MATH 6100 - HOMEWORK ASSIGNMENT 5

DUE MONDAY, OCTOBER 17 BY 6:00PM

Exercise 0.1. There does not exist a metric d on $L^\infty([0,1],\lambda)$ so that a sequence of functions $\{f_n\}_{n=1}^\infty \subset L^\infty([0,1],\lambda)$ converge almost eveywhere to a function $f \in L^\infty([0,1],\lambda)$ if and only if $d(f_n, f) \to 0$. Hint: Find a sequence $\{f_n\}_{n=1}^\infty \subset L^\infty([0,1],\lambda)$ which does not converge almost everywhere to any function but such that every subsequence has a further subsequence which does converge almost everywhere to some function.

Exercise 0.2. Let λ denote Lebesgue measure on $[0,1]$. Show that there is an integrable function $f : [0,1] \to \mathbb{R}$ so that for all $0 \leq a < b \leq 1$ and $-\infty < c < d < \infty$ we have $\lambda((a,b) \cap f^{-1}(c,d)) > 0$.

Exercise 0.3. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $f \in \mathcal{M}(X; [0,\infty))$. Define

$$I_1(f) = \sup\left\{ \int g \mid g \in \mathcal{L}^1_0(X;[0,\infty)), g \leq f \right\},$$
$$I_2(f) = \inf\left\{ \int g \mid g \in \mathcal{L}^1_0(X;[0,\infty)), f \leq g \right\}.$$

Show that f is integrable if and only if $I_1(f) < \infty$, and this this case we have $I_1(f) = \int f$. Also, show that if $\left\{ \int g \mid g \in \mathcal{L}^1_0(X;[0,\infty)), f \leq g \right\} \neq \emptyset$ then f is integrable and $I_2(f) = \int f$.

Exercise 0.4. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f \in \mathcal{M}(X, [0,\infty))$. Set $F(t) = \mu(f^{-1}([t,\infty)))$. Then F is measurable and $F \in \mathcal{L}^1([0,\infty))$ if and only if $f \in \mathcal{L}^1(X)$. Moreover, in this case we have $\int f d\mu = \int F d\lambda$.

Exercise 0.5. Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $(Y, \mathcal{N})$ is a measurable space, and $\theta : X \to Y$ is measurable. Then for all $f \in \mathcal{M}(Y)$ we have $f \circ \theta \in \mathcal{L}^1(X, \mu)$ if and only if $f \in \mathcal{L}^1(Y, \theta_*\mu)$, and in this case we have

$$\int f \circ \theta \, d\mu = \int f \, d(\theta_*\mu).$$

Exercise 0.6. Consider $\mathbb{N}$ with the counting measure, and consider the function $f : \mathbb{N}^2 \to \mathbb{C}$ given by $f(n,m) = 1$ if $n = m$, $f(n,m) = -1$ if $n = m+1$, and $f(n,m) = 0$ otherwise. Then $\sum_{n=1}^\infty \left(\sum_{m=1}^\infty f(n,m)\right)$ and $\sum_{m=1}^\infty \left(\sum_{n=1}^\infty f(n,m)\right)$ both exist but are not equal.

Exercise 0.7. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $\mathbb{R}$ be equipped with Lebesgue measure λ and the Borel σ-algebra $\mathcal{B}$. Show that if $f : X \to [0, +\infty)$ is measurable, then the set $\{(x,t) \in X \times \mathbf{R} : 0 \leq t \leq f(x)\}$ is measurable in $\mathcal{M} \times \mathcal{B}$, and

$$(\mu \times m)(\{(x,t) \in X \times \mathbf{R} : 0 \leq t \leq f(x)\}) = \int_X f(x) \; d\mu(x).$$

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