# MATH 6100 - HOMEWORK ASSIGNMENT 2

DUE FRIDAY, SEPTEMBER 9 BY 6:00PM

**Exercise 0.1.** There are two homeomorphic metric spaces $(X_1, d_1)$ and $(X_2, d_2)$ such that $(X_1, d_1)$ is complete, while $(X_2, d_2)$ is not.

A metric space $(X, d)$ is **separable** if it contains a countable dense set.

**Exercise 0.2.** A compact metric space is separable.

We let $\ell^\infty(\mathbb{N})$ denote the set of uniformly bounded sequences from $\mathbb{N}$ to $\mathbb{C}$. We consider this as a complete metric space whose metric is given by $d(f, g) = \|f - g\|_\infty = \sup_{n\in\mathbb{N}} |f(n) - g(n)|$.

**Exercise 0.3** (Kuratowski)**.** Every bounded seprable metric space is isometric to a subspace of $\ell^\infty(\mathbb{N})$.

In the following we consider vector spaces over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$, or $\mathbb{K} = \mathbb{C}$.

**Exercise 0.4.** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces, and $T : V \to W$ a linear operator. Then $T$ is bounded if and only if $T$ is continuous.

**Exercise 0.5.** Let $V$ be a finite dimensional $\mathbb{K}$-vector space, and suppose $\|\cdot\|_1$, and $\|\cdot\|_2$ are norms on $V$. Then the identity map from $(V, \|\cdot\|_1)$ to $(V, \|\cdot\|_2)$ is a homeomorphism.

**Exercise 0.6.** Let $V$ be a finite dimensional normed space. Then the closed unit ball $\overline{B}(1, 0)$ is compact, and $V$ is a Banach space.

**Exercise 0.7** (Riesz' lemma)**.** Let $(V, \|\cdot\|)$ be a normed space, $W \subset V$ a proper closed subspace, and fix $0 < \alpha < 1$. Then there exits $x \notin W$ with $\|x\| = 1$ so that $\inf_{y\in W} \|x - y\| \geq \alpha$. (Hint: Start with $x_0 \notin W$, set $d = \inf_{y\in W} \|x_0 - y\|$, take $x_1 \in W$ so that $\|x_0 - x_1\| \geq d - \varepsilon$ for some suitably chosen $\varepsilon > 0$, and show that $x = \|x_0 - x_1\|^{-1}(x_0 - x_1)$ works.)

**Exercise 0.8.** Let $V$ be a normed space such that the closed unit ball $\overline{B}(1, 0)$ is compact. Then $V$ is finite dimensional.

If $(V, \|\cdot\|)$ is a normed space, a series $\sum_{n=1}^\infty x_n$ is said to converge if the partial sums $\sum_{n=1}^k x_n$ converge as $k$ tends to infinite. A series $\sum_{n=1}^\infty x_n$ is said to converge absolutely if $\sum_{n=1}^\infty \|x_n\| < \infty$.

**Exercise 0.9.** Let $(V, \|\cdot\|)$ be a normed space over $\mathbb{K}$. Then $V$ is a Banach space if and only if every absolutely convergent series converges.

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