $1 - VV^* \geq 0$ and so we have

$$\begin{aligned} \phi(x^* x) - \phi(x)^* \phi(x) &= V^* \pi(x^* x) V - V^* \pi(x)^* V V^* \pi(x) V \\ &= V^* \pi(x^*)(1 - VV^*) \pi(x) V \geq 0. \end{aligned}$$

∎

Lemma 6.1.3. *A matrix $a = [a_{i,j}] \in \mathbb{M}_n(A)$ is positive if and only if*

$$\sum_{i,j=1}^n x_i^* a_{i,j} x_j \geq 0,$$

for all $x_1, \ldots, x_n \in A$.

Proof. For all $x_1, \ldots, x_n \in A$ we have that $\sum_{i,j=1}^n x_i^* a_{i,j} x_j$ is the conjugation of a by the $1 \times n$ column matrix with entries $x_1, \ldots, x_n$, hence if a is positive then so is $\sum_{i,j=1}^n x_i^* a_{i,j} x_j$.

Conversely, if $\sum_{i,j=1}^n x_i^* a_{i,j} x_j \geq 0$, for all $x_1, \ldots, x_n$ then for any representation $\pi : A \to \mathcal{B}(\mathcal{H})$, and $\xi \in \mathcal{H}$ we have

$$\begin{aligned} \left\langle (\mathrm{id} \otimes \pi)(a) \begin{pmatrix} \pi(x_1)\xi \\ \vdots \\ \pi(x_n)\xi \end{pmatrix}, \begin{pmatrix} \pi(x_1)\xi \\ \vdots \\ \pi(x_n)\xi \end{pmatrix} \right\rangle &= \sum_{i,j=1}^n \langle \pi(a_{i,j}) \pi(x_j) \xi, \pi(x_i) \xi \rangle \\ &= \left\langle \pi \left(\sum_{i,j=1}^n x_i^* a_{i,j} x_j \right) \xi, \xi \right\rangle \geq 0. \end{aligned}$$

Thus, if $\mathcal{H}$ has a cyclic vector, then $(\mathrm{id}\otimes\pi)(a) \geq 0$. But since every representation is decomposed into a direct sum of cyclic representations it then follows that $(\mathrm{id}\otimes\pi)(a) \geq 0$ for any representation, and hence $a \geq 0$ by considering a faithful representation. ∎

Proposition 6.1.4. *Let E be an operator system, and let B be an abelian C^*-algebras. If $\phi : E \to B$ is positive, then ϕ is completely positive.*

Proof. Since B is commutative we may assume $B = C_0(X)$ for some locally compact Hausdorff space X. If $a = [a_{i,j}] \in \mathbb{M}_n(E)$ such that $a \geq 0$, then for all $x_1, \ldots, x_n \in B$, and $\omega \in X$ we have

$$\begin{aligned}\left(\sum_{i,j} x_i^* \phi(a_{i,j}) x_j\right)(\omega) &= \left(\sum_{i,j} \phi(\overline{x_i(\omega)} x_j(\omega) a_{i,j})\right)(\omega) \\ &= \phi\left(\begin{pmatrix} x_1(\omega) \\ \vdots \\ x_n(\omega)\end{pmatrix}^* a \begin{pmatrix} x_1(\omega) \\ \vdots \\ x_n(\omega)\end{pmatrix}\right)(\omega) \geq 0.\end{aligned}$$

By Lemma 6.1.3, and since n was arbitrary, we then have that ϕ is completely positive. ∎

Proposition 6.1.5. *Let A and B be unital C^*-algebras such that A is abelian. If $\phi : A \to B$ is positive, then ϕ is completely positive.*

Proof. We may identify A with $C(K)$ for some compact Hausdorff space K, hence for $n \in \mathbb{N}$ we may identify $\mathbb{M}_n(A)$ with $C(K, \mathbb{M}_n(\mathbb{C}))$ where the norm is given by $\|f\| = \sup_{k\in K} \|f(k)\|$.

Suppose $f \in C(K, \mathbb{M}_n(\mathbb{C}))$ is positive, with $\|f\| \leq 1$, and let $\varepsilon > 0$ be given. Since K is compact f is uniformly continuous and hence there exists a finite open cover $\{U_1, U_2, \ldots, U_m\}$ of K, and $a_1, a_2, \ldots, a_m \in \mathbb{M}_n(\mathbb{C})_+$ such that $\|f(k) - a_j\| \leq \varepsilon$ for all $k \in U_j$.

For each $j \leq m$ chose $g_j \in C(K)$ such that $0 \leq g_j \leq 1$, $\sum_{j=1}^m g_j = 1$, and $g_{j|U_j^c} = 0$. If we consider $f_0 = \sum_{j=1}^m g_j a_j$ then we have $\|f - f_0\| \leq \varepsilon$. Therefore we have $\|\phi^{(n)}(f) - \phi^{(n)}(f_0)\| \leq \|\phi^{(n)}\|\varepsilon$.

Since $\phi^{(n)}(g_j a_j) = \phi(g_j)a_j \geq 0$, for all $1 \leq j \leq m$, we have that $\phi^{(n)}(f_0) \geq 0$, and hence since $\varepsilon > 0$ was arbitrary it follows that $\phi^{(n)}(f) \geq 0$, and it follows that ϕ is completely positive. ∎

The previous proposition gives us a strengthening of Kadison's inequality.

Corollary 6.1.6 (Kadison's inequality for positive maps)**.** *Let A and B be unital C^*-algebras, and $\phi : A \to B$ a unital positive map. Then for all $x \in A$ normal we have $\phi(x)^*\phi(x) \leq \phi(x^*x)$.*

Proof. Restricting ϕ to the abelian unital C^*-algebra generated by x we may then assume, by the previous proposition, that ϕ is completely positive. Hence this follows from Kadison's inequality for completely positive maps. ∎

Lemma 6.1.7. *Let A be a C^*-algebra, if* $\begin{pmatrix} 0 & x^* \\ x & y \end{pmatrix} \in \mathbb{M}_2(A)$ *is positive, then $x = 0$, and $y \geq 0$.*

Proof. We may assume A is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, hence if $\xi, \eta \in \mathcal{H}$ we have

$$2\mathrm{Re}(\langle x^*\eta, \xi\rangle) + \langle y\eta, \eta\rangle = \left\langle \begin{pmatrix} 0 & x^* \\ x & y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \geq 0.$$

The result then follows easily. ■

Theorem 6.1.8 (Choi). *If $\phi : A \to B$ is a unital 2-positive map between C^*-algebras, then for $a \in A$ we have $\phi(a^*a) = \phi(a^*)\phi(a)$ if and only if $\phi(xa) = \phi(x)\phi(a)$, and $\phi(a^*x) = \phi(a^*)\phi(x)$, for all $x \in A$.*

Proof. Applying Kadison's inequality to $\phi^{(2)}$ it follows that for all $x \in A$ we have

$$\begin{pmatrix} \phi(a^*a) & \phi(a^*x) \\ \phi(x^*a) & \phi(aa^* + x^*x) \end{pmatrix} = \phi^{(2)}\left(\left| \begin{pmatrix} 0 & a^* \\ a & x \end{pmatrix} \right|^2 \right) \geq \left| \phi^{(2)}\left(\begin{pmatrix} 0 & a^* \\ a & x \end{pmatrix} \right) \right|^2$$
$$= \begin{pmatrix} \phi(a^*)\phi(a) & \phi(a^*)\phi(x) \\ \phi(x^*)\phi(a) & \phi(a)\phi(a^*) + \phi(x^*)\phi(x) \end{pmatrix}.$$

Since $\phi(a^*a) = \phi(a)^*\phi(a)$ it follows from the previous lemma that $\phi(x^*a) = \phi(x^*)\phi(a)$, and $\phi(a^*x) = \phi(a)^*\phi(x)$. ■

If $\phi : A \to B$ is completely positive, then the **multiplicative domain** of ϕ is

$$\{a \in A \mid \phi(a^*a) = \phi(a^*)\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a^*)\}.$$

Note that by Theorem 6.1.8 the multiplicative domain is a C^*-subalgebra of A, and ϕ restricted to the multiplicative domain is a homomorphism.

Corollary 6.1.9. *If A is a unital C^*-algebra, $\phi : A \to A$ is unital and 2-positive, and $B \subset A$ is a C^*-subalgebra such that $\phi(b) = b$ for all $b \in B$ then ϕ is B-bimodular, i.e., for all $x \in A$, $b_1, b_2 \in B$ we have $\phi(b_1 x b_2) = b_1\phi(x)b_2$.*

Theorem 6.1.10 (Choi). *If A and B are unital C^*-algebras, and $\phi : A \to B$ is a unital 2-positive isometry onto B, then ϕ is an isomorphism.*

Proof. Since a self-adjoint element x of norm at most 1 in a unital C^*-algebra is positive if and only if $\|1 - x\| \leq 1$ it follows that ϕ^{-1} is positive.

Fix $a \in A$ self adjoint, and assume $\|a\| \leq 1$. Since ϕ is onto there exists $b \in A$ such that $\phi(b) = \phi(a)^2 \leq \phi(a^2)$.

Thus $b \leq a^2$, and since ϕ^{-1} is also positive we may apply the previous corollary to the map ϕ^{-1} to conclude that

$$a^2 = \phi^{-1}(\phi(a))\phi^{-1}(\phi(a)) \leq \phi^{-1}(\phi(a)^2) = b.$$

Hence, $\phi(a)^2 = \phi(b) = \phi(a^2)$.

Since a was an arbitrary self adjoint element, and since A is generated by its self adjoint elements, Theorem 6.1.8 then shows that ϕ is an isomorphism. ■

Exercise 6.1.11. Show that a C^*-algebra A is abelian if and only if for any C^*-algebra B, every positive map from B to A is completely positive.

6.2 Conditional expectations

If A is a unital C^*-algebra, and $B \subset A$ is a unital C^*-subalgebra, then a **conditional expectation** from A to B is a unital completely positive $E : A \to B$ such that $E_{|B} = \mathrm{id}$. Note that by Choi's theorem we have $E(axb) = aE(x)b$ for all $a, b \in B$, $x \in A$.

Theorem 6.2.1 (Tomiyama)**.** *Let A be a unital C^*-algebra, $B \subset A$ a unital C^*-subalgebra, and $E : A \to B$ a linear map such that $E_{|B} = \mathrm{id}$ and $\|E\| \leq 1$, then E is a conditional expectation.*

Proof. We first consider the case when A is a von Neumann algebra and $B \subset A$ is a von Neumann subalgebra. Then if $p \in \mathcal{P}(B)$ and $x \in A$ we have $(1-p)E(px) = E((1-p)E(px))$ and hence for all $t > 0$ we have

$$\begin{aligned}(1+t)^2\|pE((1-p)x)\|^2 &= \|pE((1-p)x + tpE((1-p)x))\|^2 \\ &\leq \|(1-p)x + tpE((1-p)x)\|^2 \\ &\leq \|(1-p)x\|^2 + t^2\|pE((1-p)x)\|^2\end{aligned}$$

Hence for all $t > 0$ we have $(1+2t)\|pE((1-p)x)\|^2 \leq \|(1-p)x\|^2$ which then shows that $pE((1-p)x) = 0$. Since this also holds when replacing p with $1-p$ we then have $pE(x) = pE((1-p)x + px) = pE(px) = E(px)$, and since the span of projections is norm dense in B it then follows that $E(yx) = yE(x)$ for all$y \in B$, $x \in A$. Taking adjoints shows that E is B-bimodular.

Since E is a unital contraction Lemma 6.0.8 shows that E is positive. To see that E is completely positive consider $[a_{i,j}] \in \mathbb{M}_n(A)_+$, and $x_1, \ldots, x_n \in B$. Then by Lemma 6.1.3 we have

$$\sum_{i,j=1}^n x_i^* E(a_{i,j}) x_j = E(\sum_{i,j=1}^n x_i^* a_{i,j} x_j) \geq 0.$$

Hence $E^{(n)}([a_{i,j}]) \geq 0$ and so E is completely positive.

For the general case if we consider the double dual $A \subset A^{**}$ and $B \subset B^{**}$, then these are von Neumann algebras, and the result follows by considering the map $E^{**} : A^{**} \to B^{**}$. ∎

Theorem 6.2.2 (Umegaki)**.** *Let M be a finite von Neumann algebra with normal faithful trace τ, and let $N \subset M$ be a von Neumann subalgebra, then there exists a unique normal conditional expectation $E : M \to N$ such that $\tau \circ E = \tau$.*

Proof. Let $e_N \in \mathcal{B}(L^2(M,\tau))$ be the projection onto $L^2(N,\tau) \subset L^2(M,\tau)$, and let J be the conjugation operator on $L^2(M,\tau)$ which we also view as the conjugation operator on $L^2(N,\tau)$. Note that $N' \cap \mathcal{B}(L^2(N,\tau)) = JNJ$ by

Proposition 4.7.12. Since $L^2(N,\tau)$ is invariant under JNJ we have $e_N(JyJ) = (JyJ)e_N$ for all $y \in N$.

