Math 196 - Exam 3, April 12, 2012 Name: Key

Problem 1 (20 points). Find a basis for each eigenspace of the matrix $A = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$.

$$\chi_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 3 \\ 1 & -\lambda & 1 \\ -1 & -1 & -1-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -1 & -1-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & -\lambda \\ -1 & -1 \end{vmatrix}$$

$$= (3-\lambda)(\lambda^2+\lambda+1) - (-1-\lambda+1) + 3(-1-\lambda)$$

$$= -\lambda^3 + 2\lambda^2$$

$$= \lambda^2(2-\lambda)$$

$\boxed{\lambda = 0}$:

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \xrightarrow[R_2+R_3]{-3R_2+R_1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$z = t$
$y = 0$
$x = -t$

$\boxed{\text{basis: } \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}}$

$\boxed{\lambda = 2}$:

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -2 & 1 \\ -1 & -1 & -3 \end{pmatrix} \xrightarrow[R_1+R_3]{-R_1+R_2} \begin{pmatrix} 1 & 1 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 7/3 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$z = t$
$y = -\frac{2}{3}t$
$x = -\frac{7}{3}t$

$\boxed{\text{basis: } \left\{ \begin{pmatrix} -7/3 \\ -2/3 \\ 1 \end{pmatrix} \right\}}$ or $\left\{ \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} \right\}$

Problem 2 (20 points). Consider the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, or else show that no such matrices exist.

$$\chi_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ 4 & -3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 4 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\big((2-\lambda)(-3-\lambda) + 4\big)$$
$$= (2-\lambda)(\lambda^2 + \lambda - 2)$$
$$= (2-\lambda)(\lambda+2)(\lambda-1)$$

$\lambda = 1$:

$$\begin{pmatrix} 1 & -1 & 1 \\ 4 & -4 & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} z = 0 \\ y = t \\ x = t \end{matrix} \quad \text{basis: } \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\lambda = 2$:

$$\begin{pmatrix} 0 & -1 & 1 \\ 4 & -5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 0 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} z = t \\ y = t \\ x = t \end{matrix} \quad \text{basis: } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\lambda = -2$:

$$\begin{pmatrix} 4 & -1 & 1 \\ 4 & -1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} z = 0 \\ y = t \\ x = t/4 \end{matrix} \quad \text{basis: } \left\{ \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \right\}$$

$$\boxed{P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 4 \\ 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}}$$

Problem 3 (20 points). Solve the initial value problem:

$$\mathbf{X}'(t) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & -1 \end{pmatrix}}_{P} \mathbf{X}(t), \qquad \mathbf{X}(0) = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}.$$

$$\chi_P(\lambda) = (1-\lambda)(2-\lambda)(-1-\lambda)$$

$\lambda = 1$: $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad z=0,\ y=t,\ x=-t \quad$ basis: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\lambda = -1$: $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad z=t,\ y=0,\ x=0 \quad$ basis: $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\lambda = 2$: $\begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad z=t,\ y=3t,\ x=0 \quad$ basis: $\left\{ \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$

$$\therefore \quad x(t) = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} e^{2t}$$

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = x(0) = \begin{pmatrix} -c_1 \\ c_1 + 3c_3 \\ c_2 + c_3 \end{pmatrix} \Longrightarrow \begin{array}{l} c_1 = -2 \\ c_3 = 1 \\ c_2 = 3 \end{array}$$

$$\boxed{x(t) = 2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{t} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} 2e^{t} \\ -2e^{t} + 3e^{2t} \\ 3e^{-t} + e^{2t} \end{pmatrix}}$$

Problem 4 (20 points). Find the general solution to the following system of differential equations:

$$\begin{aligned} x' &= x + y \\ y' &= -x + y \end{aligned}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{P} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\chi_P(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$$

$$\lambda = 1 \pm i$$

$\lambda = 1+i$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{aligned} y &= t \\ x &= -it \end{aligned} \quad \text{basis: } \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\vec{X}_0(t) = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{t} (\cos t + i \sin t)$$

Taking real & imaginary parts gives:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t} \cos t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} \sin t$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t} \sin t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} \cos t$$

$$\therefore \boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{t} \sin t - c_2 e^{t} \cos t \\ c_1 e^{t} \cos t + c_2 e^{t} \sin t \end{pmatrix}}$$

Problem 5 (20 points). Consider the second order system of differential equations:

$$\begin{aligned} x'' &= 2x + y \\ y'' &= 2x + 3y \end{aligned}.$$

a) Convert this into a first order system of differential equations.

b) Find a non-trivial solution to the above system. You do not need to find the general solution, any solution other than $x(t) = 0$, $y(t) = 0$ will be acceptable.

a)
$x_1 = x$
$x_2 = x'$
$x_3 = y$
$x_4 = y'$

$\therefore$ $\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= 2x_1 + x_3 \\ x_3' &= x_4 \\ x_4' &= 2x_1 + 3x_3 \end{aligned}}$ or $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 3 & 0 \end{pmatrix}}_{P} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

b):

$$\chi_P(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 2 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 2 & 0 & 3 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 3 & -\lambda \end{vmatrix} - \begin{vmatrix} 2 & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & 3 & -\lambda \end{vmatrix}$$

$$= \lambda^2 \begin{vmatrix} -\lambda & 1 \\ 3 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} -\lambda & 1 \\ 3 & -\lambda \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 2 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2 - 3) - 2(\lambda^2 - 3) + (-2)$$

$$= \lambda^4 - 5\lambda^2 + 4 = (\lambda^2 - 4)(\lambda^2 - 1)$$

$$= (\lambda - 4)(\lambda + 4)(\lambda - 1)(\lambda + 1)$$

$\lambda = 1$:

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 2 & 0 & 3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$z = t$
$y = t$
$x = -t$
$w = -t$

basis: $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\therefore$ A solution is given by

$$\boxed{\begin{aligned} x(t) &= -e^{t} \\ y(t) &= e^{t} \end{aligned}}$$