Chapter 1

Spectral theory

If A is a complex unital algebra then we denote by $G(A)$ the set of elements which have a two sided inverse. If $x \in A$, the **spectrum** of x is

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \notin G(A)\}.$$

The complement of the spectrum is called the **resolvent** and denoted $\rho_A(x)$.

Proposition 1.0.1. *Let A be a unital algebra over $\mathbb{C}$, and consider $x, y \in A$. Then $\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}$.*

Proof. If $1 - xy \in G(A)$ then we have

$$(1 - yx)(1 + y(1 - xy)^{-1}x) = 1 - yx + y(1 - xy)^{-1}x - yxy(1 - xy)^{-1}x$$
$$= 1 - yx + y(1 - xy)(1 - xy)^{-1}x = 1.$$

Similarly, we have
$$(1 + y(1 - xy)^{-1}x)(1 - yx) = 1,$$

and hence $1 - yx \in G(A)$. ∎

Knowing the formula for the inverse beforehand of course made the proof of the previous proposition quite a bit easier. But this formula is quite natural to consider. Indeed, if we just consider formal power series then we have

$$(1 - yx)^{-1} = \sum_{k=0}^{\infty}(yx)^k = 1 + y(\sum_{k=0}^{\infty}(xy)^k)x = 1 + y(1 - xy)^{-1}x.$$

1.1   Banach and C^*-algebras

A **Banach algebra** is a Banach space A, which is also an algebra such that

$$\|xy\| \leq \|x\|\|y\|.$$

A Banach algebra A is **involutive** if it possesses an anti-linear involution $*$, such that $\|x^*\| = \|x\|$, for all $x \in A$.

If an involutive Banach algebra A additionally satisfies

$$\|x^*x\| = \|x\|^2,$$

for all $x \in A$, then we say that A is a C^***-algebra**. If a Banach or C^*-algebra is unital, then we further require $\|1\| = 1$.

Note that if A is a unital involutive Banach algebra, and $x \in G(A)$ then $(x^{-1})^* = (x^*)^{-1}$, and hence $\sigma_A(x^*) = \overline{\sigma_A(x)}$.

Example 1.1.1. Let K be a locally compact Hausdorff space. Then the space $C_0(K)$ of complex valued continuous functions which vanish at infinity is a C^*-algebra when given the supremum norm $\|f\|_\infty = \sup_{x \in K} |f(x)|$. This is unital if and only if K is compact.

Example 1.1.2. Let $\mathcal{H}$ be a complex Hilbert space. Then the space of all bounded operators $\mathcal{B}(\mathcal{H})$ is a C^*-algebra when endowed with the operator norm $\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|$.

Lemma 1.1.3. *Let A be a unital Banach algebra and suppose $x \in A$ such that $\|1 - x\| < 1$, then $x \in G(A)$.*

Proof. Since $\|1 - x\| < 1$, the element $y = \sum_{k=0}^\infty (1 - x)^k$ is well defined, and it is easy to see that $xy = yx = 1$. ∎

Proposition 1.1.4. *Let A be a unital Banach algebra, then $G(A)$ is open, and the map $x \mapsto x^{-1}$ is a continuous map on $G(A)$.*

Proof. If $y \in G(A)$ and $\|x - y\| < \|y^{-1}\|$ then $\|1 - xy^{-1}\| < 1$ and hence by the previous lemma $xy^{-1} \in G(A)$ (hence also $x = xy^{-1}y \in G(A)$) and

$$\|xy^{-1}\| \leq \sum_{n=0}^\infty \|(1 - xy^{-1})\|^n$$

$$\leq \sum_{n=0}^\infty \|y^{-1}\|^n \|y - x\|^n = \frac{1}{1 - \|y\|^{-1}\|y - x\|}.$$

Hence,

$$\|x^{-1} - y^{-1}\| = \|x^{-1}(y - x)y^{-1}\|$$

$$\leq \|y^{-1}(xy^{-1})^{-1}\|\|y^{-1}\|\|y - x\| \leq \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\|\|y - x\|}\|y - x\|.$$

Thus continuity follows from continuity of the map $t \mapsto \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\|t}t$, at $t = 0$. ∎

Proposition 1.1.5. *Let A be a unital Banach algebra, and suppose $x \in A$, then $\sigma_A(x)$ is a non-empty compact set.*

Proof. If $\|x\| < |\lambda|$ then $\frac{x}{\lambda} - 1 \in G(A)$ by Lemma 1.1.3, also $\sigma_A(x)$ is closed by Proposition 1.1.4, thus $\sigma_A(x)$ is compact.

To see that $\sigma_A(x)$ is non-empty note that for any linear functional $\varphi \in A^*$, we have that $f(z) = \varphi((x-z)^{-1})$ is analytic on $\rho_A(x)$. Indeed, if $z, z_0 \in \rho_A(x)$ then we have

$$(x-z)^{-1} - (x-z_0)^{-1} = (x-z)^{-1}(z-z_0)(x-z_0)^{-1}.$$

Since inversion is continuous it then follows that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \varphi((x-z_0)^{-2}).$$

We also have $\lim_{z \to \infty} f(z) = 0$, and hence if $\sigma_A(x)$ were empty then f would be a bounded entire function and we would then have $f = 0$. Since $\varphi \in A^*$ were arbitrary this would then contradict the Hahn-Banach theorem. ∎

Theorem 1.1.6 (Gelfand-Mazur). *Suppose A is a unital Banach algebra such that every non-zero element is invertible, then $A \cong \mathbb{C}$.*

Proof. Fix $x \in A$, and take $\lambda \in \sigma(x)$. Since $x - \lambda$ is not invertible we have that $x - \lambda = 0$, and the result then follows. ∎

If $f(z) = \sum_{k=0}^n a_k z^k$ is a polynomial, and $x \in A$, a unital Banach algebra, then we define $f(x) = \sum_{k=0}^n a_k x^k \in A$.

Proposition 1.1.7. *Let A be a unital Banach algebra, $x \in A$ and f a polynomial. then $\sigma_A(f(x)) = f(\sigma_A(x))$.*

Proof. If $\lambda \in \sigma_A(x)$, and $f(z) = \sum_{k=0}^n a_k z^k$ then

$$f(x) - f(\lambda) = \sum_{k=1}^n a_k(x^k - \lambda^k)$$

$$= (x - \lambda) \sum_{k=1}^n a_k \sum_{j=0}^{k-1} x^j \lambda^{k-j-1},$$

hence $f(\lambda) \in \sigma_A(x)$. conversely if $\mu \notin f(\sigma_A(x))$ and we factor $f - \mu$ as

$$f - \mu = \alpha_n(x - \lambda_1) \cdots (x - \lambda_n),$$

then since $f(\lambda) - \mu \neq 0$, for all $\lambda \in \sigma_A(x)$ it follows that $\lambda_i \notin \sigma_A(x)$, for $1 \leq i \leq n$, hence $f(x) - \mu \in G(A)$. ∎

If A is a unital Banach algebra and $x \in A$, the **spectral radius** of x is

$$r(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|.$$

Note that by Proposition 1.1.5 the spectral radius is finite, and the supremum is attained. Also note that by Proposition 1.0.1 we have the very useful equality $r(xy) = r(yx)$ for all x and y in a unital Banach algebra A. A priori the spectral radius depends on the Banach algebra in which x lives, but we will show now that this is not the case.

Proposition 1.1.8. *Let A be a unital Banach algebra, and suppose $x \in A$. Then $\lim_{n \to \infty} \|x^n\|^{1/n}$ exists and we have*

$$r(x) = \lim_{n \to \infty} \|x^n\|^{1/n}.$$

Proof. By Proposition 1.1.7 we have $r(x^n) = r(x)^n$, and hence

$$r(x)^n \leq \|x^n\|,$$

showing that $r(x) \leq \liminf_{n \to \infty} \|x^n\|^{1/n}$.

To show that $r(x) \geq \limsup_{n \to \infty} \|x^n\|^{1/n}$, consider the domain $\Omega = \{z \in \mathbb{C} \mid |z| > r(x)\}$, and fix a linear functional $\varphi \in A^*$. We showed in Proposition 1.1.5 that $z \mapsto \varphi((x - z)^{-1})$ is analytic in Ω and as such we have a Laurent expansion

$$\varphi((z - x)^{-1}) = \sum_{n=0}^{\infty} \frac{a_n}{z^n},$$

for $|z| > r(x)$. However, we also know that for $|z| > \|x\|$ we have

$$\varphi((z - x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n}.$$

By uniqueness of the Laurent expansion we then have that

$$\varphi((z - x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n},$$

for $|z| > r(x)$.

