MATH 6100 - HOMEWORK ASSIGNMENT 4

DUE WEDNESDAY, SEPTEMBER 28 BY 6:00PM

A measure space $(X, \mathcal{M}, \mu)$ is **complete** if every subset of a null set is measurable (and hence also null).

Exercise 0.1. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and let $\mathcal{N} = \{E \in \mathcal{M} \mid \mu(E)\}$ be the space of null sets. We let $\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

The measure space $(X, \overline{\mathcal{M}}, \overline{\mu})$ from the previous theorem is called the **completion** of $(X, \mathcal{M}, \mu)$.

Exercise 0.2. If μ^* is an outer measure on X, $\mathcal{M}$ is the collection of all μ^*-measurable sets, and μ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \mu)$ is a complete measure space.

Exercise 0.3. Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $(Y, \mathcal{N})$ is a measurable space and $\theta : X \to Y$ is measurable. For each set $E \subset Y$ we set $\theta_*\mu(E) = \mu(\theta^{-1}(E))$. Then $\theta_*\mu$ is a measure on $(Y, \mathcal{N})$ called the **push forward measure of μ with respect to θ.**

We let λ denote Lebesgue measure on $\mathbb{R}$.

Exercise 0.4. For every $\varepsilon > 0$, there exists a compact set $K \subset [0, 1]$ which contains no isolated points and no non-trival open interval such that $\lambda(K) > 1 - \varepsilon$.

Exercise 0.5. Let $E \subset \mathbb{R}$ be a Borel set such that $\lambda(E) < \infty$. Then the maps $\mathbb{R} \ni t \mapsto \lambda(E \Delta tE)$, and $t \mapsto \lambda(E\Delta(E + t))$ are continuous.

Exercise 0.6. If $(X, \mathcal{M}, \mu)$ is a measure space and $\{A_j\}_{j=1}^{\infty} \subset \mathcal{M}$, we set $\liminf_{j\to\infty} A_j = \cup_{N=1}^{\infty} \cap_{k=N}^{\infty} A_k$. Then $\mu(\liminf A_j) \leq \liminf \mu(A_j)$.

Exercise 0.7. Show that there exists a Borel set $A \subset [0, 1]$ such that $0 < \lambda(A \cap I) < \lambda(I)$ for every subinterval I of $[0, 1]$.

Date: September 21, 2016.