MATH 6100 - HOMEWORK ASSIGNMENT 3

DUE MONDAY, SEPTEMBER 19 BY 6:00PM

Exercise 0.1. Suppose we have an algebra $\mathcal{C} \subset 2^X$ with the proeprty that if $E_n \in \mathcal{C}$ and $E_n \subset E_{n+1}$, for $n \geq 1$, then $\cup_{n=1}^{\infty} E_n \in \mathcal{C}$. Then $\mathcal{C}$ is a σ-algebra.

Exercise 0.2. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f, g \in \mathcal{M}(X; \mathbb{R})$. The sets

$$\{x \in X \mid f(x) < g(x)\} \qquad \text{and} \qquad \{x \in X \mid f(x) = g(x)\}$$

are measurable.

Exercise 0.3. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{M}(X; \mathbb{R})$. Set

$$\mathcal{C} = \{x \in X \mid \{f_n(x)\}_{n \in \mathbb{N}} \text{ converges}\}.$$

Then $\mathcal{C}$ is measurable.

Let $(X, \mathcal{M}, \mu)$ be a measure space. A function $f \in \mathcal{M}(X)$ is **essentially bounded** if there exists $M \in [0, \infty)$ such that $\mu(\{x \in X \mid |f(x)| > M\}) = 0$. We let $\mathcal{L}^{\infty}(X, \mu)$ denote the space of all (complex valued) essentially bounded functions, and for $f \in \mathcal{L}^{\infty}(X, \mu)$ we set

$$\|f\|_{\infty} = \inf\{M \in [0, \infty) \mid \mu(\{x \in X \mid |f(x)| > M\}) = 0\}.$$

Exercise 0.4. $\mathcal{L}^{\infty}(X, \mu)$ is an algebra and $\| \cdot \|_{\infty}$ gives a seminorm on $\mathcal{L}^{\infty}(X, \mu)$.

We let $L^{\infty}(X, \mu)$ be the normed algebra obtained from $\mathcal{L}^{\infty}(X, \mu)$ by identifying two functions f and g when $\|f - g\|_{\infty} = 0$.

Exercise 0.5. If $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^{\infty}(X, \mu)$ is Cauchy with respect to $\| \cdot \|_{\infty}$, then there exists $f \in \mathcal{L}^{\infty}(X, \mu)$ such that $\|f - f_n\|_{\infty} \to 0$, hence $L^{\infty}(X, \mu)$ is a Banach algebra.

Exercise 0.6. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and for $E, F \in \mathcal{M}$ set $\rho(E, F) = \mu(E \Delta F)$. Then ρ gives a semimetric on $\mathcal{M}$.

Exercise 0.7. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and ρ defined as above. Then ρ is a complete semimetric. Hint: If $\{E_n\}_{n \in \mathbb{N}}$ is Cauchy, by passing to a subsequence we may suppose $\mu(E_n \Delta E_m) \leq \max\{2^{-n}, 2^{-m}\}$, and in this case setting $F_m = \cup_{k \geq m} E_k$, we have that $\{F_m\}_{m \in \mathbb{N}}$ is again Cauchy, and $\mu(F_m \Delta E_n) < 2^{-n+4}$ for $m > n$.

Date: September 13, 2016.