Hence for $|z| > r(x)$ we have that $\lim_{n \to \infty} \frac{\varphi(x^{n-1})}{|z|^n} = 0$, for all linear functionals $\varphi \in A^*$. By the uniform boundedness principle we then have $\lim_{n \to \infty} \frac{\|x^{n-1}\|}{|z|^n} = 0$, hence $|z| > \limsup_{n \to \infty} \|x^n\|^{1/n}$, and thus

$$r(x) \geq \limsup_{n \to \infty} \|x^n\|^{1/n}. \qquad \blacksquare$$

Exercise 1.1.9. Suppose A is a unital Banach algebra, and $I \subset A$ is a closed two sided ideal, then A/I is again a unital Banach algebra, when given the norm $\|a + I\| = \inf_{y \in I} \|a + y\|$, and $(a + I)(b + I) = (ab + I)$.

Exercise 1.1.10. Let A be a unital Banach algebra and suppose $x, y \in A$ such that $xy = yx$. Show that $r(xy) \leq r(x)r(y)$.

1.2   The Gelfand transform

Let A be a abelian Banach algebra, the **spectrum** of A, denoted by $\sigma(A)$, is the set of continuous $*$-homomorphsims $\varphi : A \to \mathbb{C}$ such that $\|\varphi\| = 1$, which we endow with the weak*-topology as a subset of A^*.

Note that if A is unital, and $\varphi : A \to \mathbb{C}$ is a $*$-homomorphism, then it follows easily that $\ker(\varphi) \cap G(A) = \emptyset$. In particular, this shows that $\varphi(x) \in \sigma(x)$, since $x - \varphi(x) \in \ker(\varphi)$. Hence, for all $x \in A$ we have $|\varphi(x)| \leq r(x) \leq \|x\|$. Since, $\varphi(1) = 1$ this shows that the condition $\|\varphi\| = 1$ is automatic in the unital case.

It is also easy to see that when A is unital $\sigma(A)$ is closed and bounded, by the Banach-Alaoglu theorem it is then a compact Hausdorff space.

Proposition 1.2.1. *Let A be a unital Banach algebra. Then the association $\varphi \mapsto \ker(\varphi)$ gives a bijection between the spectrum of A and the space of maximal ideals.*

Proof. If $\varphi \in \sigma(A)$ then $\ker(\varphi)$ is clearly an ideal, and if we have a larger ideal I, then there exists $x \in I$ such that $\varphi(x) \neq 0$, hence $1 - x/\varphi(x) \in \ker(\varphi) \subset I$ and so $1 = (1 - x/\varphi(x)) + x/\varphi(x) \in I$ which implies $I = A$.

Conversely, if $I \subset A$ is a maximal ideal, then $I \cap G(A) = \emptyset$ and hence $\|1 - y\| \geq 1$ for all $y \in I$. Thus, $\bar{I}$ is also an ideal and $1 \notin \bar{I}$ which shows that $I = \bar{I}$ by maximality. We then have that A/I is a unital Banach algebra, and since I is maximal we have that all non-zero elements of A/I are invertible. Thus, by the Gelfand-Mazur theorem we have $A/I \cong \mathbb{C}$ and hence the projection map $\pi : A \to A/I \cong \mathbb{C}$ gives a continuous homomorphism with kernel I. $\blacksquare$

Suppose A is a unital C^*-algebra which is generated (as a unital C^*-algebra) by a single element x, if $\lambda \in \sigma_A(x)$ then we can consider the closed ideal generated by $x - \lambda$ which is maximal since x generates A. This therefore induces a map from $\sigma_A(x)$ to $\sigma(A)$. We leave it to the reader to check that this map is actually a homeomorphism.

Let A be a unital abelian Banach algebra, the **Gelfand transform** is the map $\Gamma : A \to C(\sigma(A))$ defined by

$$\Gamma(x)(\varphi) = \varphi(x).$$

Theorem 1.2.2. *Let A be a unital abelian Banach algebra, then the Gelfand transform is a contractive homomorphism, and $\Gamma(x)$ is invertible in $C(\sigma(A))$ if and only if x is invertible in A.*

Proof. It is easy to see that the Gelfand transform is a contractive homomorphism. Also, if $x \in G(A)$, then $\Gamma(a)\Gamma(a^{-1}) = \Gamma(aa^{-1}) = \Gamma(1) = 1$, hence $\Gamma(x)$ is invertible. Conversely, if $x \notin G(A)$ then since A is abelian we have that the ideal generated by x is non-trivial, hence by Zorn's lemma we see that x is contained in a maximal ideal $I \subset A$, and from Proposition 1.2.1 there exists $\varphi \in \sigma(A)$ such that $\Gamma(x)(\varphi) = \varphi(x) = 0$. Hence, in this case $\Gamma(x)$ is not invertible. $\blacksquare$

Corollary 1.2.3. *Let A be a unital abelian Banach algebra, then $\sigma(\Gamma(x)) = \sigma(x)$, and in particular $\|\Gamma(x)\| = r(\Gamma(x)) = r(x)$, for all $x \in A$.*

1.3 Continuous functional calculus

Let A be a C^*-algebra. An element $x \in A$ is:

- **normal** if $xx^* = x^*x$.

- **self-adjoint** if $x = x^*$, and **skew-adjoint** if $x = -x^*$.

- **positive** if $x = y^*y$ for some $y \in A$.

- a **projection** if $x^* = x^2 = x$.

- **unitary** if A is unital, and $x^*x = xx^* = 1$.

- **isometric** if A is unital, and $x^*x = 1$.

- **partially isometric** if x^*x is a projection.

We denote by A_+ the set of positive elements, and $a, b \in A$ are two self-adjoint elements then we write $a \leq b$ if $b - a \in A_+$. Note that if $x \in A$ then $x^*A_+x \subset A_+$, in particular, if $a, b \in A$ are self-adjoint such that $a \leq b$, then $x^*ax \leq x^*bx$.

Proposition 1.3.1. *Let A be a C^*-algebra and $x \in A$ normal, then $\|x\| = r(x)$.*

Proof. We first show this if x is self-adjoint, in which case we have $\|x^2\| = \|x\|^2$, and by induction we have $\|x^{2^n}\| = \|x\|^{2^n}$ for all $n \in \mathbb{N}$. Therefore, $\|x\| = \lim_{n \to \infty} \|x^{2^n}\|^{2^n} = r(x)$.

If x is normal then by Exercise 1.1.10 we have

$$\|x\|^2 = \|x^*x\| = r(x^*x) \leq r(x^*)r(x) = r(x)^2 \leq \|x\|^2. \qquad \blacksquare$$

Corollary 1.3.2. *Let A and B be two unital C^*-algebras and $\Phi : A \to B$ a unital $*$-homomorphism, then Φ is contractive. If Φ is a $*$-isomorphism, then Φ is isometric.*

Proof. Since Φ is a unital $*$-homomorphism we clearly have $\Phi(G(A)) \subset G(B)$, from which it follows that $\sigma_B(\Phi(x)) \subset \sigma_A(x)$, and hence $r(\Phi(x)) \leq r(x)$, for all $x \in A$. By Proposition 1.3.1 we then have

$$\|\Phi(x)\|^2 = \|\Phi(x^*x)\| = r(\Phi(x^*x)) \leq r(x^*x) = \|x^*x\| = \|x\|^2.$$

If Φ is a $*$-isomorphism then so is Φ^{-1} which then shows that Φ is isometric.
$\blacksquare$

Corollary 1.3.3. *Let A be a unital complex involutive algebra, then there is at most one norm on A which makes A into a C^*-algebra.*

Proof. If there were two norms which gave a C^*-algebra structure to A then by the previous corollary the identity map would be an isometry. $\blacksquare$

Lemma 1.3.4. *Let A be a unital C^*-algebra, if $x \in A$ is self-adjoint then $\sigma_A(x) \subset \mathbb{R}$.*

Proof. Suppose $\lambda = \alpha + i\beta \in \sigma_A(x)$ where $\alpha, \beta \in \mathbb{R}$. If we consider $y = x - \alpha + it$ where $t \in \mathbb{R}$, then we have $i(\beta + t) \in \sigma_A(y)$ and y is normal. Hence,

$$(\beta + t)^2 \le r(y)^2 = \|y\|^2 = \|y^*y\|$$
$$= \|(x - \alpha)^2 + t^2\| \le \|x - \alpha\|^2 + t^2,$$

and since $t \in \mathbb{R}$ was arbitrary it then follows that $\beta = 0$. ∎

Lemma 1.3.5. *Let A be a unital Banach algebra and suppose $x \notin G(A)$. If $x_n \in G(A)$ such that $\|x_n - x\| \to 0$, then $\|x_n^{-1}\| \to \infty$.*

Proof. If $\|x_n^{-1}\|$ were bounded then we would have that $\|1 - x x_n^{-1}\| < 1$ for some n. Thus, we would have that $x x_n^{-1} \in G(A)$ and hence also $x \in G(A)$. ∎

Proposition 1.3.6. *Let B be a unital C^*-algebra and $A \subset B$ a unital C^*-subalgebra. If $x \in A$ then $\sigma_A(x) = \sigma_B(x)$.*

Proof. Note that we always have $G(A) \subset G(B)$. If $x \in A$ is self-adjoint such that $x \notin G(A)$, then by Lemma 1.3.4 we have $it \in \rho_A(x)$ for $t > 0$. By the previous lemma we then have

$$\lim_{t \to 0} \|(x - it)^{-1}\| = \infty,$$

and thus $x \notin G(B)$ since inversion is continuous in $G(B)$.

