1.7.1 Compact representations

If $\mathcal{H}$ is a Hilbert space the strong operator topology on $\mathcal{B}(\mathcal{H})$ is such that $\lim_i T_i = T$ if and only if $\lim_i \|(T_i - T)\xi\| = 0$, for all $\xi \in \mathcal{H}$. The weak operator topology on $\mathcal{B}(\mathcal{H})$ is such that $\lim_i T_i = T$ if and only if $\lim_i \langle (T_i - T)\xi, \eta \rangle = 0$, for all $\xi, \eta \in \mathcal{H}$. The unitary group $\mathcal{U}(\mathcal{H})$ then becomes a topological group when endowed with the strong operator topology. We also note that if $U_i \in \mathcal{U}(\mathcal{H})$ is a net of unitaries such that $U_i \to U \in \mathcal{U}(\mathcal{H})$ in the weak operator topology, then we also have that $U_i \to U$ in the strong operator topology. Indeed, for any $\xi \in \mathcal{H}$ we have

$$\|(U_i - U)\xi\|^2 = 2\|\xi\|^2 - 2\Re(\langle U_i\xi, U\xi\rangle) \to 0.$$

Definition 1.7.8. Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **compact** if $\pi(\Gamma) \subset \mathcal{U}(\mathcal{H})$ is pre-compact in the strong operator topology.

Lemma 1.7.9. *Let (X, d) be a compact metric space, then* $\mathrm{Isom}(X, d)$ *with the topology of point-wise convergence is compact.*

Proof. By Tychonoff's Theorem X^X is compact, thus we need only to show that $\mathrm{Isom}(X, d) \subset X^X$ is closed. Suppose $g \in \overline{\mathrm{Isom}(X, d)}$, for all $x, y \in X$ we have that $\{f \in X^X \mid d(f(x), f(y)) = d(x, y)\}$ is closed and contains $\mathrm{Isom}(X, d)$, hence g is isometric.

For each $x \in X$ denote by d_x the distance from x to $g(X)$. Then we have that for all $m \in \mathbb{N}$, $d(x, g^m(x)) \geq d_x$, hence for all $n, m \in \mathbb{N}$, $n < m$ we have $d(g^n(x), g^m(x)) = d(x, g^{m-n}(x)) \geq d_x$. Since X is compact, it must be totally bounded and hence we must have that $d_x = 0$. Hence g is surjective and thus $\mathrm{Isom}(X, d)$ is compact. □

Lemma 1.7.10. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a group Γ. Then π is compact if and only if for each $\xi \in \mathcal{H}$, the orbit $\pi(\Gamma)\xi$ is pre-compact in $\mathcal{H}$.*

Proof. If $\pi(\Gamma)$ is pre-compact in the strong operator topology and $G = \overline{\pi(\Gamma)}$, then for each $\xi \in \mathcal{H}$ we have that $\overline{\pi(\Gamma)\xi} = G\xi$ is compact, being the continuous image of the compact set G.

Conversely, suppose that each orbit $\pi(\Gamma)\xi \subset \mathcal{H}$ is pre-compact. By Zorn's Lemma we can find a collection of vectors $\mathcal{J} \subset \mathcal{H}$ such that $\mathcal{H} = \oplus_{\xi \in \mathcal{J}} \overline{\mathrm{sp}}\pi(\Gamma)\xi$.

We therefore have a strong operator topology continuous embedding of $\pi(\Gamma)$ into the compact space $\Pi_{\xi \in \mathcal{J}} \mathrm{Isom}(\overline{\pi(\Gamma)\xi}, d_{\mathcal{H}}) \subset \mathcal{U}(\oplus_{\xi \in \mathcal{J}} \overline{\mathrm{sp}}\pi(\Gamma)\xi)$, hence $\pi(\Gamma)$ is pre-compact. □

The following is part of the Peter-Weyl Theorem.

