HOMEWORK 2, MATH 362A - FALL 2014

Due Wednesday September 10th

1. Prove that all norms on $\mathbb{R}^n$ are equivalent.

2. Show that every separable Banach space X is isometric to a quotient space of $\ell^1\mathbb{N}$.

3. A subset S of a Banach space X is called weakly bounded if for all $\varphi \in X^*$ we have $\sup_{x\in S} |\varphi(x)| < \infty$. A subset S is called strongly bounded if $\sup_{x\in S} \|x\| < \infty$. Show that a subset S is weakly bounded if and only if it is strongly bounded.

4. Let X be a locally compact Hausdorff space. For each Radon measure μ on X, denote by $\|f\|_\mu = \int |f| \, d\mu$ the induced norm on $C_c(X)$. Show that the norms $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ are equivalent, if and only if μ and ν are equivalent and have essentially bounded Radon-Nikodym derivatives.

5. Let (X, μ) be a measure space. A state on $L^\infty(X, \mu)$ is a bounded linear functional $\varphi \in (L^\infty(X, \mu))^*$ such that $\varphi(1) = 1$, and $\varphi(f) \geq 0$ whenever $f \geq 0$.

a) Show that if φ is a state on $L^\infty(X, \mu)$ then for all $f, g \in L^\infty(X, \mu)$ the Cauchy-Schwartz inequality holds: $|\varphi(\overline{g}f)|^2 \leq \varphi(|g|^2)\varphi(|f|^2)$

b) Show that $\psi \in (L^\infty(X, \mu))^*$ is a state if and only if $1 = \psi(1) = \|\psi\|$. (Hint: To show that the latter implies the former, take $f \geq 0$, write $\psi(f) = a + ib$, and estimate $|\psi(f + it)|^2$ as $t \in \mathbb{R}$ varies to show that $b = 0$. To then see that $a \geq 0$, estimate $\psi(\|f\|_\infty - f)$.)

6. Let Γ be a non-trivial discrete group. A left-invariant mean on Γ is a state $\varphi \in (\ell^\infty\Gamma)^*$ such that $\varphi(\lambda_\gamma(f)) = \varphi(f)$ for all $\gamma \in \Gamma$, where $\lambda_\gamma : \ell^\infty\Gamma \to \ell^\infty\Gamma$ is given by $(\lambda_\gamma(f))(\gamma_0) = f(\gamma^{-1}\gamma_0)$.

Suppose $S \subset \Gamma$ is a generating set. Show that Γ has a left-invariant mean if and only if the function $1 \in \ell^\infty\Gamma$ is not contained in the closure of the subspace $X_0 = \operatorname{span}\{f - \lambda_\gamma(f) \mid f \in \ell^\infty\Gamma, \gamma \in S\}$.

7. Show that $\mathbb{Z}$ has a left-invariant mean.