MATH 6100 - HOMEWORK ASSIGNMENT 6

DUE WEDNESDAY, NOVEMBER 9TH BY 6:00PM

Exercise 0.1. Consider $[0,1]$ with the Borel σ-algebra. Let ν be counting measure and μ be Lebesgue measure on $[0,1]$, then there do not exist Borel measures ν_0, ν_1 on $[0,1]$ so that $\nu = \nu_0 + \nu_1$, $\nu_0 \perp \mu$, and $\nu_1 \ll \mu$.

If μ is a measure on $(X, \mathcal{M})$ and $f \in L^1(X, \mu)$ then we obtain a complex valued measure $f\mu$ by $(f\mu)(E) = \int_E f\, d\mu$.

Exercise 0.2. If $f \in L^1(X, \mu)$ then $|f\mu| = |f|\mu$, and $\|f\mu\| = \|f\|_1$.

We let $M_b(X)$ denote the space of all complex valued measures on $(X, \mathcal{M})$.

Exercise 0.3. The map $\nu \mapsto \|\nu\|$ gives a norm on $M_b(X)$, and with this norm $M_b(X)$ is a Banach space.

Exercise 0.4. In a first countable T_1-space, singletons $\{x\}$ are G_δ.

Exercise 0.5. Every metric space is normal and first countable.

Exercise 0.6. A metric space is separable if and only if it is second countable.

Exercise 0.7. Let $X = \mathbb{R}$ and let $\mathcal{T}$ be the family of all sets of the form $U \cup (V \cap \mathbb{Q})$ where U and V are open sets in the usual sense. Then $\mathcal{T}$ gives a topology on $\mathbb{R}$ which is Hausdorff but not regular.

Exercise 0.8. Closed subsets of a metric space are G_δ.

Exercise 0.9. Let (X, d) be a metric space and consider the bounded metric $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Then (X, d') describes the same topology on X.

A topological space is **disconnected** if there exists nonempty disjoint open sets U, V which cover X; otherwise X is connected. A subset $E \subset X$ is connected or disconnected if this is the case in the relative topology.

Exercise 0.10. (a) If $\{A_i\}_{i \in I}$ is a family of connected subsets such that $\cap_{i \in I} A_i \neq \emptyset$ then $\cup_{i \in I} A_i$ is connected.
(b) If $A \subset X$ is connected then $\overline{A}$ is also connected.
(c) Every point $x \in X$ is contained in a unique maximal connected subset of X, and this subset is closed. (This is the **connected component** of x).

Exercise 0.11. The continuous image of a connected set is connected.

Date: November 8, 2016.