For general $x \in A$ we then have

$$x \in G(A) \Leftrightarrow x^*x \in G(A) \Leftrightarrow x^*x \in G(B) \Leftrightarrow x \in G(B).$$

In particular, we have $\sigma_A(x) = \sigma_B(x)$ for all $x \in A$. ∎

Because of the previous result we will henceforth write simply $\sigma(x)$ for the spectrum of an element in a C^*-algebra.

Theorem 1.3.7. *Let A be a unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C(\sigma(A))$ gives an isometric isomorphism between A and $C(\sigma(A))$.*

Proof. If x is self-adjoint then from Lemma 1.3.4 we have $\sigma(\Gamma(x)) = \sigma(x) \subset \mathbb{R}$, and hence $\overline{\Gamma(x)} = \Gamma(x^*)$. In general, if $x \in A$ we can write x as $x = a + ib$ where $a = \frac{x + x^*}{2}$ and $b = \frac{i(x^* - x)}{2}$ are self-adjoint. Hence, $\Gamma(x^*) = \Gamma(a - ib) = \Gamma(a) - i\Gamma(b) = \overline{\Gamma(a) + i\Gamma(b)} = \overline{\Gamma(x)}$ and so Γ is a $*$-homomorphism.

By Proposition 1.3.1, if $x \in A$ we then have

$$\|x\|^2 = \|x^*x\| = r(x^*x)$$
$$= r(\Gamma(x^*x)) = \|\Gamma(x^*)\Gamma(x)\| = \|\Gamma(x)\|^2,$$

and so Γ is isometric, and in particular injective.

To show that Γ is surjective note that $\Gamma(A)$ is self-adjoint, and closed since Γ is isometric. Moreover, $\Gamma(A)$ contains the constants and clearly separates points, hence $\Gamma(A) = C(\sigma(A))$ by the Stone-Weierstrauss theorem. ∎

Since we have seen above that if A is generated as a unital C^*-algebra by a single normal element $x \in A$, then we have a natural homeomorphism $\sigma(x) \cong \sigma(A)$. Thus by considering the inverse Gelfand transform we obtain an isomorphism between $C(\sigma(x))$ and A which we denote by $f \mapsto f(x)$.

Theorem 1.3.8 (Continuous functional calculus)**.** *Let A and B be a unital C^*-algebras, with $x \in A$ normal, then this functional calculus satisfies the following properties:*

(i) *The map $f \mapsto f(x)$ is a homomorphism from $C(\sigma(x))$ to A, and if $f(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial, then $f(x) = \sum_{k=0}^{n} a_k x^k$.*

(ii) *For $f \in C(\sigma(x))$ we have $\sigma(f(x)) = f(\sigma(x))$.*

(iii) *If $\Phi : A \to B$ is a C^*-homomorphism then $\Phi(f(x)) = f(\Phi(x))$.*

(iv) *If $f \in C(\sigma(x))$ and $g \in C(f(\sigma(x))$, we have $g \circ f(x) = g(f(x))$.*

(v) *If $x_n \in A$ is a sequence of normal elements such that $\|x_n - x\| \to 0$, Ω is a compact neighborhood of $\sigma(x)$, and $f \in C(\Omega)$, then for large enough n we have $\sigma(x_n) \subset \Omega$ and $\|f(x_n) - f(x)\| \to 0$.*

Proof. Parts (i), and (ii) follow easily from Theorem 1.3.7. Part (iii) is obvious for polynomials and then follows for all continuous functions by approximation. Part (iv) likewise follows by approximating g by polynomials.

For part (v), the fact that $\sigma(x_n) \subset \Omega$ for large n follows from continuity of inversion. If we write $C = \sup_n \|x_n\|$ and we have $\varepsilon > 0$ arbitrary, then we may take a polynomial $g : \Omega \to \mathbb{C}$ such that $\|f - g\|_\infty < \varepsilon$ and we have

$$\limsup_{n \to \infty} \|f(x_n) - f(x)\| \leq 2\|f - g\|_\infty C + \limsup_{n \to \infty} \|g(x_n) + g(x)\| < 2C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have $\lim_{n \to \infty} \|f(x_n) - f(x)\| = 0$. ∎

1.3.1   The non-unital case

If A is not a unital C^*-algebra then we may consider the space $\tilde{A} = A \oplus \mathbb{C}$ which is a $*$-algebra with multiplication

$$(x \oplus \alpha) \cdot (y \oplus \beta) = (xy + \alpha y + \beta x) \oplus \alpha\beta,$$

and involution $(x \oplus \alpha)^* = x^* \oplus \overline{\alpha}$. We may also place a norm on $\tilde{A}$ given by

$$\|x \oplus \alpha\| = \sup_{y \in A, \|y\| \leq 1} \|xy + \alpha y\|.$$

We call $\tilde{A}$ the **unitization** of A.

Proposition 1.3.9. *Let A be a non-unital C^*-algebra, then the unitization $\tilde{A}$ is again a C^*-algebra, and the map $x \mapsto x \oplus 0$ is an isometric $*$-isomorphism of A onto a maximal ideal in $\tilde{A}$.*

Proof. The map $x \mapsto x \oplus 0$ is indeed isometric since on one hand we have $\|x \oplus 0\| = \sup_{y \in A, \|y\| \leq 1} \|xy\| \leq \|x\|$, while on the other hand if $x \neq 0$, and we set $y = x^*/\|x^*\|$ then we have $\|x\| = \|xx^*\|/\|x^*\| = \|xy\| \leq \|x \oplus 0\|$.

The norm on $\tilde{A}$ is nothing but the operator norm when we view $\tilde{A}$ as acting on A by left multiplication and hence we have that this is at least a semi-norm such that $\|xy\| \leq \|x\|\|y\|$, for all $x, y \in \tilde{A}$. To see that this is actually a norm note that if $\alpha \neq 0$, but $\|x \oplus \alpha\| = 0$ then for all $y \in A$ we have $\|xy + \alpha y\| \leq \|x \oplus \alpha\|\|y\| = 0$, and hence $e = -x/\alpha$ is a left identity for A. Taking adjoints we see that e^* is a right identity for A, and then $e = ee^* = e^*$ is an identity for A which contradicts that A is non-unital. Thus, $\|\cdot\|$ is indeed a norm.

It is easy to see then that $\tilde{A}$ is then complete, and hence all that remains to be seen is the C^*-identity. Since, each for each $y \in A$, $\|y\| \leq 1$ we have $(y \oplus 0)^*(x \oplus \alpha) \in A \oplus 0 \cong A$ it follows that the C^*-identity holds here, and so

$$\|(x \oplus \alpha)^*(x \oplus \alpha)\| \geq \|(y \oplus 0)^*(x \oplus \alpha)^*(x \oplus \alpha)(y \oplus 0)\|$$
$$= \|(x \oplus \alpha)(y \oplus 0)\|^2.$$

Taking the supremum over all $y \in A$, $\|y\| \leq 1$ we then have

$$\|(x \oplus \alpha)^*(x \oplus \alpha)\| \geq \|x \oplus \alpha\|^2 \geq \|(x \oplus \alpha)^*(x \oplus \alpha)\|. \qquad \blacksquare$$

Lemma 1.3.10. *If A is a non-unital abelian C^*-algebra, then any norm 1 multiplicative linear functional $\varphi \in \sigma(A)$ has a unique extension $\tilde{\varphi} \in \tilde{A}$.*

Proof. If we consider $\tilde{\varphi}(x \oplus \alpha) = \varphi(x) + \alpha$ then the result follows easily. $\blacksquare$

In particular, this shows that $\sigma(A)$ is homeomorphic to $\sigma(\tilde{A}) \setminus \{\varphi_0\}$ where φ_0 is defined by $\varphi(x, \alpha) = \alpha$. Thus, $\sigma(A)$ is locally compact.

If $x \in A$ then the **spectrum** $\sigma(x)$ of x is defined to be the spectrum of $x \oplus 0 \in \tilde{A}$. Note that for a non-unital C^*-algebra A, since $A \subset \tilde{A}$ is an ideal it follows that $0 \in \sigma(x)$ whenever $x \in A$.

By considering the embedding $A \subset \tilde{A}$ we are able to extend the spectral theorem and continuous functional calculus to the non-unital setting. We leave the details to the reader.

