MATH 6100 - HOMEWORK ASSIGNMENT 1

DUE FRIDAY, SEPTEMBER 2 BY 6:00PM

Exercise 0.1. If X is countably infinite then there is a bijection from X onto $\mathbb{N}$.

Exercise 0.2. We have $|\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$.

Exercise 0.3. Let $2^{<\mathbb{N}}$ denote the set of finite sequences in $\{0, 1\}$, then $2^{<\mathbb{N}}$ is countable.

Exercise 0.4. Let X be a countably infinite set. There exists an uncountable family $\mathcal{F} \subset 2^X$ so that for any distinct pair $A, B \in \mathcal{F}$ we have that $A \cap B$ is finite. (Hint: It may help to consider the case $X = 2^{<\mathbb{N}}$.)

A complex number α is **algebraic** if it is the solution of a polynomial having rational coefficients.

Exercise 0.5. The set of algebraic numbers is countable.

Exercise 0.6. The Hausdorff maximal principle implies Zorn's lemma.

Exercise 0.7. Zorn's lemma implies the well ordering principle.

Exercise 0.8. The well ordering principle implies the axiom of choice.

Let X be a set, and $\leq$ be a linear ordering on X. We say that the linear order is **dense** if for all $x < y$ there exists $z \in X$ such that $x < z < y$.

Exercise 0.9 (Cantor's back-and-forth method)**.** Let $(X, \leq)$ and $(Y, \leq)$ be countable dense linear orderings which do not have upper or lower bounds. Enumerate $X = \{x_1, x_2, \ldots\}$, and $Y = \{y_1, y_2, \ldots\}$.

(1) There exist increasing sequences of finite sets $A_n \subset X$, $B_n \subset Y$, and order preserving bijections $f_n : A_n \to B_n$ such that $x_n \in A_n$, $y_n \in B_n$, and $f_{n+1|A_n} = f_n$, for all $n \geq 1$.
(2) There exists an order preserving bijection $f : X \to Y$.

A (undirected) **graph** consists of a pair (V, E) where V is a set (the **vertex set**) and $E \subset V \times V$ (the **edge set**) such that $(v, w) \in E$ if and only if $(w, v) \in E$. A **subgraph** is a graph (V_0, E_0) with $V_0 \subset V$, and $E_0 \subset E$.

If (V, E) is a graph, two vertices $v, w \in V$ are **adjacent** if $(v, w) \in E$. We let $N(v)$ denote the set of vertices which are adjacent to v. A graph (V, E) is **locally finite** if $|N(v)| < \infty$ for each $v \in V$. A **finite simple path** is an injective function $p : \{1, 2, \ldots, n\} \to V$, such that $(p(k), p(k + 1)) \in E$ for all $1 \leq k < n$; we say that n is the **length** of the path. A ray is an injective function $p : \mathbb{N} \to V$ such that $(p(k), p(k + 1)) \in E$ for all $1 \leq k$. A graph is **connected** if for any distinct vertices $v, w \in V$, there exists a finite simple path $p : \{1, 2, \ldots, n\} \to V$ such that $p(1) = v$ and $p(n) = w$.

Date: September 2, 2016.

Exercise 0.10 (König's lemma)**.** If a locally finite connected graph (V, E) has infinitely many vertices, then (V, E) admits a ray.