# MATH 6100 - HOMEWORK ASSIGNMENT 8

DUE WEDNESDAY, NOVEMBER 30TH BY 6:00PM

**Exercise 0.1.** Suppose that a topological space $X$ has a countable basis of clopen sets, show that $X$ embedds into $\{0,1\}^{\mathbb{N}}$.

Note that the space of irrationals $\mathbb{R} \setminus Q$ with its subspace topology is Polish, even though the usual metric is far from complete. The next two exercises give an explicit way to see this.

**Exercise 0.2.** Suppose $\{X_n\}_{n=1}^{\infty}$ is a sequence of completely metrizable spaces. Show that $\prod_{n=1}^{\infty} X_n$ is completely metrizable. Moreover, show that $\prod_{n=1}^{\infty} X_n$ is separable if each $X_n$ is separable.

**Exercise 0.3.** Show that $\mathbb{R} \setminus Q$ and **the Baire space** $\mathbb{N}^{\mathbb{N}}$ are homeomorphic. Hint: Consider continued fraction expansions.

**Exercise 0.4.** Let $X$ be a compact Hausdorff space and suppose $|X| = \infty$. Show that, as a complex vector space, $C(X)$ has no countable basis.

**Exercise 0.5.** Suppose $f_n : \mathbb{R} \to \mathbb{R}$ are continuous, and $f : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$.

(1) Show that if $I$ is an open interval of positive length then $f^{-1}(I) \cap \overline{f^{-1}(I)^c}$ is $F_\sigma$ and nowhere dense.
(2) Show that if $f$ is not continuous at a point $x$ then there exists an open interval $I$ with rational endpoints such that $x \in f^{-1}(I) \cap \overline{f^{-1}(I)^c}$.
(3) Show that $f$ is continuous on a dense set of points in $\mathbb{R}$.

**Exercise 0.6.** Show that there exists a function $f \in C([0,1])$ so that $f$ is not monotone on any interval of positive length.

**Exercise 0.7** ([?])**.** Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that at each point $x \in \mathbb{R}$ there is a derivative $f^{(n)}$ so that $f^{(n)}(x) = 0$. Let

$$Y = \{x \in \mathbb{R} \,|\, f_{|O} = p_{|O} \text{ for some polynomial } p \text{ and some neighborhood } O \text{ of } x\},$$

and let $X = Y^c$. Suppose that $X \neq \emptyset$. For each $n \geq 0$ let $S_n = \{x \in \mathbb{R} \mid f^{(n)}(x) = 0\}$.

(a) Show that $X$ is a closed set without isolated points.
(b) Show that there exists an interval $(a,b)$ such that $\emptyset \neq (a,b) \cap X \subset S_n$.
(c) Reach a contradiction by showing that $f^{(n)}(x) = 0$ for all $x \in (a,b)$.
(d) Conclude that, in fact, $X = \emptyset$, and deduce from this that $f$ agrees with a polynomial on $\mathbb{R}$.

*Date*: November 8, 2016.

**Exercise 0.8.** For each $n, m \in \mathbb{N}$ let

$$A_{n,m} = \left\{ f \in C([0,1]) \mid \text{ there exists } x \in [0,1] \text{ such that } \left| \frac{f(t) - f(x)}{t - x} \right| \leq n \text{ if } 0 < |x - t| < \frac{1}{m} \right\}.$$

(1) Show that if $f \in C([0,1])$ is differentiable at some point in $[0,1]$ then $f \in A_{n,m}$ for some $n, m \in \mathbb{N}$.
(2) Show that $A_{n,m}$ is closed in $C([0,1])$.
(3) Show that $A_{n,m}$ is nowhere dense in $C([0,1])$.
(4) Show that the set of functions $f \in C([0,1])$ which are nowhere differentiable is dense in $C([0,1])$.