If $x \in M$, $y \in N$, then

$$\begin{aligned} e_N x e_N JyJ = e_N xJyJ &= e_N JyJxe_N \\ &= JyJe_N xe_N. \end{aligned}$$

Thus, $e_N x e_N \in (JNJ)' = N$ and we denote this operator by $E(x)$. Clearly, $E : M \to N$ is normal unital completely positive, and $E_{|N} = \mathrm{id}$, thus E is a normal conditional expectation. Also, for $x \in M$ we have

$$\tau(E(x)) = \langle e_N x e_N 1_\tau, 1_\tau\rangle = \langle x1_\tau, 1_\tau\rangle = \tau(x).$$

If $\tilde{E}$ were another trace preserving conditional expectation, then for $x \in M$, and $y \in N$ we would have

$$\begin{aligned} \tau(\tilde{E}(x)y) = \tau(\tilde{E}(xy)) &= \tau(xy) \\ &= \tau(E(xy)) = \tau(E(x)y), \end{aligned}$$

from which it follows that $\tilde{E} = E$. ∎

Lemma 6.2.3 (Sakai). *Let M be a semi-finite factor, and $p \in M$ a finite projection. Then the adjoint operation is strongly continuous on bounded subsets of Mp.*

Proof. Let Tr be a semi-finite faithful normal trace on M, and let $\mathfrak{M}_{\mathrm{Tr}}$ be as in Lemma 4.9.2. Then the linear functionals of the form $x \mapsto \mathrm{Tr}(ax)$ for $a \in \mathfrak{M}_{\mathrm{Tr}}$ form a dense subset of M_*.

Suppose $\{x_i p\}$ is a net of bounded operators in M which converge strongly to 0, and consider $a \in \mathfrak{M}_{\mathrm{Tr}}$, then

$$|\,\mathrm{Tr}(ax_ipx_i^*)| = |\,\mathrm{Tr}(px_i^*ax_ip)| \leq \mathrm{Tr}(px_i^*x_ip)^{1/2}\,\mathrm{Tr}(px_i^*a^*ax_ip)^{1/2} \to 0$$

Thus since $\{x_ip\}$ is bounded, $(x_ip)(x_ip)^*$ converges σ-weakly to 0 and hence $(x_ip)^*$ converges strongly to 0. ∎

Note that the previous lemma is not true if we considered bounded subsets of pM instead. Easy counter-example can be found by considering $M = \mathcal{B}(\mathcal{H})$ and p a rank one projection. Also note that if a von Neumann algebra is not finite then the adjoint operation is not continuous on bounded sets. We leave this as an exercise.

Theorem 6.2.4 (Tomiyama). *Let M be a semi-finite factor and $N \subset M$ a purely infinite von Neumann subalgebra, then there exists no normal conditional expectation from M to N.*

Proof. Suppose $E : M \to N$ is a normal conditional expectation where M is semi-finite. Let $p \in M$ be a finite projection such that $E(p) \neq 0$, and take $q \in \mathcal{P}(N)$, $\lambda > 0$ such that $\lambda q \leq E(p)$. If $x_i \in qNq$ is a net which converges strongly to 0 then by the previous lemma $E(p)x_i^* = E(px_i^*)$ also converges strongly to 0, and hence $x_i^* = (qE(p)q)^{-1}qE(p)x_i^*$ also converges strongly to 0. Thus qNq is finite, and by a simple maximality argument it then follows that N is semi-finite. ■

Theorem 6.2.5 (Tomiyama)**.** *If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, then there exists a normal conditional expectation $E : \mathcal{B}(\mathcal{H}) \to M$ if and only if M is completely atomic.*

Proof. If M is completely atomic then $M = \oplus_{i \in I} M_i$ where each M_i is a type I factor, restricting to each corner to construct a normal conditional expectation from $\mathcal{B}(\mathcal{H})$ to M it is then enough to do so assuming M is a type I factor. In this case M' is also a type I factor by Theorem 3.3.9 and hence has a minimal projection $p \in M'$. Then for $T \in \mathcal{B}(\mathcal{H})$ we have $pTp \in (pM'p)' \cap \mathcal{B}(p\mathcal{H}) = Mp$, and since $z(p) = 1$ we have $xp \mapsto x$ is an isomorphism on M, thus composing these maps gives a normal conditional expectation.

For the converse, we first note that by Theorem 6.2.4 M must be semi-finite, and hence by restricting to corners of M it is enough to consider the case when M is finite. Similarly, by restricting to corners it is enough to consider the case when M is countably decomposable and hence we may assume that M is diffuse and has a normal faithful trace τ.

If E is a normal conditional expectation then the the map $T \mapsto \tau(E(T))$ defines a positive normal linear functional on $\mathcal{B}(\mathcal{H})$ and so must be of the form $\tau(E(T)) = \mathrm{Tr}(AT)$ for some positive trace class operator A. Moreover, for all $T \in \mathcal{B}(\mathcal{H})$ and $x \in M$ we have

$$\begin{aligned} \mathrm{Tr}(AxT) &= \tau(E(xT)) = \tau(xE(T)) \\ &= \tau(E(Tx)) = \mathrm{Tr}(ATx) = \mathrm{Tr}(xAT). \end{aligned}$$

Since this holds for all $T \in \mathcal{B}(\mathcal{H})$, and all $x \in M$ we conclude $A \in M'$. Taking a spectral projection of A we then produce a finite rank projection P such that $P \in M'$. But then if $z(P)$ is the central support of P in M' then we have $z(P)M \cong PM \subset P\,\mathcal{B}(\mathcal{H})P$, and the latter is finite dimensional contradicting the assumption that M was diffuse. ■

6.3 Injective von Neumann algebras

If X and Y are Banach spaces and $x \in X$, and $y \in Y$ then we define the linear map $x \otimes y : \mathcal{B}(X, Y^*) \to \mathbb{C}$ by $(x \otimes y)(L) = L(x)(y)$. Note that $|(x \otimes y)(L)| \leq \|L\|\|x\|\|y\|$ and hence $x \otimes y$ is bounded and indeed $\|x \otimes y\| \leq \|x\|\|y\|$. Let Z be the norm closed linear span in $\mathcal{B}(X, Y^*)^*$ of all $x \otimes y$.

Lemma 6.3.1. *The pairing $\langle L, x \otimes y \rangle = (x \otimes y)(L)$ extends to an isometric identification between Z^* and $\mathcal{B}(X, Y^*)$.*

Proof. It is easy to see that this pairing gives an isometric embedding of $\mathcal{B}(X, Y^*)$ into Z^*. To see that this is onto consider $\varphi \in Z^*$, and for each $x \in X$ define

$L_x : Y \to \mathbb{C}$ by $L_x(y) = \varphi(x \otimes y)$. Then we have $|L_x(y)| \leq \|\varphi\|\|x\|\|y\|$ and hence $L_x \in Y^*$. The mapping $x \mapsto L_x$ is easily seen to be linear and thus we have $L \in \mathcal{B}(X, Y^*)$, and clearly L is mapped to φ under this pairing. ■

Lemma 6.3.2. *If X and Y are Banach spaces and L_i is a bounded net in $\mathcal{B}(X, Y^*)$ then $L_i \to L$ in the weak*-topology described above if and only if $L_i(x) \to L(x)$ in the weak*-topology for all $x \in X$.*

Proof. If L_i converges to L in the weak*-topology then for all $x \in X$ and $y \in Y$ we have $L_i(x)(y) = (x \otimes y)(L_i) \to (x \otimes y)(L) = L(x)(y)$ showing that $L_i(x) \to L(x)$ is the weak*-topology for all $x \in X$. Conversely, if $L_i(x) \to L(x)$ in the weak*-topology for all $x \in X$ then in particular we have $(x \otimes y)(L_i) \to (x \otimes y)(L)$ for each $x \in X$ and $y \in Y$, thus this also holds for the linear span of all $x \otimes y$ and since the net is bounded we then have convergence on the closed linear span. ■

Corollary 6.3.3. *Let A be a unital C^*-algebra, then the set of contractive completely positive maps from E to $\mathcal{B}(\mathcal{H})$ is compact in the topology of point-wise weak operator topology convergence.*

Proof. First note that it is easy to see that the space of contractive completely positive maps from E to $\mathcal{B}(\mathcal{H})$ is closed in this topology. Since $\mathcal{B}(\mathcal{H})$ is a dual space, and on bounded sets the weak operator topology is the same as the weak*-topology, the result then follows from Alaoglu's theorem, together with the previous two lemmas. ■

If A is a C^*-algebra and $\phi : A \to \mathbb{M}_n(\mathbb{C})$ is a linear map, then we define the linear functional $\hat{\phi} \in \mathbb{M}_n(A)^*$ by

$$\hat{\phi}([a_{i,j}]) = \frac{1}{n} \sum_{i,j=1}^{n} \phi(a_{i,j})_{i,j}.$$

Where $\phi(a_{i,j})_{i,j}$ denotes the i,jth entry of $[\phi(a_{i,j})]$. Note that the correspondence $\phi \mapsto \hat{\phi}$ is bijective and the inverse can be computed explicitly as $\phi(a)_{i,j} = n\hat{\phi}(a \otimes E_{i,j})$ where $E_{i,j}$ is the standard elementary matrix, and $a \otimes E_{i,j}$ is the matrix with a in the i,jth entry, and zeros elsewhere.

Lemma 6.3.4. *Let A be a unital C^*-algebra. A map $\phi : A \to \mathbb{M}_n(\mathbb{C})$ is unital completely positive if and only if $\hat{\phi}$ is a state.*

Proof. Let $\{e_i\}$, $1 \leq i \leq n$ denote that standard basis for $\mathbb{C}^n$, and let $\eta = [e_1, \ldots, e_n]^T \in \mathbb{C}^{n^2}$, then for $[a_{i,j}] \in \mathbb{M}_n(A)$ a simple calculation shows

$$\hat{\phi}([a_{i,j}]) = \frac{1}{n} \langle \phi^{(n)}([a_{i,j}])\eta, \eta \rangle.$$

Thus, $\hat{\phi}$ is a state if ϕ is unital completely positive. Conversely, if $\hat{\phi}$ is a state then consider the GNS-construction $L^2(\mathbb{M}_n(A), \hat{\phi})$ with cyclic vector $1_{\hat{\phi}}$. If we

define $V : \mathbb{C}^n \to L^2(\mathbb{M}_n(A), \hat{\phi})$ by $Ve_j = \pi(e_{1,j})1_{\hat{\phi}}$ then for $a \in A$ it follows easily that
$$\phi(a) = V^* \pi(a\mathrm{I})V,$$
Hence ϕ is completely positive. ■

Lemma 6.3.5. *Let A be a unital C^*-algebra, and $E \subset A$ an operator system. If $\phi : E \to \mathbb{M}_n(\mathbb{C})$ is a completely positive map then there exists a completely positive extension $\tilde{\phi} : A \to \mathbb{M}_n(\mathbb{C})$.*

Proof. If $\phi : E \to \mathbb{M}_n(\mathbb{C})$ is completely positive then the same argument as in Lemma 6.3.4 shows that $\hat{\phi}$ defines a state on $\mathbb{M}_n(E)$. By the Hahn-Banach theorem we can then extend this to norm 1 linear functional on $\mathbb{M}_n(A)$ which then must also be a state and so by Lemma 6.3.4 this corresponds to a unital completely positive extension $\tilde{\phi} : A \to \mathbb{M}_n(\mathbb{C})$. ■

Theorem 6.3.6 (Arveson's extension theorem)**.** *Let A be a unital C^*-algebra, and $E \subset A$ an operator system. If $\phi : E \to \mathcal{B}(\mathcal{H})$ is a completely positive map then there exists a completely positive extension $\tilde{\phi} : A \to \mathcal{B}(\mathcal{H})$.*

Proof. For each finite rank projection $P \in \mathcal{B}(\mathcal{H})$ we may consider the compression $E \ni x \mapsto P\phi(x)P$ and by Lemma 6.3.5, this has a completely positive extension $\tilde{\phi}_P : A \to P\mathcal{B}(\mathcal{H})P \subset \mathcal{B}(\mathcal{H})$. If we consider the net $\{\phi_P\}$ which is ordered by the usual order on projections, then by Corollary 6.3.3 this net must have a completely positive cluster point $\tilde{\phi}$. Since $\tilde{\phi}_P(x) = P\phi(x)P$ for all $x \in E$ it is then easy to see that $\tilde{\phi}$ is an extension of ϕ. ■

An operator system F is **injective** if for any C^*-algebra A and any operator system $E \subset A$, whenever $\phi : E \to F$ is completely positive then there exists a completely positive extension $\tilde{\phi} : A \to E$. Arveson's extension theorem states that $\mathcal{B}(\mathcal{H})$ is injective.

