# HOMEWORK 1, MATH 362A - FALL 2014

Due Friday August 29th

1. Let $\mathcal{H}$ be a Hilbert space, and $K \subset \mathcal{H}$, a subspace. Prove that $K^\perp$ is closed, and $(K^\perp)^\perp = \overline{K}$.

2. Let $\mathcal{H}$ be a Hilbert space.

(a). Prove that if $\mu$ is a probability measure on $\mathbb{T}$ such that $\int \lambda \, d\mu(\lambda) = 0$, then for all $\xi, \eta \in \mathcal{H}$ we have the following **parallelogram identity**:

$$\|\xi\|^2 + \|\eta\|^2 = \int \|\xi + \lambda\eta\|^2 \, d\mu(\lambda).$$

(b). Prove that if $\mu$ is a probability measure on $\mathbb{T}$ such that $\int \lambda^2 \, d\mu(\lambda) = \int \lambda \, d\mu(\lambda) = 0$, then for all $\xi, \eta \in \mathcal{H}$ we have the following **polarization identity**:

$$\langle \xi, \eta \rangle = \int \lambda \|\xi + \lambda\eta\|^2 \, d\mu(\lambda).$$

3. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. A linear map $V : \mathcal{H}_1 \to \mathcal{H}_2$ is an **isometry** if $\|V\xi\| = \|\xi\|$ for all $\xi \in \mathcal{H}_1$. Show that an isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ satisfies $\langle V\xi, V\eta \rangle = \langle \xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}_1$.

4. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. A map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is an **isometric affine transformation** if $\|T\xi - T\eta\| = \|\xi - \eta\|$ for all $\xi, \eta \in \mathcal{H}_1$.

(a). If $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces, and if $T : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometric affine transformation, then show that there exists a unique isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$, and a unique vector $\xi \in \mathcal{H}_2$ such that $T = S_\xi \circ V$, where $S_\xi : \mathcal{H}_2 \to \mathcal{H}_2$ is the translation map $S_\xi(\eta) = \eta + \xi$.

(b). Show by example that the above decomposition does not always hold when considering complex Hilbert spaces.

5. Let $X$ be a locally compact Hausdorff space, and suppose $\mu$ is a Radon measure on $X$. Prove that $C_c(X)$ is dense in $L^2(X, \mu)$.

6. Let $\mathcal{H}$ be an infinite dimensional Hilbert space, prove that there does not exist a translation invariant Borel measure on $\mathcal{H}$ which assigns finite positive measure to the unit ball.

7. Let $\mathcal{H}$ be a Hilbert space. Show that any two orthonormal bases have the same cardinality.