MATH 6100 - HOMEWORK ASSIGNMENT 9

DUE WEDNESDAY, DECEMBER 7TH BY 6:00PM

Exercise 0.1. Give an explicit homoemorphism between the Cantor space $\{0,1\}^{\mathbb{N}}$ and the usual Cantor set $C \subset [0,1]$.

A **Souslin scheme** on a set X is a family $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of X. A **Lusin scheme** on X is a Souslin scheme such that

(1) $A_{s\hat{\ }i} \cap A_{s\hat{\ }j} = \emptyset$, for $s \in \mathbb{N}^{<\mathbb{N}}$, $i \neq j$.
(2) $A_{s\hat{\ }i} \subset A_s$, for $s \in \mathbb{N}^{<\mathbb{N}}$.

If (X,d) is a metric space and we additionally have $\lim_{n\to\infty} \mathrm{diam}(A_{s|n}) = 0$ for any $s \in \{0,1\}^{\mathbb{N}}$ then we say that $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ has **vanishing diameter**. In this case we let $D = \{s \mid \cap_{n\in\mathbb{N}} A_{s|n} \neq \emptyset\}$, and for $s \in D$ we define $f(s) \in X$ so that $\{f(s)\} = \cap_{n\in\mathbb{N}} A_{s|n}$. The map $f : D \to X$ is the **associated map**.

Exercise 0.2. Suppose (X,d) is a metric space and we have a Souslin scheme $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ which has vanishing diameter, and associated map $f : D \to X$.

(1) Show that f is continuous.
(2) Show that f is open if each A_s is open and $A_s \subset \cup_{n\in\mathbb{N}} A_{s\hat{\ }n}$, for all $s \in \mathbb{N}^{<\mathbb{N}}$.
(3) Show that f is injective if $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ is a Lusin scheme.
(4) Show that f is surjective if $A_\emptyset = X$, and $A_s = \cup_{n\in\mathbb{N}} A_{s\hat{\ }n}$, for all $s \in \mathbb{N}^{<\mathbb{N}}$.

Exercise 0.3. Show that every nonempty Polish space without isolated points X is the continuous image of the Baire space $\mathbb{N}^{\mathbb{N}}$.

Exercise 0.4. Let (X,d) be a complete metric space such that every compact subset of X has empty interior. Show that for each nonempty open set $A \subset X$, there exists $\varepsilon > 0$, so that if $A \subset \cup_{n\in\mathbb{N}} B_n$ and $\mathrm{diam}(B_n) < \varepsilon$, then $B_n \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.

Exercise 0.5. Prove the Alexandrov-Urysohn Theorem: If X is a Polish space which has a countable basis of clopen sets and such that any compact subset of X has empty interior, then X is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

Exercise 0.6. Show that if (X,d) is a nonempty countable metric space without isolated points, then there is an embedding $F : X \to \mathbb{N}^{\mathbb{N}}$ which has dense image. Hint: Find a Lusin scheme on X so that the associated map $f : D \to X$ is open, and bijective, with D dense, and then let $F = f^{-1}$.

Exercise 0.7. Prove Sierpiński's Theorem: Let (X,d) be a nonempty countable metric space without isolated points, then X is homeomorphic to $\mathbb{Q}$ with its usual topology. Hint: Combine Problem 1 on Homework 7, Problem 2 on Homework 8, and the previous problem.

Date: November 8, 2016.

Exercise 0.8 (Benyamini)**.** Show that there exists $f \in C_b(\mathbb{R})$ such that given any doubly infinite sequence $\{y_n\}_{n\in\mathbb{Z}}$ there is a point $t \in \mathbb{R}$ so that $y_n = f(t+n)$ for $n \in \mathbb{Z}$. Hint: If $C \subset [0, 1/2]$ is homeomorphic to the Cantor set, first construct a continuous surjection $f : C \to [0, 1]^{\mathbb{Z}}$. Then define $g : \cup_{n\in\mathbb{Z}} C + n \to \mathbb{R}$ by $g(t + n) = \pi_n(f(t))$ for $t \in C$. Then extend g to a continuous function in $C_b(\mathbb{R})$.