Theorem 1.3.11. *Let A be a non-unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C_0(\sigma(A))$ gives an isometric isomorphism between A and $C_0(\sigma(A))$.*

Theorem 1.3.12. *Let A be a C^*-algebra, and $x \in A$ a normal element, then if $f \in C(\sigma(x))$ such that $f(0) = 0$, then $f(x) \in A \subset \tilde{A}$.*

Exercise 1.3.13. Suppose K is a non-compact, locally compact Hausdorff space, and $K \cup \{\infty\}$ is the one point compactification. Show that we have a natural isomorphism $C(K \cup \{\infty\}) \cong \widetilde{C_0(K)}$.

1.4   Applications of functional calculus

Given any element x in a C^*-algebra A, we can decompose x uniquely as a sum of a self-adoint and skew-adjoint elements $\frac{x+x^*}{2}$ and $\frac{x-x^*}{2}$. We refer to the self-adjoint elements $\frac{x+x^*}{2}$ and $i\frac{x^*-x}{2}$ the **real** and **imaginary** parts of x.

Also, if $x \in A$ is self-adjoint then from above we know that $\sigma(x) \subset \mathbb{R}$, hence by considering $x_+ = (0 \vee t)(x)$ and $x_- = -(0 \wedge t)(x)$ it follows easily from functional calculus that $\sigma(x_+), \sigma(x_-) \subset [0, \infty)$, $x_+ x_- = x_- x_+ = 0$, and $x = x_+ - x_-$. We call x_+ and x_- the **positive** and **negative** parts of x.

1.4.1   The positive cone

Lemma 1.4.1. *Suppose we have self-adjoint elements $x, y \in A$ such that $\sigma(x), \sigma(y) \subset [0, \infty)$ then $\sigma(x + y) \subset [0, \infty)$.*

Proof. Let $a = \|x\|$, and $b = \|y\|$. Since x is self-adjoint and $\sigma(x) \subset [0, \infty)$ we may use the spectral radius formula to see that $\|a - x\| = r(a - x) = a$. Similarly we have $\|b - y\| = b$ and since $\|x + y\| \leq a + b$ we have

$$\sup_{\lambda \in \sigma(x+y)} \{a + b - \lambda\} = r((a + b) - x) = \|(a + b) - (x + y)\|$$

$$\leq \|x - a\| + \|y - b\| = a + b.$$

Therefore, $\sigma(x + y) \subset [0, \infty)$.   ∎

Proposition 1.4.2. *Let A be a C^*-algebra. A normal element $x \in A$ is*

  (*i*) *self-adjoint if and only if $\sigma(x) \subset \mathbb{R}$.*

  (*ii*) *positive if and only if $\sigma(x) \subset [0, \infty)$.*

  (*iii*) *unitary if and only if $\sigma(x) \subset \mathbb{T}$.*

  (*iv*) *a projection if and only if $\sigma(x) \subset \{0, 1\}$.*

Proof. Parts (i), (iii), and (iv) all follow easily by applying continuous functional calculus. For part (ii) if x is normal and $\sigma(x) \subset [0, \infty)$ then $x = (\sqrt{x})^2 = (\sqrt{x})^* \sqrt{x}$ is positive. It also follows easily that if $x = y^* y$ where y is normal then $\sigma(x) \subset [0, \infty)$. Thus, the difficulty arises only when $x = y^* y$ where y is perhaps not normal.

Suppose $x = y^* y$ for some $y \in A$, to show that $\sigma(x) \subset [0, \infty)$, decompose x into its positive and negative parts $x = x_+ - x_-$ as described above. Set $z = yx_-$ and note that $z^* z = x_-(y^* y)x_- = -x_-^3$, and hence $\sigma(zz^*) \subset \sigma(z^* z) \subset (-\infty, 0]$.

If $z = a + ib$ where a and b are self-adjoint, then we have $zz^* + z^* z = 2a^2 + 2b^2$, hence we also have $\sigma(zz^* + z^* z) \subset [0, \infty)$ and so by Lemma 1.4.1 we have $\sigma(z^* z) = \sigma((2a^2 + 2b^2) - zz^*) \subset [0, \infty)$. Therefore $\sigma(-x_-^3) = \sigma(z^* z) \subset \{0\}$ and since x_- is normal this shows that $x_-^3 = 0$, and consequently $x_- = 0$.   ∎

Corollary 1.4.3. *Let A be a C^*-algebra. An element $x \in A$ is a partial isometry if and only if x^* is a partial isometry.*

Proof. Since x^*x is normal, it follows from the previous proposition that x is a partial isometry if and only if $\sigma(x^*x) \subset \{0, 1\}$. Since $\sigma(x^*x) \cup \{0\} = \sigma(xx^*) \cup \{0\}$ this gives the result. ∎

Corollary 1.4.4. *Let A be a C^*-algebra, then the set of positive elements forms a closed cone. Moreover, if $a \in A$ is self-adjoint, and A is unital, then we have $a \leq \|a\|$.*

Note that if $x \in A$ is an arbitrary element of a C^*-algebra A, then from above we have that x^*x is positive and hence we may define the **absolute value** of x as the unique element $|x| \in A$ such that $|x|^2 = x^*x$.

Proposition 1.4.5. *Let A be a unital C^*-algebra, then every element is a linear combination of four unitaries.*

Proof. If $x \in A$ is self-adjoint and $\|x\| \leq 1$, then $u = x + i(1 - x^2)^{1/2}$ is a unitary and we have $x = u + u^*$. In general, we can decompose x into its real and imaginary parts and then write each as a linear combination of two unitaries. ∎

Proposition 1.4.6. *Let A be a C^*-algebra, and suppose $x, y \in A_+$ such that $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$. Moreover, if A is unital and $x, y \in A$ are invertible, then $y^{-1} \leq x^{-1}$.*

Proof. First consider the case that A is unital and x and y are invertible, then we have
$$y^{-1/2}xy^{-1/2} \leq 1,$$
hence
$$x^{1/2}y^{-1}x^{1/2} \leq \|x^{1/2}y^{-1}x^{1/2}\| = r(x^{1/2}y^{-1}x^{1/2})$$
$$= r(y^{-1/2}xy^{-1/2}) \leq 1.$$

Conjugating by $x^{-1/2}$ gives $y^{-1} \leq x^{-1}$.

We also have
$$\|y^{-1/2}x^{1/2}\|^2 = \|y^{-1/2}xy^{-1/2}\| \leq 1,$$
therefore
$$y^{-1/4}x^{1/2}y^{-1/4} \leq \|y^{-1/4}x^{1/2}y^{-1/4}\| = r(y^{-1/4}x^{1/2}y^{-1/4})$$
$$= r(y^{-1/2}x^{1/2}) \leq \|y^{-1/2}x^{1/2}\| \leq 1.$$

Conjugating by $y^{1/4}$ gives $x^{1/2} \leq y^{1/2}$.

In the general case we may consider the unitization of A, and note that if $\varepsilon > 0$, then we have $0 \leq x + \varepsilon \leq y + \varepsilon$, where $x + \varepsilon$, and $y + \varepsilon$ are invertible, hence from above we have
$$(x + \varepsilon)^{1/2} \leq (y + \varepsilon)^{1/2}.$$

Taking the limit as $\varepsilon \to 0$ we obtain the result. ∎

Corollary 1.4.7. *Let A be a C^*-algebra, then for $x, y \in A$ we have $|xy| \leq \|x\|\|y\|$.*

Proof. Since $|xy|^2 = y^*x^*xy \leq \|x\|^2 y^*y$, this follows from the previous proposition. ∎

1.4.2 Extreme points

Given a involutive normed algebra A, we denote by $(A)_1$ the unit ball of A, and $A_{\mathrm{s.a.}}$ the subspace of self-adjoint elements.

Proposition 1.4.8. *Let A be a C^*-algebra.*

(i) *The extreme points of $(A_+)_1$ are the projections of A.*

(ii) *The extreme points of $(A_{\mathrm{s.a.}})_1$ are the self-adjoint unitaries in A.*

(iii) *Every extreme point of $(A)_1$ is a partial isometry in A.*

Proof. (i) If $x \in (A_+)_1$, then we have $x^2 \leq 2x$, and $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$. Hence if x is an extreme point then we have $x = x^2$ and so x is a projection. For the converse we first consider the case when A is abelian, and so we may assume $A = C_0(K)$ for some locally compact Hausdorff space K. If x is a projection then $x = 1_E$ is the characteristic function on some open and closed set $E \subset K$, hence the result follows easily from the fact that 0 and 1 are extreme points of $[0, 1]$.

For the general case, suppose $p \in A$ is a projection, if $p = \frac{1}{2}(a + b)$ then $\frac{1}{2}a = p - b \leq p$, and hence $0 \leq (1 - p)a(1 - p) \leq 0$, thus $a = ap = pa$. We therefore have that a, b, and p commute and hence the result follows from the abelian case.