Corollary 6.3.7. *Let $F \subset \mathcal{B}(\mathcal{H})$ be an operator system, then F is injective if and only if there exists a completely positive map $E : \mathcal{B}(\mathcal{H}) \to F$ such that $E_{|F} = \mathrm{id}$. In particular, the existence of such a completely positive map does not depend on the representation $F \subset \mathcal{B}(\mathcal{H})$.*

Proof. If F is injective, then the identity map from F to F has an extension E to $\mathcal{B}(\mathcal{H})$. Conversely, if $E : \mathcal{B}(\mathcal{H}) \to F$ is completely positive such that $E_{|F} = \mathrm{id}$, and if A is a C^*-algebra and $F' \subset A$ is an operator space, such that $\phi : F' \to F \subset \mathcal{B}(\mathcal{H})$ is unital completely positive, then by Arveson's extension theorem there exists an extension $\tilde{\phi} : A \to \mathcal{B}(\mathcal{H})$, and $E \circ \tilde{\phi} : A \to F$ then gives an extension of ϕ showing that F is injective. ■

Let M be a von Neumann algebra, and $N \subset M$ a finite von Neumann subalgebra with normal faithful trace τ. A **hypertrace** for the inclusion $N \subset M$ is a state φ on M such that $\varphi_{|N} = \tau$, and $\varphi(xA) = \varphi(Ax)$ for all $x \in N$, $A \in M$. A finite von Neumann algebra N with normal faithful trace τ is **hyperfinite** if there exists a hypertrace for the inclusion $N \subset \mathcal{B}(\mathcal{H})$. This does not depend on the representation $N \subset \mathcal{B}(\mathcal{H})$, nor on the normal faithful trace τ, as we see from Corollary 6.3.7, and the following non-normal analogue of Umegaki's theorem.

Theorem 6.3.8. *Let M be a von neumann algebra, and $N \subset M$ a finite von Neumann subalgebra with normal faithful trace τ. Then there exists a hypertrace for $N \subset M$ if and only if there exists a (possibly non-normal) conditional expectation $E : M \to N$. In particular, considering $M = \mathcal{B}(\mathcal{H})$, N is hyperfinite if and only if N is injective.*

Proof. First suppose there exists a conditional expectation $E : M \to N$. If we consider $\varphi \in M^*$ given by $\varphi(T) = \tau(E(T))$ then for $T \in M$ and $x \in N$ we have

$$\varphi(Tx) = \tau(E(Tx)) = \tau(E(T)x) = \tau(xE(T)) = \varphi(xT),$$

thus φ is a hypertrace.

Conversely, suppose $\varphi \in M^*$ is a hypertrace. If we consider the GNS-construction $L^2(M, 1_\varphi)$, then since $\varphi_{|N} = \tau$ the map $x \mapsto x1_\varphi$ extends to give an isometric embedding $L^2(N, \tau) \subset L^2(M, \varphi)$. We denote by e_N the orthogonal projection onto this subspace, and we denote by $E : M \to \mathcal{B}(L^2(N, \tau))$ the unital completely positive map $E(T) = e_N T e_N$. A simple check shows that $E_{|N} = \mathrm{id}$, and if $T \in M$, and $x, y, z \in N$ then we have

$$\begin{aligned}
\langle E(T)(JxJ)y1_\tau, z1_\tau\rangle &= \langle ze_M T e_M yx^* 1_\varphi, 1_\varphi\rangle \\
&= \langle e_M zTyx^* e_M 1_\varphi, 1_\varphi\rangle \\
&= \varphi(z^* Tyx^*) \\
&= \varphi(x^* z^* Ty) \\
&= \langle e_M x^* z^* Tye_M 1_\varphi, 1_\varphi\rangle \\
&= \langle E(T)y1_\tau, zx1_\tau\rangle = \langle (JxJ)E(T)y1_\tau, z1_\tau\rangle.
\end{aligned}$$

By Proposition 4.7.12, $(JNJ)' = N$, and hence $E : M \to N$ is a conditional expectation. ∎

Example 6.3.9. Let Γ be a countable group which is locally finite, i.e., every finitely generated subgroup is finite. Then we can write $\Gamma = \cup_{n\in\mathbb{N}}\Gamma_n$ where Γ_n forms an increasing sequence of finite subgroups. For each $n \in \mathbb{N}$ we define the unital completely positive map $\phi_n : \mathcal{B}(\ell^2\Gamma) \to \mathcal{B}(\ell^2\Gamma)$ by $\phi_n(T) = \frac{1}{|\Gamma_n|}\sum_{g\in\Gamma_n} \rho_g T \rho_{g^{-1}}$. By Corollary 6.3.3 there exists a cluster point $E : \mathcal{B}(\ell^2\Gamma) \to \mathcal{B}(\ell^2\Gamma)$ for this sequence in the topology of point-wise weak convergence.

Note that $E_{|L\Gamma} = \mathrm{id}$ since this holds for each ϕ_n. Also, for each $n \in \mathbb{N}$, $h \in \Gamma_n$ and $T \in \mathcal{B}(\ell^2\Gamma)$ we have $\rho_h\phi_n(T)\rho_{h^{-1}} = \phi_n(T)$, and since the sequence Γ_n is increasing we then have that the range of ϕ_m is in $\rho(\Gamma_n)'$ whenever $m \geq n$. Since $\rho(\Gamma_n)'$ is closed in the weak operator topology we have that the range of E is in $\rho(\Gamma_n)'$ for every $n \in \mathbb{N}$. However, $\rho(\cup_{n\in\mathbb{N}}\Gamma_n)' = L\Gamma$ and hence E is a conditional expectation from $\mathcal{B}(\ell^2\Gamma)$ to $L\Gamma$ showing that $L\Gamma$ is injective.

Chapter 7

Group representations and approximation properties

To motivate our investigation into finite von Neumann algebras, as well as to provide us with examples, we will make a detour into approximation properties for representations of discrete groups.

7.1 Almost invariant vectors

Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ contains **invariant vectors** if there exists a non-zero vector $\xi \in \mathcal{H}$ such that $\pi_g \xi = \xi$ for all $g \in \Gamma$. The representation contains **almost invariant vectors** if for each $F \subset \Gamma$, and $\varepsilon > 0$, there exists $\xi \in \mathcal{H}$, such that

$$\|\pi_g \xi - \xi\| < \varepsilon \|\xi\|, \text{ for all } g \in F.$$

Proposition 7.1.1. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c > 0$ such that $\mathrm{Re}(\langle \pi_g \xi, \xi \rangle) > c\|\xi\|^2$ for all $g \in \Gamma$, then π contains an invariant vector ξ_0 such that $\mathrm{Re}(\langle \xi_0, \xi \rangle) \geq c\|\xi\|^2$.*

Proof. Let K be the closed convex hull of the orbit $\pi(\Gamma)\xi$. We therefore have that K is Γ-invariant and $\mathrm{Re}(\langle \eta, \xi \rangle) \geq c\|\xi\|^2$ for every $\eta \in K$. Let $\xi_0 \in K$ be the unique element of minimal norm, then since Γ acts isometrically we have that for each $g \in \Gamma$, $\pi_g \xi_0$ is the unique element of minimal norm for $\pi_g K = K$, and hence $\pi_g \xi_0 = \xi_0$ for each $g \in \Gamma$. Since $\xi_0 \in K$ we have that $\mathrm{Re}(\langle \xi_0, \xi \rangle) \geq c\|\xi\|^2$. ∎

Corollary 7.1.2. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c < \sqrt{2}$ such that $\|\pi_g \xi - \xi\| < c\|\xi\|$ for all $g \in \Gamma$, then π contains an invariant vector.*

Proof. For each $g \in \Gamma$ we have

$$2\mathrm{Re}(\langle \pi_g \xi, \xi \rangle) = 2\|\xi\|^2 - \|\pi_g \xi - \xi\|^2 \geq (2 - c^2)\|\xi\|^2.$$

Hence, we may apply Proposition 7.1.1. ■

Lemma 7.1.3. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. Then π contains almost invariant vectors if and only if $\pi^{\oplus n}$ contains almost invariant vector, where $n \geq 1$ is any cardinal number.*

Proof. If π does not contain almost invariant vectors then there exists $c > 0$, and $S \subset \Gamma$ finite, such that for all $\xi \in \mathcal{H}$ we have

$$c\|\xi\|^2 \leq \sum_{g \in S} \|\pi_g \xi - \xi\|^2.$$

If I is a set $|I| = n$, and $\xi_i \in \mathcal{H}$ for $i \in I$, such that $\sum_{i \in I} \|\xi_i\|^2 < \infty$, then

$$c\|\bigoplus_{i \in I} \xi_i\|^2 = \sum_{i \in I} c\|\xi_i\|^2$$

$$\leq \sum_{i \in I} \sum_{g \in S} \|\pi_g \xi_i - \xi_i\|^2 = \sum_{g \in S} \|\pi^{\oplus n}(g)(\bigoplus_{i \in I} \xi_i) - \bigoplus_{i \in I} \xi_i\|^2.$$

Hence, $\pi^{\oplus n}$ does not contain almost invariant vectors. The converse is trivial since π is contained in $\pi^{\oplus \infty}$. ■

If Γ is a group and $\mu \in \mathrm{Prob}(\Gamma) \subset \ell^1\Gamma$, then for a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ the μ-**gradient** operator $\nabla_\mu : \Gamma \to \mathcal{H}^{\oplus \Gamma}$ is given by

$$\nabla_\mu \xi = \bigoplus_{g \in \Gamma} \mu(g)^{1/2}(\xi - \pi_g \xi).$$

Note that by Hölder's inequality we have $\|\nabla_\mu\| \leq \sqrt{2}$. The μ-**divergence** operator $\mathrm{div}_\mu : \mathcal{H}^{\oplus \Gamma} \to \mathcal{H}$ is given by

$$\mathrm{div}_\mu(\bigoplus_{g \in \Gamma} \xi_g) = \sum_{g \in \Gamma} \mu(g)^{1/2}(\xi_g - \pi_{g^{-1}} \xi_g).$$

If $\xi \in \mathcal{H}$, and $\eta = \oplus_{g \in \Gamma} \eta_g \in \mathcal{H}^{\oplus \Gamma}$ then we have

$$\begin{aligned}
\langle \nabla_\mu \xi, \eta \rangle &= \sum_{g \in \Gamma} \mu(g)^{1/2} \langle \xi - \pi_g \xi, \eta_g \rangle \\
&= \sum_{g \in \Gamma} \mu(g)^{1/2} \langle \xi, \eta_g - \pi_{g^{-1}} \eta_g \rangle \\
&= \langle \xi, \mathrm{div}_\mu \eta \rangle,
\end{aligned}$$

hence $\mathrm{div}_\mu = \nabla_\mu^*$. The μ-**Laplacian** is defined to be $\Delta_\mu = \mathrm{div}_\mu \nabla_\mu$, which we can compute directly as

$$\Delta_\mu = \sum_{g \in \Gamma} \mu(g)(2 - \pi_g - \pi_{g^{-1}}) = (2 - \pi(\mu) - \pi(\mu^*)).$$

Note that if μ is **symmetric**, i.e., $\mu^* = \mu$, then we have $\Delta_\mu = 2(1 - \sum_{g \in \Gamma} \mu(g)\pi_g) = 2(1 - \pi(\mu))$.

Proposition 7.1.4 (Kesten)**.** *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. Then the following conditions are equivalent:*

(i) *π contains almost invariant vectors;*

(ii) *For every $\mu \in \mathrm{Prob}(\Gamma)$ we have $0 \in \sigma(\Delta_\mu)$;*

(iii) *For some $\mu \in \mathrm{Prob}(\Gamma)$ with support generating Γ we have $0 \in \sigma(\Delta_\mu)$;*

(iv) *For every $\mu \in \mathrm{Prob}(\Gamma)$ we have $\|\pi(\mu)\| = 1$;*

(v) *For some $\mu \in \mathrm{Prob}(\Gamma)$ with support generating Γ, and $e \in \mathrm{supp}(\mu)$, we have $\|\pi(\mu)\| = 1$.*

Proof. If π contains almost invariant vectors then there is a net $\{\xi_i\} \subset \mathcal{H}$ such that $\|\xi_i\| = 1$, and $\|\xi_i - \pi_g \xi_i\| \to 0$ for all $g \in \Gamma$. If $\mu \in \mathrm{Prob}(\Gamma)$ then we see that

$$\|\nabla_\mu \xi_i\|^2 = \sum_{g\in\Gamma} \mu(g)\|\xi_i - \pi_g\xi_i\|^2 \to 0,$$

and hence also $\|\Delta_\mu \xi_i\|^2 \to 0$. Thus, $0 \in \sigma(\Delta_\mu)$ showing that (i) $\implies$ (ii).

