MATH 6100 - HOMEWORK ASSIGNMENT 7

DUE WEDNESDAY, NOVEMBER 16TH BY 6:00PM

Exercise 0.1 (Fréchet)**.** Let $A, B \subset \mathbb{R}$ be two countable dense sets, show that there is a homeomorphism $\theta : \mathbb{R} \to \mathbb{R}$ so that $\theta(A) = B$. Hint: Use Cantor's back and forth method (Problem 9 on Homework 1).

Exercise 0.2. Show that a metric space X is compact if and only if every real valued function on X is bounded.

Exercise 0.3. Let $(X, \mathcal{T})$ be a compact Hausdorff space. Show that if $\mathcal{T}'$ is any weaker topology then $(X, \mathcal{T}')$ is not Hausdorff. Show that if $\mathcal{T}'$ is any stronger topology then $(X, \mathcal{T}')$ is not compact.

Let X be a locally compact topological space, and fix a point $\omega \notin X$. On the space $\tilde{X} = X \cup \{\omega\}$ we define a new topology whose open sets consist of the open sets in X, together with the compliments (in $\tilde{X}$) of compact subsets of X. The space $\tilde{X}$ is called the **one-point compactification** of X.

Exercise 0.4. Show that $\tilde{X}$ is a compact Hausdorff space and that the relative topology from $X \subset \tilde{X}$ agrees with the topology on X.

Let X be a locally compact topological space, a function $f : X \to \mathbb{C}$ is said to **vanish at infinity** if for every $\varepsilon > 0$, there exists a compact set $K \subset X$ so that $|f(x)| < \varepsilon$ for all $x \in K^c$. We denote by $C_0(X)$ the space of all continuous functions which vanish at infinity.

Exercise 0.5. Show that $C_0(X)$ is a closed subspace of $C_b(X)$.

Exercise 0.6 (Compare this with Exercise 4 from Homework 6)**.** Suppose X is a compact Hausdorff space such that singletons $\{x\}$ are G_δ.

(1) For each $x \in X$ find a countable open cover $\mathcal{O}$ of $X \setminus \{x\}$ so that $x \notin \overline{O}$ for all $O \in \mathcal{O}$.
(2) Show that X is first countable.

If (V, E) is a graph, and $k \in \mathbb{N}$, a **k-coloring** of the graph (V, E) is an assignment $f \in \{1, 2, \dots, k\}^V$ such that for all $(v, w) \in E$ we have $f(v) \neq f(w)$.

Exercise 0.7. Prove the De Bruijn-Erdös theorem: If (V, E) is a graph such that a k-coloring exists for every finite subgraph, then a k-coloring exists for (V, E). Hint: For each finite subgraph (V_0, E_0) consider

$$F_{(V_0,E_0)} = \{f \in \{1, \dots, k\}^V \mid f_{|V_0} \text{ gives a } k-\text{coloring of } (V_0, E_0)\},$$

then show that the family of all such $F_{(V_0,E_0)}$ has the finite intersection property. (This approach is due to Gottschalk.)

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Exercise 0.8. Suppose X and Y are compact Hausdorff spaces and $f \in C(X \times Y)$. Show that for all $\varepsilon > 0$ there exist $g_1, \ldots, g_n \in C(X)$ and $h_1, \ldots, h_n \in C(Y)$ so that $|f(x,y) - \sum_{i=1}^n g_i(x)h_i(y)| < \varepsilon$ for all $(x,y) \in X \times Y$.

Exercise 0.9. Let X be a compact Hausdorff space. An **ideal** in $C(X)$ is a subalgebra $I \subset C(X)$, such that $fg \in I$ whenever $f \in C(X)$ and $g \in I$.

(1) If $I \subset C(X)$ is an ideal, let $h(I) = \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$, the **hull** of I. Show that $h(I)$ is closed.
(2) If $A \subset X$, let $k(A) = \{f \in C(X) \mid f(x) = 0 \text{ for all } x \in A\}$, the **kernel** of A. Show that $k(A)$ is a closed ideal in $C(X)$ which is closed under conjugation.
(3) Show that $k(h(I)) = \overline{I}$ for any ideal $I \subset C(X)$ which is closed under conjugation, and $h(k(A)) = \overline{A}$ for any subset $A \subset X$.