(ii) First note that if A is unital then 1 is an extreme point in the unit ball. Indeed, if $1 = \frac{1}{2}(a + b)$ where $a, b \in (A)_1$, then we have the same equation when replacing a and b by their real parts. Thus, assuming a and b are self-adjoint we have $\frac{1}{2}a = 1 - \frac{1}{2}b$ and hence a and b commute. By considering the unital C^*-subalgebra generated by a and b we may assume $A = C(K)$ for some compact Hausdorff space K, and then it is an easy exercise to conclude that $a = b = 1$.

If u is a unitary in A, then the map $x \mapsto ux$ is a linear isometry of A, thus since 1 is an extreme point of $(A)_1$ it follows that u is also an extreme point. In particular, if u is self-adjoint then it is an extreme point of $(A_{\mathrm{s.a.}})_1$.

Conversely, if $x \in (A_{\mathrm{s.a.}})_1$ is an extreme point then if $x_+ = \frac{1}{2}(a + b)$ for $a, b \in (A_+)_1$, then $0 = x_- x_+ x_- = \frac{1}{2}(x_- a x_- + x_- b x_-) \geq 0$, hence we have $(a^{1/2}x_-)^*(a^{1/2}x_-) = x_- a x_- = 0$. We conclude that $ax_- = x_- a = 0$, and similarly $bx_- = x_- b = 0$. Thus, $a - x_-$ and $b - x_-$ are in $(A_{\mathrm{s.a.}})_1$ and $x = \frac{1}{2}((a - x_-) + (b - x_-))$. Since x is an extreme point we conclude that $x = a - x_- = b - x_-$ and hence $a = b = x_+$.

We have shown now that x_+ is an extreme point in $(A_+)_1$ and thus by part (i) we conclude that x_+ is a projection. The same argument shows that x_- is also a projection, and thus x is a self-adjoint unitary.

(iii) If $x \in (A)_1$ such that x^*x is not a projection then by applying functional calculus to x^*x we can find an element $y \in A_+$ such that $x^*xy = yx^*x \neq 0$, and $\|x(1 \pm y)\|^2 = \|x^*x(1 \pm y)^2\| \leq 1$. Since $xy \neq 0$ we conclude that $x = \frac{1}{2}((x + xy) + (x - xy))$ is not an extreme point of $(A)_1$. ∎

1.4.3 Ideals and quotients

Theorem 1.4.9. *Let A be a C^*-algebra, and let $I \subset A$ be a left ideal, then there exists an increasing net $\{a_\lambda\}_\lambda \subset I$ of positive elements such that for all $x \in I$ we have*
$$\|xa_\lambda - x\| \to 0.$$
Moreover, if A is separable then the net can be taken to be a sequence.

Proof. Consider Λ to be the set of all finite subsets of $I \subset A \subset \tilde{A}$, ordered by inclusion. If $\lambda \in \Lambda$ we consider
$$h_\lambda = \sum_{x \in \lambda} x^*x, \qquad a_\lambda = |\lambda| h_\lambda (1 + |\lambda| h_\lambda)^{-1}.$$

Then we have $a_\lambda \in I$ and $0 \leq a_\lambda \leq 1$. If $\lambda \leq \lambda'$ then we clearly have $h_\lambda \leq h_{\lambda'}$ and hence by Proposition 1.4.6 we have that
$$\frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_\lambda\right)^{-1}.$$

Therefore
$$a_\lambda = 1 - \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_\lambda\right)^{-1} \leq 1 - \frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} = a_{\lambda'}.$$

If $y \in \lambda$ then we have
$$(y(1 - a_\lambda))^*(y(1 - a_\lambda)) \leq \sum_{x \in \lambda}(x(1 - a_\lambda))^*(x(1 - a_\lambda)) = (1 - a_\lambda)h_\lambda(1 - a_\lambda).$$

But $\|(1 - a_\lambda)h_\lambda(1 - a_\lambda)\| = \|h_\lambda(1 + |\lambda|h_\lambda)^{-2}\| \leq \frac{1}{4|\lambda|}$, from which it follows easily that $\|y - ya_\lambda\| \to 0$, for all $y \in I$.

If A is separable then so is $\bar{I}$, hence there exists a countable subset $\{x_n\}_{n \in \mathbb{N}} \subset I$ which is dense in I. If we take $\lambda_n = \{x_1, \ldots, x_n\}$, then clearly $a_n = a_{\lambda_n}$ also satisfies
$$\|y - ya_n\| \to 0. \qquad \blacksquare$$

We call such a net $\{a_\lambda\}$ a **right approximate identity** for I. If I is self-adjoint then we also have $\|a_\lambda x - x\| = \|x^*a_\lambda - x^*\| \to 0$ and in this case we call $\{a_\lambda\}$ an **approximate identity**. Using the fact that the adjoint is an isometry we also obtain the following corollary.

Corollary 1.4.10. *Let A be a C^*-algebra, and $I \subset A$ a closed two sided ideal. Then I is self-adjoint. In particular, I is a C^*-algebra.*

Exercise 1.4.11. Show that if A is a C^*-algebra such that $x \leq y \implies x^2 \leq y^2$, for all $x, y \in A_+$, then A is abelian.

Exercise 1.4.12. Let A be a C^*-algebra and $I \subset A$ a non-trivial closed two sided ideal. Show that A/I is again a C^*-algebra.

Chapter 2

Operators on Hilbert space

Recall that if $\mathcal{H}$ is a Hilbert space then $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators is a C^*-algebra with norm

$$\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|,$$

and involution given by the adjoint, i.e., x^* is the unique bounded linear operator such that

$$\langle \xi, x^*\eta \rangle = \langle x\xi, \eta \rangle,$$

for all $\xi, \eta \in \mathcal{H}$.

Lemma 2.0.13. *Let $\mathcal{H}$ be a Hilbert space and consider $x \in \mathcal{B}(\mathcal{H})$, then $\ker(x) = R(x^*)^\perp$.*

Proof. If $\xi \in \ker(x)$, and $\eta \in \mathcal{H}$, then $\langle \xi, x^*\eta \rangle = \langle x\xi, \eta \rangle = 0$, hence $\ker(x) \subset R(x^*)^\perp$. If $\xi \in R(x^*)^\perp$ then for any $\eta \in \mathcal{H}$ we have $\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle = 0$, hence $\xi \in \ker(x)$. $\blacksquare$

Lemma 2.0.14. *Let $\mathcal{H}$ be a Hilbert space, then an operator $x \in \mathcal{B}(\mathcal{H})$ is*

  (i) *normal if and only if $\|x\xi\| = \|x^*\xi\|$, for all $\xi \in \mathcal{H}$.*

 (ii) *self-adjoint if and only if $\langle x\xi, \xi \rangle \in \mathbb{R}$, for all $\xi \in \mathcal{H}$.*

(iii) *positive if and only if $\langle x\xi, \xi \rangle \geq 0$, for all $\xi \in \mathcal{H}$.*

 (iv) *an isometry if and only if $\|x\xi\| = \|\xi\|$, for all $\xi \in \mathcal{H}$.*

  (v) *a projection if and only if x is the orthogonal projection onto some closed subspace of $\mathcal{H}$.*

 (vi) *a partial isometry if and only if there is a closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $x_{|\mathcal{K}}$ is an isometry while $x_{|\mathcal{K}^\perp} = 0$.*

Proof.

(i) If x is normal than for all $\xi \in \mathcal{H}$ we have $\|x\xi\|^2 = \langle x^*x\xi, \xi \rangle = \langle xx^*\xi, \xi \rangle = \|x^*\xi\|^2$. Conversely, is $\langle (x^*x - xx^*)\xi, \xi \rangle = 0$, for all $\xi \in \mathcal{H}$, then for all $\xi, \eta \in \mathcal{H}$, by polarization we have

$$\langle (x^*x - xx^*)\xi, \eta \rangle = \sum_{k=0}^{3} i^k \langle (x^*x - xx^*)(\xi + i^k\eta), (\xi + i^k\eta) \rangle = 0.$$

Hence $x^*x = xx^*$.

(ii) If $x = x^*$ then $\overline{\langle x\xi, \xi \rangle} = \langle \xi, x\xi \rangle = \langle x\xi, \xi \rangle$. The converse follows again by a polarization argument.