Clearly, (ii) $\implies$ (iii), and (iv) $\implies$ (v). We next show, (ii) $\implies$ (iv), and (iii) $\implies$ (v). If $\mu \in \mathrm{Prob}(\Gamma)$ such that $0 \in \sigma(\Delta_\mu)$ then for some net $\{\xi_i\} \subset \mathcal{H}$ with $\|\xi_i\| = 1$ we have

$$\begin{aligned}\|\pi(\mu^*)\xi_i\| + \|\pi(\mu)\xi_i\| &\geq \|(\pi(\mu^*) + \pi(\mu))\xi_i\| \\ &\geq 2 - \|\Delta_\mu \xi_i\| \to 2.\end{aligned}$$

Since $\|\pi(\mu)\|, \|\pi(\mu^*)\| \leq 1$ we then have $\|\pi(\mu)\| = \|\pi(\mu^*)\| = 1$. Note that by replacing μ with $\tilde{\mu} = \frac{\delta_e + \mu}{2}$ then we again have $0 \in \sigma(\Delta_{\tilde{\mu}})$, and $\mathrm{supp}(\tilde{\mu}) = \mathrm{supp}(\mu) \cup \{e\}$.

It remains to show (v) $\implies$ (i). For this just notice that if $\mu \in \mathrm{Prob}(\Gamma)$ such that $\|\pi(\mu)\| = 1$, then there exists some net $\{\xi_i\} \subset \mathcal{H}$ such that $\|\xi_i\| = 1$, and $\|\pi(\mu)\xi_i\| \to 1$. We then have

$$\begin{aligned}1 - \|\pi(\mu)\xi_i\|^2 &= \sum_{g,h\in\Gamma} \mu(g)\mu(h)(1 - \langle \pi(g)\xi_i, \pi(h)\xi_i\rangle) \\ &= \sum_{g,h\in\Gamma} \mu(g)\mu(h)(1 - \mathrm{Re}\langle \pi(g)\xi_i, \pi(h)\xi_i\rangle)\end{aligned}$$

Since $\|\pi(\mu)\xi_i\| \to 1$, and $|\langle \pi(g)\xi_i, \pi(h)\xi_i\rangle| \leq 1$ we then have $\langle \pi(g)\xi_i, \pi(h)\xi_i\rangle \to 1$ for all $g, h \in \mathrm{supp}(\mu)$.

If $e \in \mathrm{supp}(\mu)$ then this shows $\|\pi(g)\xi_i - \xi_i\|^2 = 2(1 - \mathrm{Re}\langle \pi(g)\xi_i, \xi_i\rangle) \to 0$ for all $g \in \mathrm{supp}(\mu)$ and since $\mathrm{supp}(\mu)$ generates Γ it then follows that π has almost invariant vectors. ∎

7.2 Amenability

A (left) **invariant mean** m on a discrete group Γ is a finitely additive probability measure on 2^Γ, which is invariant under the action of left multiplication, i.e., $m : 2^\Gamma \to [0,1]$ such that $m(\Gamma) = 1$, if $A_1, \ldots, A_n \subset \Gamma$ are disjoint then $m(\cup_{j=1}^n A_n) = \sum_{j=1}^n m(A_n)$, and $m(gA) = m(A)$ for all $g \in \Gamma$. If Γ possesses an invariant mean then Γ is **amenable**. We can similarly define right invariant means, and in fact if m is a left invariant mean then $m^*(A) = m(A^{-1})$ defines a right invariant mean. Amenable groups were first introduced by von Neumann in his investigations of the Banach-Tarski paradox, the term amenability was later coined by Day.

Given a right invariant mean m on Γ it is possible to define an integral over Γ just as in the case if m were a measure. We therefore obtain a state $\phi_m \in (\ell^\infty\Gamma)^*$ by the formula $\phi_m(f) = \int_\Gamma f\, dm$, and this state is left invariant, i.e., $\phi_m(f \circ g) = \phi_m(f)$ for all $g \in \Gamma$, $f \in \ell^\infty\Gamma$. Conversely, if $\phi \in (\ell^\infty\Gamma)^*$ is a left invariant state, then restricting ϕ to characteristic functions defines a right invariant mean.

Example 7.2.1. Let $\mathbb{F}_2$ be the free group on two generators a, and b. Let A^+ be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a, let A^- be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a^{-1}, let B^+, and B^- be defined analogously, and consider $C = \{e, b, b^2, \ldots\}$. Then we have that

$$\begin{aligned} \mathbb{F}_2 &= A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C) \\ &= A^+ \sqcup aA^- \\ &= b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C). \end{aligned}$$

If m were a left-invariant mean on $\mathbb{F}_2$ then we would have

$$\begin{aligned} m(\mathbb{F}_2) &= m(A^+) + m(A^-) + m(B^+ \setminus C) + m(B^- \cup C) \\ &= m(A^+) + m(aA^-) + m(b^{-1}(B^+ \setminus C)) + m(B^- \cup C) \\ &= m(A^+ \sqcup aA^-) + m(b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C)) = 2m(\mathbb{F}_2). \end{aligned}$$

Hence, $\mathbb{F}_2$ is non-amenable.

An **approximately invariant mean** on Γ is a net $\mu_i \in \mathrm{Prob}(\Gamma)$ such that $\|g_*\mu_i - \mu_i\|_1 \to 0$, for all $g \in \Gamma$.

A **Følner net** is a net of non-empty finite subsets $F_i \subset \Gamma$ such that $|F_i \Delta g F_i|/|F_i| \to 0$, for all $g \in \Gamma$. Note that we do not require that $\Gamma = \cup_i F_i$, nor do we require that F_i are increasing, however, if $|\Gamma| = \infty$ then it is easy to see that any Følner net $\{F_i\}_i$ must satisfy $|F_i| \to \infty$.

Theorem 7.2.2. *Let Γ be a discrete group, then the following conditions are equivalent.*

(*i*) *Γ is amenable.*

(*ii*) Γ *has an approximate invariant mean.*

(*iii*) Γ *has a Følner net.*

(*iv*) *The left regular representation* $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ *has almost invariant vectors.*

(*v*) *For any* $\mu \in \mathrm{Prob}(\Gamma)$ *we have* $0 \in \sigma(\lambda(\Delta_\mu))$.

(*vi*) *Any* $\mu \in \mathrm{Prob}(\Gamma)$ *satisfies* $\|\lambda(\mu)\| = 1$.

(*vii*) *There exists a state* $\varphi \in (\mathcal{B}(\ell^2\Gamma))^*$ *such that* $\varphi(\lambda(g)T) = \varphi(T\lambda(g))$ *for all* $g \in \Gamma$, $T \in \mathcal{B}(\ell^2\Gamma)$.

(*viii*) *The continuous action of* Γ *on its Stone-Čech compactification* $\beta\Gamma$ *which is induced by left-multiplication admits an invariant Radon probability measure.*

(*ix*) *Any continuous action* $\Gamma \curvearrowright K$ *on a compact Hausdorff space* K *admits an invariant Radon probability measure.*

Proof. We show (i) $\implies$ (ii) using the method of Day: Since $\ell^\infty\Gamma = (\ell^1\Gamma)^*$, the unit ball in $\ell^1\Gamma$ is weak*-dense in the unit ball of $(\ell^\infty\Gamma)^* = (\ell^1\Gamma)^{**}$. It follows that $\mathrm{Prob}(\Gamma) \subset \ell^1\Gamma$ is weak*-dense in the state space of $\ell^\infty\Gamma$.

Let $S \subset \Gamma$, be finite and let $K \subset \bigoplus_{g\in S} \ell^1\Gamma$ be the weak-closure of the set $\{\bigoplus_{g\in S}(g_*\mu - \mu) \mid \mu \in \mathrm{Prob}(\Gamma)\}$. Since Γ has a left invariant state on $\ell^\infty\Gamma$, and since $\mathrm{Prob}(\Gamma)$ is weak*-dense in the state space of $\ell^\infty\Gamma$, we have that $0 \in K$. However, K is convex and so by the Hahn-Banach separation theorem the weak-closure coincides with the norm closure. Thus, for any $\varepsilon > 0$ there exists $\mu \in \mathrm{Prob}(\Gamma)$ such that

$$\sum_{g\in S} \|g_*\mu - \mu\|_1 < \varepsilon.$$

We show (ii) $\implies$ (iii) using the method of Namioka: Let $S \subset \Gamma$ be a finite set, and denote by E_r the characteristic function on the set (r, ∞). If $\mu \in \mathrm{Prob}(\Gamma)$ then we have

$$\begin{aligned}
\sum_{g\in S} \|g_*\mu - \mu\|_1 &= \sum_{g\in S}\sum_{x\in\Gamma} |g_*\mu(x) - \mu(x)| \\
&= \sum_{g\in S}\sum_{x\in\Gamma} \int_{\mathbb{R}_{\geq 0}} |E_r(g_*\mu(x)) - E_r(\mu(x))|\, dr \\
&= \sum_{g\in S} \int_{\mathbb{R}_{\geq 0}} \sum_{x\in\Gamma} |E_r(g_*\mu(x)) - E_r(\mu(x))|\, dr \\
&= \sum_{g\in S} \int_{\mathbb{R}_{\geq 0}} \|E_r(g_*\mu) - E_r(\mu)\|_1\, dr.
\end{aligned}$$

By hypothesis, if $\varepsilon > 0$ then there exists $\mu \in \mathrm{Prob}(\Gamma)$ such that $\sum_{g \in S} \|g_*\mu - \mu\|_1 < \varepsilon$, and hence for this μ we have

$$\sum_{g \in S} \int_{\mathbb{R}_{\geq 0}} \|E_r(g_*\mu) - E_r(\mu)\|_1 \, dr < \varepsilon = \varepsilon \int_{\mathbb{R}_{\geq 0}} \|E_r(\mu)\|_1 \, dr.$$

Hence, if we denote by $F_r \subset \Gamma$ the (finite) support of $E_r(\mu)$, then for some $r > 0$ we must have

$$\sum_{g \in S} |gF_r \Delta F_r| = \sum_{g \in S} \|E_r(g_*\mu) - E_r(\mu)\|_1 < \varepsilon \|E_r(\mu)\|_1 = \varepsilon |F_r|.$$

For (iii) $\implies$ (iv) just notice that if $F_i \subset \Gamma$ is a Følner net, then $\frac{1}{|F_i|^{1/2}} 1_{F_i} \in \ell^2\Gamma$ is a net of almost invariant vectors.

(iv) $\iff$ (v) $\iff$ (vi) follows from Proposition 7.1.4.

For (iv) $\implies$ (vii) let $\xi_i \in \ell^2\Gamma$ be a net of almost invariant vectors for λ. We define states φ_i on $\mathcal{B}(\ell^2\Gamma)$ by $\varphi_i(T) = \langle T\xi_i, \xi_i \rangle$. By weak compactness of the state space, we may take a subnet and assume that this converges in the weak topology to $\varphi \in \mathcal{B}(\ell^2\Gamma)^*$. We then have that for all $T \in \mathcal{B}(\ell^2\Gamma)$ and $g \in \Gamma$,

$$\begin{aligned} |\varphi(\lambda_g T - T\lambda_g)| &= \lim_i |\langle (\lambda_g T - T\lambda_g)\xi_i, \xi_i \rangle| \\ &= \lim_i |\langle T\xi_i, \lambda_{g^{-1}}\xi_i \rangle - \langle T\lambda_g \xi_i, \xi_i \rangle| \\ &\leq \lim_i \|T\| (\|\lambda_{g^{-1}}\xi_i - \xi_i\| + \|\lambda_g \xi_i - \xi_i\|) = 0. \end{aligned}$$

For (vii) $\implies$ (i), we consider the usual embedding $M : \ell^\infty\Gamma \to \mathcal{B}(\ell^2\Gamma)$ by point-wise multiplication. For $f \in \ell^\infty\Gamma$ and $g \in \Gamma$ we have $\lambda_g M_f \lambda_{g^{-1}} = M_{f \circ g^{-1}}$. Thus, if $\varphi \in \mathcal{B}(\ell^2\Gamma)^*$ is a state which is invariant under the conjugation by λ_g, then restricting this state to $\ell^\infty\Gamma$ gives a state on $\ell^\infty\Gamma$ which is Γ-invariant.

(i) $\iff$ (viii), follows from the Γ-equivariant identification $\ell^\infty\Gamma \cong C(\beta\Gamma)$, together with the Riesz representation theorem.

For (viii) $\iff$ (ix), suppose Γ acts continuously on a compact Hausdorff space K, and fix a point $x_0 \in K$. Then the map $f(g) = gx_0$ on Γ extends uniquely to a continuous map $\beta f : \beta\Gamma \to K$, moreover since f is Γ-equivariant, so is βf. If μ is an invariant Radon probability measure for the action on $\beta\Gamma$ then we obtain the invariant Radon probability measure $f_*\mu$ on K. Since $\beta\Gamma$ itself is compact, the converse is trivial. ■

The previous theorem is the combined work of many mathematicians, including von Neumann, Følner, Day, Namioka, Hulanicki, Reiter, and Kesten.