Theorem 1.7.11. *Let G be a compact group, and let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a strong operator topology continuous unitary representation. Then π decomposes as a direct sum of finite dimensional representations.*

Proof. Let λ denote the Haar measure on G. Since every representation decomposes into a direct sum of cyclic representations we may assume that the representation π has a cyclic vector $\zeta \in \mathcal{H}$. We then define an operator $K \in \mathcal{B}(\mathcal{H})$ such that for all $\xi, \eta \in \mathcal{H}$ we have

$$\langle K\xi, \eta \rangle = \int_G \langle \pi(g)\xi, \zeta \rangle \langle \zeta, \pi(g)\eta \rangle \, d\lambda(g).$$

First, note that from left invariance of the Haar measure we have that $\pi(g)K\pi(g^{-1}) = K$, for all $g \in G$. Also, if $\xi \in \mathcal{H}$, then we have

$$\langle K\xi, \xi \rangle = \int_G |\langle \pi(g)\xi, \zeta \rangle|^2 \, d\lambda(g) \geq 0.$$

Thus, K is positive.

Moreover, if $\langle K\xi, \xi \rangle = 0$ then we have that $\langle \pi(g)\xi, \zeta \rangle = \langle \xi, \pi(g^{-1})\zeta \rangle = 0$, for all $g \in G$. This then implies that $\xi = 0$ since ζ is a cyclic vector. Thus K is strictly positive.

If $\xi_n \in \mathcal{H}$ is a bounded sequence which weakly converges to 0, then

$$\lim_{n\to\infty} \|K\xi_n\|^2 = \lim_{n\to\infty} \int_G \langle \pi(g)\xi_n, \zeta \rangle \langle \zeta, \pi(g)K\xi_n \rangle \, d\lambda(g)$$

$$= \lim_{n\to\infty} \int_G \int_G \langle \pi(g)\xi_n, \zeta \rangle \langle \zeta, \pi(h)\xi_n \rangle \langle \pi(h)\pi(g^{-1})\zeta, \zeta \rangle \, d\lambda(g) \, d\lambda(h) = 0.$$

We therefore have shown that K is a compact operator. Hence, $\mathcal{H}$ decomposes as a direct sum of the (finite dimensional) eigenspaces of K, each of which is G-invariant. □

Lemma 1.7.10 and Theorem 1.7.11 then give us a structural result for compact representations.

Corollary 1.7.12. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a representation of a group Γ. Then π is compact if and only if π decomposes as a direct sum of finite dimensional representations.*

Proposition 1.7.13. *Let Γ be a group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then there is a unique Γ-invariant closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $\pi_{|\mathcal{K}}$ is compact and $\pi_{|\mathcal{K}^\perp}$ is weak mixing.*

Proof. Let $\mathcal{Z}$ be the set of all orthonormal sets $\mathcal{J} \subset \mathcal{H}$ such that $\mathrm{sp}(\pi(\Gamma)\xi)$ is finite dimensional for all $\xi \in \mathcal{J}$ and $\mathrm{sp}(\pi(\Gamma)\xi) \perp \mathrm{sp}(\pi(\Gamma)\eta)$ for all $\xi, \eta \in \mathcal{J}$, $\xi \neq \eta$. If we order $\mathcal{Z}$ by inclusion then it is easy to see that the union of any increasing chain in $\mathcal{Z}$ is again in $\mathcal{Z}$.

If π is not weakly mixing then by Proposition 1.7.5 we have that $\mathcal{Z} \neq \emptyset$, hence by Zorn's Lemma there is a maximal element $\mathcal{J} \in \mathcal{Z}$. Let $\mathcal{K} = \Sigma_{\xi \in \mathcal{J}} \mathrm{sp}(\pi(\Gamma)\xi) \subset \mathcal{H}$, then $\pi_{|\mathcal{K}} \cong \oplus_{\xi \in \mathcal{J}} \pi_{|\mathrm{sp}(\pi(\Gamma)\xi)}$ is compact, and by maximality of $\mathcal{J}$ we have that $\pi_{|\mathcal{K}^\perp}$ contains no finite dimensional sub-representation, and hence is weakly mixing.

If $\mathcal{K}_0 \subset \mathcal{H}$ is a finite dimensional Γ-invariant subspace, then $\mathrm{Proj}_{\mathcal{K}^\perp}(\mathcal{K}_0) \subset \mathcal{K}^\perp$ is also a finite dimensional Γ-invariant subspace. Since $\pi_{|\mathcal{K}^\perp}$ is weak mixing it then follows that $\mathrm{Proj}_{\mathcal{K}^\perp}(\mathcal{K}_0) = \{0\}$ and hence $\mathcal{K}_0 \subset \mathcal{K}$. This then implies uniqueness of the decomposition. $\square$

jesse.d.peterson@vanderbilt.edu