(iii) If $x = y^*y$, then $\langle x\xi, \xi \rangle = \|y\xi\|^2 \geq 0$. Conversely, if $\langle x\xi, \xi \rangle \geq 0$, for all $\xi \in \mathcal{H}$ then we know from part (b) that x is self-adjoint, and for all $a > 0$ we have $\langle (x + a)\xi, \xi \rangle \geq a\|\xi\|^2$. This shows that $x + a$ is an injective operator with dense image (since the orthogonal complement of the range is trivial). Moreover, by the Cauchy-Schwartz inequality we have

$$a\|\xi\|^2 \leq \langle (x + a)\xi, \xi \rangle \leq \|(x + a)\xi\|\|\xi\|,$$

and hence $a\|\xi\| \leq \|(x + a)\xi\|$, for all $\xi \in \mathcal{H}$. In particular this shows that the image of $x + a$ is closed since if $\{(x + a)\xi_n\}$ is Cauchy then $\{\xi_n\}$ is also Cauchy. Therefore $(x + a)$ is invertible and $a\|(x + a)^{-1}\xi\| \leq \|\xi\|$, for all $\xi \in \mathcal{H}$, showing that $(x + a)^{-1}$ is bounded. Since $a > 0$ was arbitrary this shows that $\sigma(x) \subset [0, \infty)$ and hence x is positive.

(iv) If x is an isometry then $x^*x = 1$ and hence $\|x\xi\|^2 = \langle x^*x\xi, \xi \rangle = \|\xi\|^2$ for all $\xi \in \mathcal{H}$. The converse again follows from the polarization identity.

(v) If x is a projection then let $\mathcal{K} = \overline{R(x)} = \ker(x)^\perp$, and note that for all $\xi \in \mathcal{K}, \eta \in \ker(x), x\zeta \in R(x)$ we have $\langle x\xi, \eta + x\zeta \rangle = \langle \xi, x\zeta \rangle$, hence $x\xi \in \mathcal{K}$, and $x\xi = \xi$. This shows that x is the orthogonal projection onto the subspace $\mathcal{K}$.

(vi) This follows directly from iv and v.   ∎

Proposition 2.0.15 (Polar decomposition). *Let $\mathcal{H}$ be a Hilbert space, and $x \in \mathcal{B}(\mathcal{H})$, then there exists a partial isometry v such that $x = v|x|$, and $\ker(v) = \ker(|x|) = \ker(x)$. Moreover, this decomposition is unique, in that if $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$ then $y = |x|$, and $w = v$.*

Proof. We define a linear operator $v_0 : R(|x|) \to R(x)$ by $v_0(|x|\xi) = x\xi$, for $\xi \in \mathcal{H}$. Since $\||x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$ it follows that v_0 is well defined and extends to a partial isometry v from $\overline{R(|x|)}$ to $\overline{R(x)}$, and we have $v|x| = x$. We also have $\ker(v) = R(|x|)^\perp = \ker(|x|) = \ker(x)$.

To see the uniqueness of this decomposition suppose $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$. Then $|x|^2 = x^*x = yw^*wy = y^2$, and hence $|x| = (|x|^2)^{1/2} = (y^2)^{1/2} = y$. We then have $\ker(w) = \overline{R(|x|)}^\perp$, and $\|w|x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$, hence $w = v$.   ∎

2.1 Trace class operators

Given a Hilbert space $\mathcal{H}$, an operator $x \in \mathcal{B}(\mathcal{H})$ has finite rank if $\overline{R(x)} = \ker(x^*)^\perp$ is finite dimensional, the **rank** of x is $\dim(\overline{R(x)})$. We denote the space of finite rank operators by $\mathcal{FR}(\mathcal{H})$. If x is finite rank than $R(x^*) = R(x^*_{|\ker(x^*)^\perp})$ is also finite dimensional being the image of a finite dimensional space, hence we see that x^* also has finite rank. If $\xi, \eta \in \mathcal{H}$ are vectors we denote by $\xi \otimes \overline{\eta}$ the operator given by

$$(\xi \otimes \overline{\eta})(\zeta) = \langle \zeta, \eta \rangle \xi.$$

Note that $(\xi \otimes \overline{\eta})^* = \eta \otimes \overline{\xi}$, and if $\|\xi\| = \|\eta\| = 1$ then $\xi \otimes \overline{\eta}$ is a rank one partial isometry from $C\eta$ to $\mathbb{C}\xi$. Also note that if $x, y \in \mathcal{B}(\mathcal{H})$, then we have $x(\xi \otimes \overline{\eta})y = (x\xi) \otimes (\overline{y^*\eta})$.

From above we see that any finite rank operator is of the form pxq where $p, q \in \mathcal{B}(\mathcal{H})$ are projections onto finite dimensional subspaces. In particular this shows that $\mathcal{FR}(\mathcal{H}) = \mathrm{sp}\{\xi \otimes \overline{\eta} \mid \xi, \eta \in \mathcal{H}\}$

Lemma 2.1.1. *Suppose $x \in \mathcal{B}(\mathcal{H})$ has polar decomposition $x = v|x|$. Then for all $\xi \in \mathcal{H}$ we have*

$$2|\langle x\xi, \xi \rangle| \leq \langle |x|\xi, \xi \rangle + \langle |x|v^*\xi, v^*\xi \rangle.$$

Proof. If $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, then we have

$$0 \leq \|(|x|^{1/2} - \lambda|x|^{1/2}v^*)\xi\|^2$$
$$= \||x|^{1/2}\xi\|^2 - 2\mathrm{Re}(\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi \rangle) + \||x|^{1/2}v^*\xi\|^2.$$

Taking λ such that $\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi \rangle \geq 0$, the inequality follows directly. ∎

If $\{\xi_i\}$ is an orthonormal basis for $\mathcal{H}$, and $x \in \mathcal{B}(\mathcal{H})$ is positive, then we define the trace of x to be

$$\mathrm{Tr}(x) = \sum_i \langle x\xi_i, \xi_i \rangle.$$

Lemma 2.1.2. *If $x \in \mathcal{B}(\mathcal{H})$ then $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$.*

Proof. By Parseval's identity and Fubini's theorem we have

$$\sum_i \langle x^*x\xi_i, \xi_i \rangle = \sum_i \sum_j \langle x\xi_i, \xi_j \rangle \overline{\langle \xi_j, x\xi_i \rangle}$$
$$= \sum_j \sum_i \langle \xi_i, x^*\xi_j \rangle \overline{\langle \xi_i, x^*\xi_j \rangle} = \sum_j \langle xx^*\xi_j, \xi_j \rangle. \quad ∎$$

Corollary 2.1.3. *If $x \in \mathcal{B}(\mathcal{H})$ is positive and u is a unitary, then $\mathrm{Tr}(u^*xu) = \mathrm{Tr}(x)$. In particular, the trace is independent of the chosen orthonormal basis.*

Proof. If we write $x = y^*y$, then from the previous lemma we have

$$\mathrm{Tr}(y^*y) = \mathrm{Tr}(yy^*) = \mathrm{Tr}((yu)(u^*y^*)) = \mathrm{Tr}(u^*(y^*y)u). \quad ∎$$

An operator $x \in \mathcal{B}(\mathcal{H})$ is said to be of **trace class** if $\|x\|_1 := \mathrm{Tr}(|x|) < \infty$. We denote the set of trace class operators by $L^1(\mathcal{B}(\mathcal{H}))$ or $L^1(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$.

Given an orthonormal basis $\{\xi_i\}$, and $x \in L^1(\mathcal{B}(\mathcal{H}))$ we define the **trace** of x by

$$\mathrm{Tr}(x) = \Sigma_i \langle x\xi_i, \xi_i \rangle.$$

By Lemma 2.1.1 this is absolutely summable, and

$$2|\mathrm{Tr}(x)| \leq \mathrm{Tr}(|x|) + \mathrm{Tr}(v|x|v^*) \leq 2\|x\|_1.$$

Lemma 2.1.4. *$L^1(\mathcal{B}(\mathcal{H}))$ is a two sided self-adjoint ideal in $\mathcal{B}(\mathcal{H})$ which co-incides with the span of the positive operators with finite trace. The trace is independent of the chosen basis, and $\| \cdot \|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.*

Proof. If $x, y \in L^1(\mathcal{B}(\mathcal{H}))$ and we let $x+y = w|x+y|$ be the polar decomposition, then we have $w^*x, w^*y \in L^1(\mathcal{B}(\mathcal{H}))$, therefore $\sum_i \langle |x+y|\xi_i, \xi_i \rangle = \sum_i \langle w^*x\xi_i, \xi_i \rangle + \langle w^*y\xi_i, \xi_i \rangle$ is absolutely summable. Thus $x + y \in L^1(\mathcal{B}(\mathcal{H}))$ and

$$\|x + y\|_1 \leq \|w^*x\|_1 + \|w^*y\|_1 \leq \|x\|_1 + \|y\|_1.$$

Thus, it follows that $L^1(\mathcal{B}(\mathcal{H}))$ is a linear space which contains the span of the positive operators with finite trace, and $\| \cdot \|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.