Note that if $L\Gamma$ is injective, then by Theorem 6.3.8, and part (vii) of Theorem 7.2.2, we see that Γ is amenable (we will see later that the converse holds as well). In particular, when we combine this with Examples 6.3.9 and 7.2.1 then we see that LS_∞ and $L\mathbb{F}_2$ are non-isomorphic II_1 factors. In fact, we see that $L\mathbb{F}_2$ is also not isomorphic to any subfactor of LS_∞.

Example 7.2.3. Any finite group is amenable, and from part (iii) of Theorem 7.2.2 we see that any group which is locally amenable (each finitely generated subgroup is amenable) is also amenable. The group $\mathbb{Z}^n$ is amenable (consider the Følner sequence $F_k = \{1, \ldots, k\}^n$ for example). From this it then follows easily that all abelian groups are amenable.

It is also easy to see from Lemma 7.1.3 and part (v) in Theorem 7.2.2 that subgroups of amenable groups are amenable (hence any group containing $\mathbb{F}_2$ is non-amenable). If Γ is amenable and $\Sigma \lhd \Gamma$ then it follows directly from the definition that Γ/Σ is again amenable.

From part (ix) in Theorem 7.2.2 it follows that if $1 \to \Sigma \to \Gamma \to \Lambda \to 1$ is an exact sequence of groups then Γ is amenable if both Σ and Λ are amenable. Indeed, if $\Gamma \curvearrowright K$ is a continuous action on a compact Hausdorff space, then if we consider $\tilde{K} \subset \mathrm{Prob}(K)$ the set of Σ-invariant probability measures, then $\tilde{K}$ is a non-empty compact set on which $\Lambda = \Gamma/\Sigma$ acts continuously. Thus there is a Λ-invariant probability measure $\tilde{\mu} \in \mathrm{Prob}(\tilde{K})$ and if we consider the barycenter $\mu = \int \nu \, d\tilde{\mu}(\nu)$, then μ is a Γ invariant probability measure on K.

From the above we then see that all nilpotent groups, and even all solvable groups are amenable.

7.3 Mixing properties

Let Γ be a discrete group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **weak mixing** if for each finite set $\mathcal{F} \subset \mathcal{H}$, and $\varepsilon > 0$ there exists $g \in \Gamma$ such that

$$|\langle \pi_g \xi, \xi \rangle| < \varepsilon,$$

for all $\xi \in \mathcal{F}$.

The representation π is (**strong**) **mixing** if $|\Gamma| = \infty$, and for each finite set $\mathcal{F} \subset \mathcal{H}$, we have

$$\lim_{g \to \infty} |\langle \pi_g \xi, \xi \rangle| = 0.$$

Note that mixing implies weak mixing, which in turn implies that there are no invariant vectors. It is also easy to see that if $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is mixing (resp. weak mixing) then so is $\pi^{\oplus\infty}$, and if π is mixing then so is $\pi \otimes \rho$ for any representation ρ. We'll see below in Corollary 7.3.3 that weak mixing is also stable under tensoring.

Lemma 7.3.1. *Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is weak mixing if and only if for each finite set $\mathcal{F} \subset \mathcal{H}$, and $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that*

$$|\langle \pi_g \xi, \eta \rangle| < \varepsilon,$$

for all $\xi, \eta \in \mathcal{F}$.

The representation π is mixing if $|\Gamma| = \infty$ and for each finite set $\mathcal{F} \subset \mathcal{H}$, we have

$$\lim_{g \to \infty} |\langle \pi_g \xi, \eta \rangle| = 0,$$

for all $\xi, \eta \in \mathcal{F}$.

Proof. This follows from the polarization identity

$$\langle \pi(\gamma)\xi, \eta \rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle \pi(\gamma)(\xi + i^k\eta), (\xi + i^k\eta)\rangle. \qquad \blacksquare$$

Theorem 7.3.2 (Dye)**.** *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. The following are equivalent:*

(*i*) *π is not weak mixing.*

(*ii*) *$\pi \otimes \overline{\pi}$ contains invariant vectors.*

(*iii*) *$\pi \otimes \rho$ contains invariant vectors for some unitary representation $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$.*

(*iv*) *π contains a finite dimensional sub-representation.*

Proof. To show (i) $\implies$ (ii) suppose $\pi \otimes \overline{\pi}$ does not contain invariant vectors. If $\mathcal{F} \subset \mathcal{H}$ is finite, and $\varepsilon > 0$, then setting $\zeta = \sum_{\xi\in\mathcal{F}} \xi \otimes \overline{\xi}$ it then follows from Proposition 7.1.1 that there exists $g \in \Gamma$ such that

$$\sum_{\xi,\eta\in\mathcal{F}} |\langle \pi_g\xi, \eta\rangle|^2 = \sum_{\xi,\eta\in\mathcal{F}} \langle \pi_g\xi, \eta\rangle\langle \overline{\pi}_g\overline{\xi}, \overline{\eta}\rangle = \mathrm{Re}(\langle (\pi\overline{\otimes}\overline{\pi})(g)\zeta, \zeta\rangle) < \varepsilon.$$

Thus, π is weak mixing.

(ii) $\implies$ (iii) is obvious. To show (iii) $\implies$ (iv) suppose $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ is a unitary representation such that $\pi \otimes \rho$ contains invariant vectors. Identifying $\mathcal{H}\overline{\otimes}\mathcal{K}$ with the space of Hilbert-Schmidt operators $\mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$ we then have that there exists $T \in \mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$, non-zero, such that $\pi_g T \overline{\rho}_{g^{-1}} = T$, for all $g \in \Gamma$. Then $TT^* \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ is positive, non-zero, compact, and $\pi_g TT^* \pi_{g^{-1}} = TT^*$, for all $g \in \Gamma$. By taking the range of a non-trivial spectral projection of TT^* we then obtain a finite dimensional invariant subspace of π.

For (iv) $\implies$ (i), if π is weak mixing then for $\mathcal{L} \subset \mathcal{H}$ any non-trivial finite dimensional subspace with orthonormal basis $\mathcal{F} \subset \mathcal{H}$, there exists $g \in \Gamma$ such that $|\langle \pi_g\xi, \eta\rangle| < 1/\sqrt{\dim(\mathcal{L})}$, for all $\xi, \eta \in \mathcal{F}$. Hence, if $\xi \in \mathcal{F}$ then

$$\|[\mathcal{L}](\pi_g\xi)\|^2 = \sum_{\eta\in\mathcal{F}} |\langle \pi_g\xi, \eta\rangle|^2 < 1 = \|\xi\|^2$$

showing that $\mathcal{L}$ is not an invariant subspace. $\blacksquare$

Corollary 7.3.3. *Let Γ be a group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then π is weak mixing if and only if $\pi \otimes \overline{\pi}$ is weak mixing, if and only if $\pi\overline{\otimes}\rho$ is weak mixing for all unitary representations ρ.*

Corollary 7.3.4. *Let Γ be a group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a weak mixing unitary representation. If $\Sigma < \Gamma$ is a finite index subgroup then $\pi_{|\Sigma}$ is also weak mixing.*

Proof. Let $D \subset \Gamma$ be a set of coset representatives for Σ. If $\pi_{|\Sigma}$ is not mixing, then by Theorem 7.3.2 there is a finite dimensional subspace $\mathcal{L} \subset \mathcal{H}$ which is Σ-invariant. We then have that $\sum_{\gamma\in D} \pi_g\mathcal{L} \subset \mathcal{H}$ is finite dimensional and Γ-invariant. Hence, again by Theorem 7.3.2, π is not weak mixing. $\blacksquare$

7.4 Cocycles and affine actions

If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then a 1**-cocycle** for the representation is a map $c : \Gamma \to \mathcal{H}$ such that $c(gh) = c(g) + \pi_g c(h)$ for all $g, h \in \Gamma$. A cocycle c is **inner** if it is of the form $c(g) = \xi - \pi_g \xi$ for some $\xi \in \mathcal{H}$. Note that the space of cocycles is a linear space which is closed in the topology of point-wise convergence. Also note that the inner cocycles form a linear subspace and a cocycle is constant 0 if and only if it is 0 on some generating set of Γ. Two cocycles c_1 and c_2 are **cohomologous** or are in the same cohomology class if $c_1 - c_2$ is inner.

We denote by $Z^1(\Gamma, \pi)$ the space of cocycles (for a fixed representation π), and by $B^1(\Gamma, \pi)$ the space of inner-cocycles. We also denote by $H^1(\Gamma, \pi) = Z^1(\Gamma, \pi)/B^1(\Gamma, \pi)$, the space of cohomology classes. The space of inner-cocycles need not be closed in general and so we denote by $\overline{B^1}(\Gamma, \pi)$ its closure, and $\overline{H^1}(\Gamma, \pi) = Z^1(\Gamma, \pi)/\overline{B^1}(\Gamma, \pi)$. A cocycle $c \in \overline{B^1}(\Gamma, \pi)$ is **approximately inner**.

In the sequel we will have occasion to restrict our attention to *real* Hilbert spaces and we use the same definitions as above when considering orthogonal representations.

If $\mathcal{H}$ is a real Hilbert space we let $\mathrm{Isom}(\mathcal{H})$ be the group of isometric (but not necessarily linear) bijections. Note that if $\alpha \in \mathrm{Isom}(\mathcal{H})$ such that $\alpha(0) = 0$, then α preserves the Hilbert space norm since for $\xi \in \mathcal{H}$ we have $\|\alpha(\xi)\| = \|\alpha(\xi) - \alpha(0)\| = \|\xi - 0\| = \|\xi\|$. In this case for $\xi, \eta \in \mathcal{H}$ we have

$$\begin{aligned} 2\langle \alpha(\xi), \alpha(\eta)\rangle &= \|\alpha(\xi) - \alpha(\eta)\|^2 - \|\alpha(\xi)\|^2 - \|\alpha(\eta)\|^2 \\ &= \|\xi - \eta\|^2 - \|\xi\|^2 - \|\eta\|^2 = 2\langle \xi, \eta\rangle, \end{aligned}$$

showing that α preserves the inner-product. Thus, for $\xi, \eta \in \mathcal{H}$ and $\lambda \in \mathbb{R}$ we have

$$\langle \alpha(\lambda\xi), \eta\rangle = \langle \lambda\xi, \alpha^{-1}(\eta)\rangle = \lambda\langle \xi, \alpha^{-1}(\eta)\rangle = \lambda\langle \alpha(\xi), \eta\rangle.$$

Therefore α is a linear orthogonal operator.

In general, if $\alpha \in \mathrm{Isom}(\mathcal{H})$ then from above we see that $\xi \mapsto \alpha(\xi) - \alpha(0)$ is an orthogonal operator, and hence α consists of an orthogonal operation followed by a translation. This shows that the natural inclusion $\mathcal{O}(\mathcal{H}) \ltimes \mathcal{H} \to \mathrm{Isom}(\mathcal{H})$ is surjective.

An **affine isometric representation** of a discrete group Γ on a real Hilbert space $\mathcal{H}$ is given by a homomorphism $\alpha : \Gamma \to \mathrm{Isom}(\mathcal{H})$. From above we see that every affine isometric representation gives rise to an orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$, together with a map $c : \Gamma \to \mathcal{H}$. Note that since α is a homomorphism we have that c is a cocycle. Indeed, for $g, h \in \Gamma$, we have

$$c(gh) = \alpha_{gh}(0) = \alpha_g \alpha_h(0) = \pi_g \alpha_h(0) + c(g) = \pi_g c(h) + c(g).$$

Conversely, if $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ is a representation and $c : \Gamma \to \mathcal{H}$ is a cocycle, then we see just as easily that we obtain an affine isometric representation $\alpha : \Gamma \to \mathrm{Isom}(\mathcal{H})$ by the formula $\alpha_g \xi = \pi_g \xi + c(g)$.

Lemma 7.4.1. *Let $\mathcal{H}$ be a real or complex Hilbert space. If $X \subset \mathcal{H}$ is a bounded set, then there exists a unique element $\hat{\xi} \in \mathcal{H}$ which realizes the infimum of the function $\xi \mapsto \sup_{\eta \in X} \|\xi - \eta\|$.*

Proof. Let d be the infimum of the above function and take $\{\xi_n\} \subset \mathcal{H}$ a sequence such that $\sup_{\eta \in X} \|\xi_n - \eta\| < d + 1/n$. For $\eta \in X$, the parallelogram identity gives

$$\left\| \frac{\xi_n + \xi_m}{2} - \eta \right\|^2 = \frac{1}{2}\|\xi_n - \eta\|^2 + \frac{1}{2}\|\xi_m - \eta\|^2 - \left\| \frac{\xi_n - \xi_m}{2} \right\|^2 .$$

Thus,

$$\begin{aligned} d^2 \leq \sup_{\eta \in X} \left\| \frac{\xi_n + \xi_m}{2} - \eta \right\|^2 &\leq \sup_{\eta \in X} \frac{1}{2}\|\xi_n - \eta\|^2 + \sup_{\eta \in X} \frac{1}{2}\|\xi_m - \eta\|^2 - \left\| \frac{\xi_n - \xi_m}{2} \right\|^2 \\ &\leq (d + 1/n)^2 - \left\| \frac{\xi_n - \xi_m}{2} \right\|^2 . \end{aligned}$$

Hence, it follows that $\{\xi_n\}$ is Cauchy and so must converge to a vector $\hat{\xi}$ which then realizes this infimum.

