Math 6100 - Final Exam, December 17, 2016 Name:____________________________________

Instructions: Work on any 5 problems. Circle the problems you want to be graded:

1 2 3 4 5 6 7

Problem 1 (20 points). Suppose K is a compact Hausdorff space and $\{f_n\}_n$ is a sequence of continuous complex-valued functions on K such that f_n converges pointwise to a continuous function f. Does it follow that f_n converges to f uniformly. Prove or give a counterexample.

Problem 2 (20 points). Let $(X, \mathcal{M}, \mu)$ be a finite measure space and suppose $f \in L^\infty(X, \mu)$. Set $a_n = \int |f|^n \, d\mu$. Show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \|f\|_\infty$.

Problem 3 (20 points). Let μ be a finite, positive, Borel measure on $\mathbb{R}^2$, and let $\mathcal{G}$ be the family of finite unions of squares of the form

$$S = \{(x, y) \mid j2^{-n} \leq x \leq (j+1)2^{-n}; k2^{-n} \leq y \leq (k+1)2^{-n}\},$$

where j, k, and n are integers. Prove that the set of linear combinations of characteristic functions of elements from $\mathcal{G}$ is dense in $L^1(\mathbb{R}^2, \mu)$.

Problem 4 (20 points). Let $f : [0,1] \to \mathbb{C}$ be Borel. Assume $fg \in L^1([0,1],\lambda)$ whenever $g \in L^1([0,1],\lambda)$, where λ is Lebesgue measure on $[0,1]$. Prove that $f \in L^\infty([0,1],\lambda)$.

Problem 5 (20 points). Is the Banach space $\ell^\infty(\mathbb{N})$ separable? Prove your answer is correct.

Problem 6 (20 points). Suppose $E \subset \mathbb{R}$ is a Borel set which has positive Lebesgue measure. Show that the set $E - E := \{x - y \mid x, y \in E\}$ contains an open interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Hint: Suppose $\lambda(E) < \infty$ and consider the function $f(y) = \int_{\mathbb{R}} 1_E(x) 1_E(y + x)\, d\lambda(x)$.

Problem 7 (20 points). Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and suppose $\mathcal{F} \subset L^\infty(X, \mu)$ is a family such that for each $g \in L^1(X, \mu)$ we have $\sup_{f \in \mathcal{F}} |\int fg \, d\mu| < \infty$, prove that $\sup_{f \in \mathcal{F}} \|f\|_\infty < \infty$. Hint: First show that for some n the set $X_n = \{g \in L^1(X, \mu) \mid \sup_{f \in \mathcal{F}} |\int fg \, d\mu| \leq n\}$, contains an L^1-open ball $B(g_0, \varepsilon)$. Then, writing $g = \frac{1}{\varepsilon}(\varepsilon g + g_0 - g_0)$, use this to show that $\sup_{f \in \mathcal{F}} \|f\|_\infty \leq 2n/\varepsilon$.