If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ then

$$4a|x| = \sum_{k=0}^{3} i^k(a + i^k)|x|(a + i^k)^*,$$

and for each k we have

$$\mathrm{Tr}((a + i^k)|x|(a + i^k)^*) = \mathrm{Tr}(|x|^{1/2}|a + i^k|^2|x|^{1/2}) \leq \|a + i^k\|^2 \mathrm{Tr}(|x|).$$

Thus if we take a to be the partial isometry in the polar decomposition of x we see that x is a linear combination of positive operators with finite trace, (in particular, the trace is independent of the basis). This also shows that $L^1(\mathcal{B}(\mathcal{H}))$ is a self-adjoint left ideal, and hence is also a right ideal. ∎

Theorem 2.1.5. *If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a, b \in \mathcal{B}(\mathcal{H})$ then*

$$\|x\| \leq \|x\|_1$$

$$\|axb\|_1 \leq \|a\| \ \|b\| \ \|x\|_1,$$

and

$$\mathrm{Tr}(ax) = \mathrm{Tr}(xa).$$

Proof. Since the trace is independent of the basis, and $\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|$ it follows easily that $\|x\| \leq \|x\|_1$.

Since for $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ we have $|ax| \leq \|a\||x|$ it follows that $\|ax\|_1 \leq \|a\|\|x\|_1$. Since $\|x\|_1 = \|x^*\|_1$ we also have $\|xb\|_1 \leq \|b\|\|x\|_1$.

Since the definition of the trace is independent of the chosen basis, if $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $u \in \mathcal{U}(\mathcal{H})$ we have

$$\mathrm{Tr}(xu) = \sum_i \langle xu\xi_i, \xi_i \rangle = \sum_i \langle uxu\xi_i, u\xi_i \rangle = \mathrm{Tr}(ux).$$

Since every operator $a \in \mathcal{B}(\mathcal{H})$ is a linear combination of four unitaries this also gives

$$\mathrm{Tr}(xa) = \mathrm{Tr}(ax). \qquad \blacksquare$$

We also remark that for all $\xi, \eta \in \mathcal{H}$, the operators $\xi \otimes \overline{\eta}$ satisfy $\mathrm{Tr}(\xi \otimes \overline{\eta}) = \langle \xi, \eta \rangle$. Also, it's easy to check that $\mathcal{FR}(\mathcal{H})$ is a dense subspace of $L^1(\mathcal{B}(\mathcal{H}))$, endowed with the norm $\| \cdot \|_1$.

Proposition 2.1.6. *The space of trace class operators $L^1(\mathcal{B}(\mathcal{H}))$, with the norm $\| \cdot \|_1$ is a Banach space.*

Proof. From Lemma 2.1.4 we know that $\| \cdot \|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$ and hence we need only show that $L^1(\mathcal{B}(\mathcal{H}))$ is complete. Suppose x_n is Cauchy in $L^1(\mathcal{B}(\mathcal{H}))$. Since $\|x_n - x_m\| \leq \|x_n - x_m\|_1$ it follows that x_n is also Cauchy in $\mathcal{B}(\mathcal{H})$, therefore we have $\|x - x_n\| \to 0$, for some $x \in \mathcal{B}(\mathcal{H})$, and by continuity of functional calculus we also have $\|\,|x| - |x_n|\,\| \to 0$. Thus for any finite orthonormal set $\eta_1, \ldots, \eta_k$ we have

$$\sum_{i=1}^{k} \langle |x|\eta_i, \eta_i \rangle = \lim_{n \to \infty} \sum_{i=1}^{k} \langle |x_n|\eta_i, \eta_i \rangle$$
$$\leq \lim_{n \to \infty} \|x_n\|_1 < \infty.$$

Hence $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $\|x\|_1 \leq \lim_{n\to\infty} \|x_n\|_1$.

If we let $\varepsilon > 0$ be given and consider $N \in \mathbb{N}$ such that for all $n > N$ we have $\|x_n - x_N\|_1 < \varepsilon/3$, and then take $\mathcal{H}_0 \subset \mathcal{H}$ a finite dimensional subspace such that $\|x_N P_{\mathcal{H}_0^\perp}\|_1, \|x P_{\mathcal{H}_0^\perp}\|_1 < \varepsilon/3$. Then for all $n > N$ we have

$$\|x - x_n\|_1$$
$$\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|x P_{\mathcal{H}_0^\perp} - x_N P_{\mathcal{H}_0^\perp}\|_1 + \|(x_N - x_n)P_{\mathcal{H}_0^\perp}\|_1$$
$$\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \varepsilon.$$

Since $\|x - x_n\| \to 0$ it follows that $\|(x - x_n)P_{\mathcal{H}_0}\|_1 \to 0$, and since $\varepsilon > 0$ was arbitrary we then have $\|x - x_n\|_1 \to 0$. $\blacksquare$

Theorem 2.1.7. *The map $\psi : \mathcal{B}(\mathcal{H}) \to L^1(\mathcal{B}(\mathcal{H}))^*$ given by $\psi_a(x) = \mathrm{Tr}(ax)$, for $a \in \mathcal{B}(\mathcal{H})$, $x \in L^1(\mathcal{B}(\mathcal{H}))$, is a Banach space isomorphism.*

Proof. From Theorem 2.1.5 we have that ψ is a linear contraction.

Suppose $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$, then $(\xi, \eta) \mapsto \varphi(\xi \otimes \overline{\eta})$ defines a bounded sesquilinear form on $\mathcal{H}$ and hence there exists a bounded operator $a \in \mathcal{B}(\mathcal{H})$ such that $\langle a\xi, \eta \rangle = \varphi(\xi \otimes \overline{\eta})$, for all $\xi, \eta \in \mathcal{H}$. Since the finite rank operators is dense in

$L^1(\mathcal{B}(\mathcal{H}))$, and since operators of the form $\xi \otimes \overline{\eta}$ span the finite rank operators we have $\varphi = \psi_a$, thus we see that ψ is bijective.

We also have

$$
\begin{aligned}
\|a\| &= \sup_{\substack{\xi,\eta\in\mathcal{H},\\ \|\xi\|,\|\eta\|\leq 1}} |\langle a\xi, \eta\rangle| \\
&= \sup_{\substack{\xi,\eta\in\mathcal{H},\\ \|\xi\|,\|\eta\|\leq 1}} |\operatorname{Tr}(a(\xi \otimes \overline{\eta}))| \leq \|\psi_a\|.
\end{aligned}
$$

Hence ψ is isometric.   ∎

2.2   Hilbert-Schmidt operators

Given a Hilbert space $\mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$, we say that x is a Hilbert-Schmidt operator on $\mathcal{H}$ if $|x|^2 \in L^1(\mathcal{B}(\mathcal{H}))$. We define the set of Hilbert-Schmidt operators by $L^2(\mathcal{B}(\mathcal{H}))$, or $L^2(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

Lemma 2.2.1. $L^2(\mathcal{B}(\mathcal{H}))$ *is a self-adjoint ideal in* $\mathcal{B}(\mathcal{H})$*, and if* $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ *then* $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$*, and*

$$
\operatorname{Tr}(xy) = \operatorname{Tr}(yx).
$$

Proof. Since $|x+y|^2 \leq |x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$ we see that $L^2(\mathcal{B}(\mathcal{H}))$ is a linear space, also since $|ax|^2 \leq \|a\|^2|x|^2$ we have that $L^2(\mathcal{B}(\mathcal{H}))$ is a left ideal. Moreover, if $x = v|x|$ is the polar decomposition of x then we have $xx^* = v|x|^2v^*$, and thus $x^* \in L^2(\mathcal{B}(\mathcal{H}))$ and $\operatorname{Tr}(xx^*) = \operatorname{Tr}(x^*x)$. In particular, $L^2(\mathcal{B}(\mathcal{H}))$ is also a right ideal.

By the polarization identity

$$
4y^*x = \sum_{k=0}^{3} i^k |x + i^k y|^2,
$$

we have that $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$ for $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, and

$$
\begin{aligned}
4\operatorname{Tr}(y^*x) &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x + i^k y)^*(x + i^k y)) \\
&= \sum_{k=0}^{3} i^k \operatorname{Tr}((x + i^k y)(x + i^k y)^*) = 4\operatorname{Tr}(xy^*).
\end{aligned}
$$
∎

From the previous lemma we see that the sesquilinear form on $L^2(\mathcal{B}(\mathcal{H}))$ give by

$$
\langle x, y\rangle_2 = \operatorname{Tr}(y^*x)
$$

is well defined and positive definite. We again have $\|axb\|_2 \leq \|a\|\,\|b\|\,\|x\|_2$, and any $x \in L^2(\mathcal{B}(\mathcal{H}))$ can be approximated in $\|\cdot\|_2$ by operators px where p is a

finite rank projection. Thus, the same argument as for the trace class operators shows that the Hilbert-Schmidt operators is complete in the Hilbert-Schmidt norm.