Since the sequence $\{\xi_n\}$ realizing this infimum was arbitrary it then follows that $\hat{\xi}$ is the unique such vector. ■

The vector ξ_0 in the previous lemma is called the **Chebyshev center** of X.

Proposition 7.4.2. *Let Γ be a discrete group, and $\alpha : \Gamma \to \mathrm{Isom}(\mathcal{H})$ an affine representation with linear part $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$, and affine part $c : \Gamma \to \mathcal{H}$. The following conditions are equivalent.*

(i) α has a fixed point.

(ii) c is inner.

(iii) c is bounded.

(iv) All orbits of α are bounded.

(v) Some orbit of α is bounded.

Proof. For (i) $\Longrightarrow$ (ii), if $\xi \in \mathcal{H}$, is fixed by α then for $g \in \Gamma$ we have $\xi = \alpha_g \xi = \pi_g \xi + c(g)$ showing that c is inner.

(ii) $\Longrightarrow$ (iii) is obvious. For (iii) $\Longrightarrow$ (iv), if c is bounded and $\xi \in \mathcal{H}$, then for $g \in \Gamma$ we have $\|\alpha_g \xi\| = \|\pi_g \xi + c(g)\| \leq \|\xi\| + \|c(g)\|$.

(iv) $\Longrightarrow$ (v) is also obvious so only (v) $\Longrightarrow$ (i) remains. For this, suppose that $\xi \in \mathcal{H}$ such that $X = \{\alpha_g \xi \mid g \in \Gamma\}$ is bounded. If $\hat{\xi} \in \mathcal{H}$ is the Chebyshev center of X then for $h \in \Gamma$ we have

$$\sup_{g \in \Gamma} \|\alpha_h \hat{\xi} - \alpha_g \xi\| = \sup_{g \in \Gamma} \|\hat{\xi} - \alpha_{h^{-1}g} \xi\| = \sup_{g \in \Gamma} \|\hat{\xi} - \alpha_g \xi\|$$

and so $\alpha_h \hat{\xi} = \hat{\xi}$ by uniqueness of the Chebyshev center. Thus $\hat{\xi}$ is a fixed point for α. ■

If Γ is a discrete group, $\mu \in \mathrm{Prob}(\Gamma)$ is a symmetric probability measure with finite support, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a representation, then a cocycle $c : \Gamma \to \mathcal{H}$ is **μ-harmonic** if $\sum_{g \in \Gamma} \mu(g)c(g) = 0$. Note that if c is a harmonic cocycle then for all $h \in \Gamma$ we have

$$\sum_{g \in \Gamma} \mu(g)c(gh) = \sum_{g \in \Gamma} \mu(g)(\pi(g)c(h) + c(g)) = \pi(\mu)c(h),$$

and

$$\sum_{g \in \Gamma} \mu(g)c(hg) = \sum_{g \in \Gamma} \mu(g)(\pi(h)c(g) + c(h)) = c(h).$$

Theorem 7.4.3. *Let Γ be a finitely generated group, let $\mu \in \mathrm{Prob}(\Gamma)$ be a symmetric probability measure with finite support wich generates Γ, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a representation. Then for each cocycle $c \in Z^1(\Gamma, \pi)$ there is a unique μ-harmonic cocycle $\tilde{c} \in Z^1(\Gamma, \pi)$ such that $c - \tilde{c} \in \overline{B^1}(\Gamma, \pi)$.*

Proof. We endow $Z^1(\Gamma, \pi)$ with the inner-product

$$\langle c_1, c_2 \rangle = \sum_{g \in \Gamma} \mu(g)\langle c_1(g), c_2(g)\rangle.$$

Note that $Z^1(\Gamma, \pi)$ is complete with respect to this inner-product, and that this inner-product has no kernel since the support of μ generates Γ.

If we have a cocycle $\tilde{c} : \Gamma \to \mathcal{H}$ then we can compute its inner-product with respect to an inner cocycle as:

$$\begin{aligned}
\sum_{g \in \Gamma} \mu(g)\langle \tilde{c}(g), \xi - \pi(g)\xi \rangle &= \sum_{g \in \Gamma} \mu(g)\langle \tilde{c}(g) - \pi(g^{-1})\tilde{c}(g), \xi \rangle \\
&= \sum_{g \in \Gamma} \mu(g)\langle \tilde{c}(g) + \tilde{c}(g^{-1}), \xi \rangle \\
&= 2\langle \sum_{g \in \Gamma} \tilde{c}(g), \xi \rangle.
\end{aligned}$$

It follows that the space of μ-harmonic cocycles is precisely $B^1(\Gamma, \pi)^{\perp}$. In particular, if a μ-harmonic cocycle $\tilde{c}$ is approximately inner then $\tilde{c} = 0$.

Given $c \in Z^1_{\mu}(\Gamma, \pi)$ we let $\tilde{c}$ be the orthogonal projection of c onto $B^1(\Gamma, \pi)^{\perp}$. Then $c - \tilde{c} \in \overline{B^1}(\Gamma, \pi)$, and $\tilde{c}$ is μ-harmonic. If c' is another harmonic representative in the reduced cohomology class then $\tilde{c} - c' = (\tilde{c} - c) + (c - c')$ is approximately inner and also harmonic, hence $\tilde{c} - c' = 0$. ∎

7.5 Functions of conditionally negative type

A **kernel of positive type** on a set X is a function $\varphi : X \times X \to \mathbb{C}$ such that for all $x_1, \ldots, x_n \in X$ the matrix $[\varphi(x_i, x_j)]$ is non-negative definite, i.e., for all

$\alpha_1, \dots, \alpha_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \varphi(x_i, x_j) = \langle [\varphi(x_i, x_j)][\alpha_1, \dots, \alpha_n]^T, [\alpha_1, \dots, \alpha_n]^T \rangle \geq 0.$$

The space of positive type kernels is clearly closed under convex combinations and under pointwise limits. Note that for a group Γ, a function $\varphi : \Gamma \to \mathbb{C}$ is of positive type if and only if $(g, h) \mapsto \varphi(h^{-1}g)$ is of positive type. The GNS-construction can also be applied to kernels of positive type:

Proposition 7.5.1. *Let X be a set, and $\varphi : X \times X \to \mathbb{C}$, then φ is of positive type if and only if there is a Hilbert space $\mathcal{H}$ and a function $\xi : X \to \mathcal{H}$ such that for all $x, y \in X$ we have*

$$\varphi(x, y) = \langle \xi(x), \xi(y) \rangle.$$

Proof. First note that if $\xi : X \to \mathcal{H}$, then for all $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, and $x_1, \dots, x_n \in X$ we have

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \langle \xi(x_i), \xi(y_j) \rangle = \| \sum_{i=1}^{n} \alpha_i \xi(x_i) \|^2 \geq 0,$$

and hence $(x, y) \mapsto \langle \xi(x), \xi(y) \rangle$ is of positive type.

For the converse we define an inner-product on $\mathbb{C}X$ by

$$\langle \sum_{i=1}^{n} \alpha_i \delta_{x_i}, \sum_{i=1}^{n} \beta_i \delta_{x_i} \rangle = \sum_{i,j=1}^{n} \alpha_i \overline{\beta_j} \varphi(x_i, x_j).$$

Since φ is of positive type, this inner-product is non-negative definite and hence we obtain a Hilbert space $\mathcal{H}$ by separation and completion. If we let $\xi(x)$ be the equivalence class of δ_x in $\mathcal{H}$ then we have $\langle \xi(x), \xi(y) \rangle = \varphi(x, y)$. ∎

Corollary 7.5.2. *If $\varphi_1, \varphi_2 : X \times X \to \mathbb{C}$ are of positive type then so is $\varphi_1 \varphi_2$.*

Proof. If for $i = 1, 2$ we have $\xi_i : X \to \mathcal{H}_i$ such that $\varphi_i(x, y) = \langle \xi_i(x), \xi_i(y) \rangle$, then we have $\varphi_1(x, y)\varphi_2(x, y) = \langle \xi_1(x) \otimes \xi_2(x), \xi_1(y) \otimes \xi_2(y) \rangle$. ∎

A **kernel of conditionally negative type** on a set X is a function $\psi : X \times X \to \mathbb{R}$ such that $\psi(x, x) = 0$, and $\psi(x, y) = \psi(y, x)$, for all $x, y \in X$, and such that for all $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ with $\sum_{i=1}^n \alpha_i = 0$, we have $\sum_{i,j=1}^n \alpha_i \overline{\alpha_j} \psi(x_i, x_j) \leq 0$. A **function of conditionally negative type** on a group Γ is a function $\psi : \Gamma \to \mathbb{R}$ such that $(g, h) \mapsto \psi(h^{-1}g)$ is of conditionally negative type. There is also a GNS type construction for kernels of conditionally negative type:

Proposition 7.5.3. *Let X be a set, and $\psi : X \times X \to \mathbb{R}$, then ψ is of conditionally negative type if and only if there is a Hilbert space $\mathcal{H}$ and a function $\xi : X \to \mathcal{H}$ such that $\psi(x, y) = \|\xi(x) - \xi(y)\|^2$, for all $x, y \in X$.*

Proof. First, suppose that $\xi : X \to \mathcal{H}$. Then $\|\xi(x) - \xi(x)\| = 0$ for all $x \in X$, and if $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that $\sum_{i=1} \alpha_i = 0$ then we have

$$\begin{aligned}\sum_{i,j=1}^n \alpha_i \overline{\alpha_j} \|\xi(x_i) - \xi(x_j)\|^2 &= - \sum_{i,j=1}^n \alpha_i \overline{\alpha_j} 2\mathrm{Re}(\langle \xi(x_i), \xi(x_j) \rangle) \\ &= -\| \sum_{i=1}^n \alpha_i \xi(x_i) \|^2 - \| \sum_{i=1}^n \overline{\alpha_i} \xi(x_i) \|^2 \leq 0.\end{aligned}$$

Conversely, if $\psi : X \times X \to \mathbb{R}$ is of conditionally negative type, then consider the vector space $\mathbb{C}_0 X$ consisting of all finitely supported functions $f : X \to \mathbb{C}$ such that $\sum_{x \in X} f(x) = 0$. On $\mathbb{C}_0 X$ we define the inner-product

$$\langle \sum_{i=1}^n \alpha_i \delta_{x_i}, \sum_{i=1}^n \beta_i \delta_{x_i} \rangle = - \sum_{i,j=1}^n \alpha_i \overline{\beta_j} \psi(x_i, x_j), \tag{7.1}$$

which is non-negative definite since ψ is of conditionally negative type. Hence we obtain a Hilbert space $\mathcal{H}$ by separation and completion.

Fix $x_0 \in X$ and define the function $\xi : X \to \mathcal{H}$ by letting $\xi(x)$ be the equivalence class of $\frac{1}{\sqrt{2}}(\delta_{x_0} - \delta_x)$. Then for $x, y \in \mathcal{H}$ we have

$$\|\xi(x) - \xi(y)\|^2 = \frac{1}{2} \|\delta_x - \delta_y\|^2 = \psi(x, y). \qquad \blacksquare$$

Corollary 7.5.4. *If $\varphi : X \times X \to \mathbb{C}$ is of positive type, then the function $(x, y) \mapsto \varphi(x, x) + \varphi(y, y) - 2\mathrm{Re}(\varphi(x, y))$ is of conditionally negative type.*

Proof. If $\varphi : X \times X \to \mathbb{C}$ is of positive type then by Proposition 7.5.1 there exists a Hilbert space $\mathcal{H}$, and a map $\xi : X \to \mathcal{H}$ such that $\varphi(x, y) = \langle \xi(x), \xi(y) \rangle$.

We then have that $\varphi(x, x) + \varphi(y, y) - 2\mathrm{Re}(\varphi(x, y)) = \|\xi(x) - \xi(y)\|^2$ is of conditionally negative type by Proposition 7.5.3. ■

If we have a function of conditionally negative type on a group then we have a similar characterization as in Proposition 7.5.3.