Also, note that if $x \in L^2(\mathcal{B}(\mathcal{H}))$ then since $\|y\| \leq \|y\|_2$ for all $y \in L^2(\mathcal{B}(\mathcal{H}))$ it follows that

$$\|x\|_2 = \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})), \\ \|y\|_2 \leq 1}} |\operatorname{Tr}(y^*x)|$$

$$\leq \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})), \\ \|y\|_2 \leq 1}} \|y\|\|x\|_1 \leq \|x\|_1.$$

Proposition 2.2.2. *Let $\mathcal{H}$ be a Hilbert space and suppose $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then*

$$\|xy\|_1 \leq \|x\|_2\|y\|_2.$$

Proof. If we consider the polar decomposition $xy = v|xy|$, then by the Cauchy-Schwartz inequality we have

$$\|xy\|_1 = |\operatorname{Tr}(v^*xy)| = |\langle y, x^*v \rangle_2|$$

$$\leq \|x^*v\|_2\|y\|_2 \leq \|x\|_2\|y\|_2. \qquad \blacksquare$$

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then we may extend a bounded operator $x : \mathcal{H} \to \mathcal{K}$ to a bounded operator $\tilde{x} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ by $\tilde{x}(\xi \oplus \eta) = 0 \oplus x\xi$. We define $\operatorname{HS}(\mathcal{H}, \mathcal{K})$ as the bounded operators $x : \mathcal{H} \to \mathcal{K}$ such that $\tilde{x} \in L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$. In this way $\operatorname{HS}(\mathcal{H}, \mathcal{K})$ forms a closed subspace of $L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$.

Note that $\operatorname{HS}(\mathcal{H}, \mathbb{C})$ is the dual Banach space of $\mathcal{H}$, and is naturally anti-isomorphic to $\mathcal{H}$, we denote this isomorphism by $\xi \mapsto \overline{\xi}$. We call this the **conjugate Hilbert space** of $\mathcal{H}$, and denote it by $\overline{\mathcal{H}}$. Note that we have the natural identification $\overline{\overline{\mathcal{H}}} = \mathcal{H}$. Also, we have a natural anti-linear map $x \mapsto \overline{x}$ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\overline{\mathcal{H}})$ given by $\overline{x}\overline{\xi} = \overline{x\xi}$.

If we wish to emphasize that we are considering only the Hilbert space aspects of the Hilbert-Schmidt operators, we often use the notation $\mathcal{H}\overline{\otimes}\mathcal{K}$ for the Hilbert-Schmidt operators $\operatorname{HS}(\mathcal{H}, \overline{\mathcal{K}})$. In this setting we call $\mathcal{H}\overline{\otimes}\mathcal{K}$ the **Hilbert space tensor product** of $\mathcal{H}$ with $\mathcal{K}$. Note that if $\{\xi_i\}_i$ and $\{\eta_j\}_j$ form orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, then $\{\xi_i \otimes \eta_j\}_{i,j}$ forms an orthonormal basis for $\mathcal{H}\overline{\otimes}\mathcal{K}$. We see that the algebraic tensor product $\mathcal{H}\otimes\mathcal{K}$ of $\mathcal{H}$ and $\mathcal{K}$ can be realized as the subspace of finite rank operators, i.e., we have $\mathcal{H}\otimes\mathcal{K} = \operatorname{sp}\{\xi \otimes \eta \mid \xi \in \mathcal{H}, \eta \in \mathcal{K}\}$.

If $x \in \mathcal{B}(\mathcal{H})$ and $y \in \mathcal{B}(\mathcal{K})$ then we obtain an operator $x \otimes y \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ which is given by $(x \otimes y)h = xh\overline{y}^*$. We then have that $\|x \otimes y\| \leq \|x\|\|y\|$, and $(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$ for all $\xi \in \mathcal{H}$, and $\eta \in \mathcal{K}$.

If (X, μ) is a measure space then we have a particularly nice description of the Hilbert-Schmidt operators on $L^2(X, \mu)$.

Theorem 2.2.3. *For each $k \in L^2(X \times X, \mu \times \mu)$ the integral operator T_k defined by*

$$T_k\xi(x) = \int k(x,y)\xi(y)d\mu(y), \qquad \xi \in L^2(X, \mu),$$

is a Hilbert-Schmidt operator on $L^2(X,\mu)$. Moreover, the map $k \mapsto T_k$ is a unitary operator from $L^2(X \times X, \mu \times \mu)$ to $L^2(\mathcal{B}(L^2(X,\mu)))$. Moreover, if we define $k^*(x,y) = \overline{k(x,y)}$ then we have $T_k^* = T_{k^*}$.

Proof. For all $\eta \in L^2(X,\mu)$, the Cauchy-Schwartz inequality gives

$$\|k(x,y)\xi(y)\eta(x)\|_1 \leq \|k\|_2 \|\xi\|_{L^2(X,\mu)} \|\eta\|_2.$$

This shows that T_k is a well defined operator on $L^2(X,\mu)$ and $\|T_k\| \leq \|k\|_2$. If $\{\xi_i\}_i$ gives an orthonormal basis for $L^2(X,\mu)$ and $k(x,y) = \sum \alpha_{i,j}\xi_i(x)\xi_j(y)$ is a finite sum then for $\eta \in L^2(X,\mu)$ we have

$$T_k\eta = \sum \alpha_{i,j}\langle \xi, \xi_j\rangle \xi_i = \left(\sum \alpha_{i,j}\xi_i \otimes \overline{\xi_j}\right)\eta.$$

Thus, $\|T_k\|_2 = \|\sum \alpha_{i,j}\xi_i \otimes \overline{\xi_j}\|_2 = \|k\|_2$, which shows that $k \mapsto T_k$ is a unitary operator.

The same formula above also shows that $T_k^* = T_{k^*}$. ∎

2.3   Compact operators

We denote by $\mathcal{H}_1$ the unit ball in $\mathcal{H}$.

Theorem 2.3.1. *For $x \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent:*

 (i) $x \in \overline{\mathcal{FR}(\mathcal{H})}^{\|\cdot\|}$.

 (ii) x restricted to $\mathcal{H}_1$ is continuous from the weak to the norm topology.

 (iii) $x(\mathcal{H}_1)$ is compact in the norm topology.

 (iv) $x(\mathcal{H}_1)$ has compact closure in the norm topology.

Proof. (i) $\implies$ (ii) Let $\{\xi_\alpha\}_\alpha$ be net in $\mathcal{H}_1$ which weakly converges to ξ. By hypothesis for every $\varepsilon > 0$ there exists $y \in \mathcal{FR}(\mathcal{H})$ such that $\|x - y\| < \varepsilon$. We then have

$$\|x\xi - x\xi_\alpha\| \leq \|y\xi - y\xi_\alpha\| + 2\varepsilon.$$

Thus, it is enough to consider the case when $x \in \mathcal{FR}(\mathcal{H})$. This case follows easily since then the range of x is then finite dimensional where the weak and norm topologies agree.

   (ii) $\implies$ (iii) $\mathcal{H}_1$ is compact in the weak topology and hence $x(\mathcal{H}_1)$ is compact being the continuous image of a compact set.

   (iii) $\implies$ (iv) This implication is obvious.

   (iv) $\implies$ (i) Let P_α be a net of finite rank projections such that $\|P_\alpha\xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}$. Then $P_\alpha x$ are finite rank and if $\|P_\alpha x - x\| \not\to 0$ then there exists $\varepsilon > 0$, and $\xi_\alpha \in \mathcal{H}_1$ such that $\|x\xi_\alpha - P_\alpha x\xi_\alpha\| \geq \varepsilon$. By hypothesis we may pass

to a subnet and assume that $x\xi_\alpha$ has a limit ξ in the norm topology. We then have

$$\varepsilon \leq \|x\xi_\alpha - P_\alpha x\xi_\alpha\| \leq \|\xi - P_\alpha \xi\| + \|(1 - P_\alpha)(x\xi_\alpha - \xi)\|$$
$$\leq \|\xi - P_\alpha \xi\| + \|x\xi_\alpha - \xi\| \to 0,$$

which gives a contradiction. ∎

If any of the above equivalent conditions are satisfied we say that x is a **compact operator**. We denote the space of compact operators by $\mathcal{K}(\mathcal{H})$. Clearly $\mathcal{K}(\mathcal{H})$ is a norm closed two sided ideal in $\mathcal{B}(\mathcal{H})$.

Exercise 2.3.2. Show that the map $\psi : L^1(\mathcal{B}(\mathcal{H})) \to \mathcal{K}(\mathcal{H})^*$ given by $\psi_x(a) = \mathrm{Tr}(ax)$ implements a Banach space isomorphism between $L^1(\mathcal{B}(\mathcal{H}))$ and $\mathcal{K}(\mathcal{H})^*$.

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