Proposition 7.5.5. *Let Γ be a group, and suppose $\psi : \Gamma \to \mathbb{R}$, then ψ is of conditionally negative type if and only if there is a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ and a cocycle $c : \Gamma \to \mathcal{H}$ such that $\psi(g) = \|c(g)\|^2$ for all $g \in \Gamma$.*

Proof. If we have such a representation and cocycle, then for $g, h \in \Gamma$ we have

$$\|c(h^{-1}g)\|^2 = \|c(h^{-1}) + \pi(h^{-1})c(g)\|^2 = \|c(g) - c(h)\|^2.$$

Hence, ψ is of conditionally negative type by Proposition 7.5.3.

Conversely, if $\psi : \Gamma \to \mathbb{R}$ is of conditionally negative type, then so is the kernel $(g, h) \mapsto \psi(h^{-1}g)$ and so we may consider on $\mathbb{C}_0\Gamma$ the inner-product described by (7.1), and we let $\mathcal{H}$ be the corresponding Hilbert space.

We define a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ by extending the left multiplication structure on the group. Note that

$$\begin{aligned}\|\pi(g)\sum_{i=1}^n \alpha_i\delta_{x_i}\|^2 &= \|\sum_{i=1}^n \alpha_i\delta_{gx_i}\|^2 \\ &= -\sum_{i,j=1}^n \alpha_i\overline{\alpha_j}\psi(x_j^{-1}g^{-1}gx_i) = \|\sum_{i=1}^n \alpha_i\delta_{x_i}\|^2,\end{aligned}$$

hence π is indeed a unitary representation.

The map $c(g) = \frac{1}{\sqrt{2}}(\delta_e - \delta_g)$ is then a cocycle for this representation and we have $\psi(g) = \|c(g)\|^2$. ■

Lemma 7.5.6. *Let $\psi : X \times X \to \mathbb{R}$ be a kernel of conditionally negative type on a set X, and fix $x_0 \in X$. Then the function $\varphi : X \times X \to \mathbb{R}$ defined by*

$$\varphi(x, y) = \psi(x_0, x) - \psi(x, y) + \psi(y, x_0)$$

is of positive type.

Proof. By Proposition 7.5.3 there exists a Hilbert space $\mathcal{H}$ and a function $\xi : X \to \mathcal{H}$ such that $\psi(x, y) = \|\xi(x) - \xi(y)\|^2$. We then have

$$\begin{aligned}\varphi(x, y) &= \|\xi(x_0) - \xi(x)\|^2 - \|\xi(x) - \xi(y)\|^2 + \|\xi(y) - \xi(x_0)\|^2 \\ &= 2\mathrm{Re}(\langle \xi(x) - \xi(x_0), \xi(y) - \xi(x_0)\rangle).\end{aligned}$$

Hence φ is of positive type by Proposition 7.5.1. ■

Theorem 7.5.7 (Schoenberg)**.** *Let X be a set, and let $\psi : X \times X \to \mathbb{R}$ be a function such that $\psi(x, x) = 0$, and $\psi(x, y) = \psi(y, x)$, for all $x, y \in X$. Then ψ is of conditionally negative type if and only if $\varphi^t(x, y) = \exp(-t\psi(x, y))$ is of positive type for all $t > 0$.*

Proof. If $\exp(-t\psi)$ is of positive type for all $t > 0$, then by Corollary 7.5.4 we have that $\frac{1}{t}(1 - \exp(-t\psi))$ is of conditionally negative type for all $t > 0$, and hence taking a limit as t approaches 0 it follows that ψ is of conditionally negative type.

Conversely, if ψ is of conditionally negative type, and if we fix $x_0 \in X$, then by Lemma 7.5.6 the function $\varphi(x, y) = \psi(x_0, x) - \psi(x, y) + \psi(y, x_0)$ is of positive type. Since the space of kernels of positive type is preserved under products and pointwise limits it then follows that $\exp(\varphi)$ is also of positive type.

Also, the kernel

$$(x, y) \mapsto \exp(-\psi(x_0, x)) \exp(-\psi(y, x_0))$$

is of positive type, (just consider $\xi(x) = \exp(-\psi(x_0, x))$ as a map into the one dimensional Hilbert space). Hence

$$\exp(-\psi(x, y)) = \exp(\varphi(x, y)) \exp(-\psi(x_0, x)) \exp(-\psi(y, x_0))$$

is of positive type. By considering $t\psi$ instead we see that $\exp(-t\psi)$ is of positive type for all $t > 0$. ■

Corollary 7.5.8. *If Γ is a group and $\psi : \Gamma \to \mathbb{R}$, such that $\psi(e) = 0$, and $\psi(g^{-1}) = \psi(g)$ for all $g \in \Gamma$. Then ψ is of conditionally negative type if and only if* $\exp(-t\psi)$ *is of positive type for all $t > 0$.*

7.6 Kazhdan's property (T)

Let Γ be a discrete group, and $\Sigma < \Gamma$ a subgroup, the pair (Γ, Σ) has **relative property (T)** if every representation of Γ which has almost invariant vectors has a non-zero Σ-invariant vector. The group Γ has **Kazhdan's property (T)** if the pair (Γ, Γ) has relative property (T). Finite groups of course have property (T) but to provide examples of infinite groups with property (T) we will first build some equivalences. We first note that infinite amenable groups cannot have property (T), i.e., a group is finite if and only if it is amenable and has property (T). Perhaps surprisingly, this simple observation has been used quite successfully, first by Margulis, to prove a number of striking rigidity results in geometric group theory.

Proposition 7.6.1. *A group Γ is finite if and only if Γ is amenable and has property (T).*

Proof. Since Γ is amenable the left-regular representation has almost invariant vectors, by property (T) this implies the left-regular representation has a non-zero invariant vector. This trivially implies that Γ is finite. ∎

Note that a property (T) group is necessarily finitely generated since if we consider the family $\mathcal{F}$ of finitely generated subgroups then it is easy to see that the representation on $\oplus_{\Lambda \in \mathcal{F}} \ell^2(\Gamma/\Lambda)$ has almost invariant vectors, and this will have an invariant vector only if $[\Gamma : \Lambda] < \infty$ for some $\Lambda \in \mathcal{F}$, in which case we have that Γ is finitely generated. Also, note that quotients of property (T) groups again have property (T) since we may view a representation of the quotient as a representation of the whole group. In particular, a property (T) group Γ has finite abelianization $\Gamma/[\Gamma, \Gamma]$.

Theorem 7.6.2. *Let Γ be a countable group, and let $\mu \in \mathrm{Prob}(\Gamma)$ be a symmetric probability measure with support generating Γ, and such that $e \in \mathrm{supp}(\mu)$. Suppose $\Sigma < \Gamma$ is a subgroup. For a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ we denote by P_Σ the projection onto the space of Σ-invariant vectors. The following conditions are equivalent:*

- (*i*) *The pair (Γ, Σ) has relative property (T).*
- (*ii*) *Every representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ with almost invariant vectors has a finite dimensional Σ-invariant subspace.*
- (*iii*) *Every function $\psi : \Gamma \to \mathbb{C}$ of conditionally negative type is bounded on Σ.*
- (*iv*) *For every representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and every $c \in Z^1(\Gamma, \pi)$ we have $c_{|\Sigma} \in B^1(\Sigma, \pi)$.*

(*v*) *Every affine isometric action of* Γ *on a real Hilbert space has a* Σ*-fixed point.*

(*vi*) *For every sequence* $\varphi_n : \Gamma \to \mathbb{C}$ *of functions of positive type which converge pointwise to* 1*, we must have that* φ_n *converges uniformly to* 1 *on* Σ.

(*vii*) *There exists a constant* $0 < c < 1$ *such that for any representation* $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ *we have* $\|\pi(\mu)(\xi - P_\Sigma \xi)\| < c\|\xi - P_\Sigma \xi\|$.

(*viii*) *There exists a constant* $K > 0$ *such that for any representation* $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$*, we have* $\|\xi - P_\Sigma \xi\| \leq K\|\nabla_\mu(\xi - P_\Sigma \xi)\|$.

Proof. (i) $\implies$ (ii) is trivial. For (ii) $\implies$ (iii) we proceed by contraposition. Suppose $\psi : \Gamma \to \mathbb{C}$ is a function of conditionally negative type, and $h_n \in \Sigma$ such that $\psi(h_n) \to \infty$. By Proposition 7.5.5 ψ is of the form $\psi(g) = \|c(g)\|^2$, where c is a cocycle for some representation π. Since

$$c(g_1 h_n g_2) = c(g_1) + \pi(g_1)c(h_n) + \pi(g_1 h_n)c(g_2)$$

it then follows that $\psi(g_1 h_n g_2) \to \infty$ for any fixed $g_1, g_2 \in \Gamma$.

By Schoenberg's theorem we have that $\varphi_n = \exp(-\psi/n)$ is of positive type for all $n \in \mathbb{N}$, and hence by the GNS-construction there exists a sequence of representations $\pi_n : \Gamma \to \mathcal{U}(\mathcal{H}_n)$, with unit cyclic vectors $\xi_n \in \mathcal{H}_n$, such that $\varphi_n(g) = \langle \pi_n(g)\xi_n, \xi_n \rangle$ for all $g \in \Gamma$, $n \in \mathbb{N}$. Since

$$\lim_{k \to \infty} \langle \pi_n(g_k)\pi(g_2)\xi_n, \pi(g_1^{-1})\xi_n \rangle = \lim_{k \to \infty} \varphi_n(g_1 h_k g_2) = 0$$

for all $n \in \mathbb{N}$, and since ξ_n is a cyclic vector for $\mathcal{H}_n$ it then follows that $\pi_n(g_k)\xi$ converges weakly to 0, for all $\xi \in \mathcal{H}_n$, and thus π_n is weakly mixing when restricted to Σ, hence so is $\oplus_{n \in \mathbb{N}} \pi_n$. Therefore by Theorem 7.3.2 we have that $\oplus_{n \in \mathbb{N}} \mathcal{H}_n$ has no finite dimensional Σ-invariant subspace.

However, for all $g \in \Gamma$ we have

$$\|\pi_n(g)\xi_n - \xi_n\|^2 = 2 - 2\mathrm{Re}(\varphi_n(g)) \to 0,$$

and thus ξ_n is a sequence of almost invariant vectors for $\oplus_{n \in \mathbb{N}} \pi_n$.

The equivalence between (iii), (iv), and (v) follow easily from Propositions 7.5.5 and 7.4.2.

To show (iv) $\implies$ (vi) suppose that there is a sequence of positive definite functions $\varphi_n : \Gamma \to \mathbb{C}$, which converge pointwise to 1, but such that no subsequence converges uniformly on Σ. By replacing φ_n with $\varphi_n(e)^{-1}\varphi_n$ we may assume that $\varphi_n(e) = 1$. Let $\pi_n : \Gamma \to \mathcal{U}(\mathcal{H}_n)$ be the corresponding GNS-representations, and $\xi_n \in \mathcal{H}_n$ unit cyclic vectors such that $\varphi_n(g) = \langle \pi_n(g)\xi_n, \xi_n \rangle$ for all $g \in \Gamma$. Since φ_n converges to 1 pointwise it follows easily that ξ_n is a seqnece of almost invariant vectors in $\oplus_{n \in \mathbb{N}} \mathcal{H}_n$.

If we enumerate Γ as $\{g_n\}_{n \in \mathbb{N}}$, then taking a subsequence we may assume that $\|\xi_n - \pi(g_k)\xi_n\| < 2^n$ for all $1 \leq k \leq n$.

We define the cocycle $c : \Gamma \to \oplus_{n\in\mathbb{N}}\mathcal{H}_n$ by

$$c(g) = \oplus_{n\in\mathbb{N}}(\xi_n - \pi_n(g)\xi_n).$$

Since $\|\xi_n - \pi(g_k)\xi_n\| < 2^n$ for all $1 \leq k \leq n$, it follows that c is well defined. Note that

$$\sup_{h\in\Sigma} |1 - \varphi_n(h)| = \sup_{h\in\Sigma} |\langle \xi_n - \pi_n(h)\xi_n, \xi_n| \leq 2\|\xi_n - P_\Sigma \xi_n\|,$$

and so since φ_n does not converge uniformly on Σ we must have $\sum_{n\in\mathbb{N}} \|\xi_n - P_\Sigma\xi_n\|^2 = \infty$.

It follows from Corollary 7.1.2 applied to the representation $\oplus_{k=1}^n \pi_k$, that there exists $h_n \in \Sigma$ such that

$$\| \oplus_{k=1}^n (\xi_k - \pi_k(h_n)\xi_k)\|^2 > \| \oplus_{k=1}^n (\xi_k - P_\Sigma\xi_k)\|^2,$$

hence we have $\|c(h_n)\| \to \infty$ and so $c_{|\Sigma} \notin B^1(\Sigma, \pi)$.

For (vi) $\implies$ (vii) suppose there exists a sequence of representations $\pi_n : \Gamma \to \mathcal{U}(\mathcal{H}_n)$, and unit vectors $\xi_n \in \mathcal{H}$ such that $P_\Sigma\xi_n = 0$, and $\|\xi_n\|^2 - \|\pi(\mu)\xi_n\|^2 \to 0$. Then as in the proof of (v) $\implies$ (i) from Proposition 7.1.4, we have that ξ_n is a sequence of almost invariant vectors for $\oplus_{n=1}^\infty \pi_n$. Thus $\varphi_n(g) = \langle \pi_n(g)\xi_n, \xi_n\rangle$ is a sequence of functions of positive type which converge pointwise to 1. If φ_n converges uniformly to 1 on Σ, then for some $n \in \mathbb{N}$ we have

$$\sup_{h\in\Sigma} \mathrm{Re}(\langle \pi_n(h)\xi_n, \xi_n\rangle) = \sup_{h\in\Sigma} \mathrm{Re}(\varphi_n(h)) \geq 1/2.$$

By Proposition 7.1.1 we then have $\|P_\Sigma\xi\| \neq 0$ giving a contradiction.

(vii) $\implies$ (viii) follows easily since for any representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and any vector $\xi \in \mathcal{H}$, we have

$$\|\xi\| - \|\pi(\mu)\xi\| \leq \|\Delta_\mu\xi\| \leq \|\nabla_\mu\xi\|.$$

Hence, if $c < 1$ such that $\|\pi(\mu)(\xi - P_\Sigma\xi)\| < \|\xi - P_\Sigma\xi\|$, then setting $K = (1-c)^{-1}$ shows (viii).

To see (viii) $\implies$ (i) suppose that $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a representation which does not have Σ-invariant vectors. Then from (viii) there does not exists a sequence of unit vectors ξ_n such that $\|\nabla_\mu\xi_n\| \to 0$. Hence it follows that $0 \notin \sigma(|\nabla_\mu|)$ and so $0 \notin \sigma(\Delta_\mu)$ since $\Delta_\mu = |\nabla_\mu|^2$. By Proposition 7.1.4 it then follows that π does not have almost invariant vectors. ∎

The previous theorem, like Theorem 7.2.2, is the combined work of many mathematicians, including Kazhdan, Delorme, Guichardet, Akemann, Walters, Bates, Robertson, and Jolissaint.

We also have further equivalences of property (T) which do not adapt as easily to the relative case. For this we first introduce another tool. An **ultrafilter** ω on a locally compact Hausdorff space X is a point in the Stone-Čech compactification βX.[1] The ultrafilter is **principal** if $\omega \in X \subset \beta X$, and non-principle

[1] This is not the usual definition, but it is equivalent and will be easier to work with for our situation.

otherwise. By the universal property of the Stone-Čech compactification any bounded continuous function $f : X \to \mathbb{C}$ has a unique continuous extension to βX. Thus, by considering the Gelfand transform, an ultrafilter ω corresponds uniquely to a norm 1 multiplicative linear functional on $C_b(X)$ which we denote by $f \mapsto \lim_{x\to\omega} f(x)$. We will usually be interested in the case when ω is a non-principal ultrafilter on $\mathbb{N}$ endowed with the discrete topology.

Constructions with non-principle ultrafilters give a convenient way to exploit compactness properties without having to restrict ourselves to subsequences. As an example, if $\mathcal{H}_n$ is a sequence of Hilbert spaces then we may consider the Banach space $\ell^\infty(\mathcal{H}_n)$ of all bounded sequences $(\xi_n)_n$ with $\xi_n \in \mathcal{H}_n$ for each $n \in \mathbb{N}$. If $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a non-principal ultrafilter, then for each pair $(\xi_n)_n, (\eta_n)_n \in \ell^\infty(\mathcal{H}_n)$ we obtain a bounded function on $\mathbb{N}$ by $n \mapsto \langle \xi_n, \eta_n \rangle$, and hence we may apply our linear functional to this function to obtain a non-negative definite inner product on $\ell^\infty(\mathcal{H}_n)$ given by $\langle (\xi_n)_n, (\eta_n)_n \rangle = \lim_{n\to\omega} \langle \xi_n, \eta_n \rangle$. The kernel of this inner-product is a closed linear subspace and hence taking a quotient we obtain a Hilbert space $\mathcal{H}_\omega$ which is the **ultraproduct** of the sequence of Hilbert spaces $\mathcal{H}_n$.

If $T_n \in \mathcal{B}(\mathcal{H}_n)$ for each $n \in \mathbb{N}$ and we have a uniform bound $\|T_n\| \leq K$, then we obtain a bounded operator $T_\omega \in \mathcal{B}(\mathcal{H}_\omega)$, with $\|T_\omega\| \leq K$, by setting $T_\omega(\xi_n)_n = (T_n\xi_n)_n$. Note that since ω is non-principle, many properties which are approximate for the sequence become exact in the limit. As an example, if $v_n, u_n, w_n \in \mathcal{U}(\mathcal{H}_n)$ for each $n \in \mathbb{N}$, such that $\lim_{n\to\omega} \|v_n u_n - w_n\| = 0$, then we have $v_\omega u_\omega = w_\omega$.

If Γ is a discrete group, and $\mu \in \mathrm{Prob}(\Gamma)$. Then for any map $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ (not necessarily a homomorphism), we again define $\nabla_\mu : \Gamma \to \mathcal{H}^{\oplus\Gamma}$ by

$$\nabla_\mu \xi = \bigoplus_{g\in\Gamma} \mu(g)^{1/2}(\xi - \pi_g \xi).$$

We also define Δ_μ, as before

$$\Delta_\mu = \nabla_\mu^* \nabla_\mu = \sum_{g\in\Gamma} \mu(g)(1 - \pi(g)^* - \pi(g) + \pi(g)^*\pi(g)).$$

Theorem 7.6.3 (Shalom). *Let Γ be a finitely generated group, and let $\mu \in \mathrm{Prob}(\Gamma)$ be a symmetric finitely supported probability measure with support S generating Γ, and such that $e \in \mathrm{supp}(\mu)$. The following conditions are equivalent:*

(i) *Γ has property (T).*

(ii) *$\overline{H^1}(\Gamma, \pi) = \{0\}$ for any representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$.*

(iii) *For all $\varepsilon > 0$ there exists $n \in \mathbb{N}$, and $0 < \kappa < \varepsilon$, such that for all $\delta \geq 0$, if $\pi : S^n \to \mathcal{U}(\mathcal{H})$ such that $\pi(g^{-1}) = \pi(g)^*$, and $\|\pi(gh) - \pi(g)\pi(h)\| \leq \delta$ whenever $g, h, gh \in S^n$, then $\sigma(\Delta_\mu) \cap (\delta/\kappa, \kappa] = \emptyset$.*

(*iv*) *There exists* $n \in \mathbb{N}$*, and* $\kappa > 0$ *such that if* $\pi : S^n \to \mathcal{U}(\mathcal{H})$ *such that* $\pi(g^{-1}) = \pi(g)^*$*, and* $\pi(gh) = \pi(g)\pi(h)$*, for all* $g, h, gh \in S^n$ *then* $\sigma(\Delta_\mu) \cap (0, \kappa] = \emptyset$*.*

(*v*) *There exists a constant* $\kappa > 0$ *such that for any representation* $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ *we have* $\sigma(\Delta_\mu) \cap (0, \kappa] = \emptyset$*.*

Proof. (i) $\implies$ (ii) follows Part (iv) of Theorem 7.6.2. Also note that (iii) $\implies$ (iv) by setting $\varepsilon = 1$ and $\delta = 0$ in Part (iii), and clearly we have (iii) $\implies$ (iv). Also, since for a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ we have that $\ker(\Delta_\mu)$ is the space of invariant vectors, it then follows from Part (viii) of Theorem 7.6.2 that (v) $\implies$ (i).

Thus, most of the effort in the proof will go into establishing (ii) $\implies$ (iii), which we will achieve by contraposition. So suppose that (iii) does not hold. Then there exists a sequence $\delta_n > 0$, and a sequence $\pi_n : S^n \to U(\mathcal{H}_n)$, such that $\pi(g^{-1}) = \pi(g)^*$, and $\|\pi(gh) - \pi(g)\pi(h)\| \leq \delta_n$ whenever $g, h, gh \in S^n$, and such that $\sigma(\Delta_\mu) \cap (n\delta_n, \frac{1}{n}] \neq \emptyset$, for large enough $n \in \mathbb{N}$.

If we let $\xi_n \in \mathcal{H}_n$ be a non-zero vector in the range of $1_{(n\delta_n, \frac{1}{n}]}(\Delta_\mu)$, such that $\|\nabla_\mu \xi_n\|^2 = \langle \Delta_\mu \xi_n, \xi_n \rangle = 1$, then we have $\|\xi_n\|^2 \leq \frac{1}{n\delta_n}$, and $\|\Delta_\mu \xi_n\|^2 \leq \frac{1}{n}$.

Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a non-principal ultrafilter, and consider the ultraproduct representation $\pi_\omega : \Gamma \to U(\mathcal{H}_\omega)$. Note that this indeed gives a unitary representation since $\delta_n \leq 1/n^2 \to 0$. Consider the map $c_n : S^n \to \mathcal{H}_n$ given by $c_n(g) = \xi_n - \pi_n(g)\xi_n$. For $g, h, gh \in S^n$ we have

$$\begin{aligned}\|c_n(g) + \pi_n(g)c_n(h) - c_n(gh)\| &= \|(\pi_n(gh) - \pi_n(g)\pi_n(h))\xi_n\| \\ &\leq \delta_n \|\xi_n\| \leq 1/n.\end{aligned}$$

Hence, for each $g \in \Gamma$ we have

$$\begin{aligned}\limsup_{n\to\infty} \|c_n(g)\| &\leq |g|_S \limsup_{n\to\infty} \sum_{s\in S} \|c_n(s)\| \\ &\leq (|S| + |g|_S) \limsup_{n\to\infty} \|\nabla \xi_n\|^2 = |S| + |g|_S.\end{aligned}$$

Thus, the sequence $(c_n(g))_n$ is bounded and so defines a vector $c_\omega(g)$ in the ultraproduct Hilbert space $\mathcal{H}_\omega$. Moreover, we see that $g \mapsto c_\omega(g)$ is a cocycle.

The cocycle c_ω is μ-harmonic since

$$\|\sum_{s\in S} c_\omega(s)\| = \lim_{n\to\omega} \|\sum_{s\in S} c_n(s)\| = \lim_{n\to\omega} \|\Delta_\mu \xi_n\| \leq \lim_{n\to\omega} \frac{1}{n} = 0,$$

and is non-zero since

$$\sum_{s\in S} \|c_\omega(s)\|^2 = \lim_{n\to\omega} \sum_{s\in S} \|c_n(s)\|^2 = \lim_{n\to\omega} |S| \|\nabla_\mu \xi_n\|^2 = |S|.$$

Thus, $c_\omega \notin \overline{Z^1}(\Gamma, \pi_\omega)$ by Theorem 7.4.3, showing that (ii) does not hold. ∎

The above proof is based on ideas of Gromov, Kleiner, Shalom, and Tao. Note that in (ii) of Theorem 7.6.3 the assumption of finite generation is necessary. Indeed, it's not hard to see that any locally finite group Γ satisfies $\overline{H^1}(\Gamma, \pi) = \{0\}$, for every representation.

Corollary 7.6.4 (Shalom)**.** *Every property (T) group is a quotient of a finitely presented property (T) group.*

Proof. Let Γ be a property (T) group generated by a finite symmetric set S, with $e \in S$, and let $\mu \in \mathrm{Prob}(\Gamma)$ be a symmetric probability measure with support S, e.g., $\mu = \frac{1}{|S|} \sum_{s \in S} \delta_s$. By Part (iv) in Theorem 7.6.3, there exists $\kappa > 0$ and $n \in \mathbb{N}$ such that if $\pi : S^n \to \mathcal{U}(\mathcal{H})$ with $\pi(g^{-1}) = \pi(g)^*$, and $\pi(gh) = \pi(g)\pi(h)$ for all $g, h, gh \in S^n$ then $\sigma(\Delta_\mu) \cap (0, \kappa] = \emptyset$.

Consider the group Λ with presentation $\langle S_0, R \rangle$, where S_0 is a copy of S, and R is the set of relations of the form $s_1 s_2 \cdots s_k = e$ if $s_i \in S_0$, this relation holds in Γ, and $k \leq 3n$. Then R is a finite set since S^n is finite and Γ is a quotient of Λ by considering the canonical map which takes elements in S_0 to their corresponding elements in S.

For any representation $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$, we may also view this as a mapping from S_0^n into $\mathcal{U}(\mathcal{H})$, and since the Γ-relations in S^n also hold in Λ we may then view it as a map from S^n into $\mathcal{U}(\mathcal{H})$ and so we must have $\sigma(\Delta_\mu) \cap (0, \kappa] = \emptyset$. Thus Λ has property (T) by Theorem 7.6.3. ■

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