1.7 Mixing properties

Definition 1.7.1. Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **weak mixing** if for each finite set $\mathcal{F} \subset \mathcal{H}$, and $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that

$$|\langle \pi(\gamma)\xi, \xi \rangle| < \varepsilon,$$

for all $\xi \in \mathcal{F}$.

The representation π is (**strong**) **mixing** if $|\Gamma| = \infty$, and for each finite set $\mathcal{F} \subset \mathcal{H}$, we have

$$\lim_{\gamma \to \infty} |\langle \pi(\gamma)\xi, \xi \rangle| = 0.$$

These definitions should be compared with the characterization of ergodicity found in Proposition 1.5.2. Hence we see that mixing implies weak mixing, which in turn implies ergodic. It is also easy to see that if $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is mixing (resp. weak mixing) then so is $\pi^{\oplus\infty}$, and if π is mixing then so is $\pi \otimes \rho$ for any representation ρ. We'll see below in Corollary 1.7.6 that weak mixing is also stable under tensoring.

Exercise 1.7.2. Let Γ be a group and $\Sigma < \Gamma$ a subgroup. Show that the quasi-regular representation of Γ on $\ell^2(\Gamma/\Sigma)$ is weak mixing if and only if it is ergodic, if and only if $[\Gamma : \Sigma] = \infty$. Show that it is mixing if and only if $|\Gamma| = \infty$ and $|\Sigma| < \infty$.

Exercise 1.7.3. Let $\Gamma \curvearrowright X$ be an action of a group Γ on a set X, let Λ be a group and let $\alpha : \Gamma \times X \to \Lambda$ be a cocycle. Suppose $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ is a unitary representation. Find necessary and sufficient conditions for the induced representation of Section 1.4 to be mixing.

Lemma 1.7.4. *Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is weak mixing if and only if for each finite set $\mathcal{F} \subset \mathcal{H}$, and $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that*

$$|\langle \pi(\gamma)\xi, \eta \rangle| < \varepsilon,$$

for all $\xi, \eta \in \mathcal{F}$.

The representation π is mixing if $|\Gamma| = \infty$ and for each finite set $\mathcal{F} \subset \mathcal{H}$, we have

$$\lim_{\gamma \to \infty} |\langle \pi(\gamma)\xi, \eta \rangle| = 0,$$

for all $\xi, \eta \in \mathcal{F}$.

Proof. This follows from the polarization identity. For each $\xi, \eta \in \mathcal{H}$, and $\gamma \in \Gamma$ we have

$$\langle \pi(\gamma)\xi, \eta \rangle = \frac{1}{4}\Sigma_{k=0}^{3} i^k \langle \pi(\gamma)(\xi + i^k\eta), (\xi + i^k\eta) \rangle.$$

□

We now add another equivalent condition to Proposition 1.5.6.

Proposition 1.7.5. *Let Γ be a group, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then π is weak mixing if and only if π contains no finite dimensional sup-representations.*

Proof. If π is weak mixing then if $\mathcal{L} \subset \mathcal{H}$ is a non-trivial, finite dimensional subspace with orthonormal basis $\mathcal{F} \subset \mathcal{H}$, there exists $\gamma \in \Gamma$ such that $|\langle \pi(\gamma)\xi, \eta \rangle| < 1/\sqrt{\dim(\mathcal{L})}$, for all $\xi, \eta \in \mathcal{F}$. Hence, if $\xi \in \mathcal{F}$ then $\| \operatorname{Proj}_{\mathcal{L}}(\pi(\gamma)\xi)\| < 1 = \|\xi\|$. Thus, $\mathcal{L}$ is not an invariant subspace.

Conversely, If π has no finite dimensional invariant subspaces, $\mathcal{L} \subset \mathcal{H}$ is a finite dimensional subspace, and $\varepsilon > 0$, then there exists $\gamma \in \Gamma$ such that for all $\xi \in \mathcal{L}$ we have that

$$\| \operatorname{Proj}_{\mathcal{L}}(\pi(\gamma)\xi)\| < \varepsilon.$$

Indeed, if this is not the case then we would have that there exists $c > 0$ such that

$$\langle \pi(\gamma^{-1}) \operatorname{Proj}_{\mathcal{L}} \pi(\gamma), \operatorname{Proj}_{\mathcal{L}} \rangle_{\mathrm{HS}} \geq \sup_{\xi \in \mathcal{L}, \|\xi\|=1} \| \operatorname{Proj}_{\mathcal{L}}(\pi(\gamma)\xi)\|^2 \geq c.$$

It would then follow Proposition 1.5.2 that there is a non-zero Hilbert-Schmidt operator T such that $\pi(\gamma)T\pi(\gamma^{-1}) = T$, for all $\gamma \in \Gamma$. Proposition 1.5.6 would then give a contradiction.

Thus, if $\mathcal{F} \subset \mathcal{H}$ is finite such that $\|\xi\| \leq 1$ for each $\xi \in \mathcal{F}$, then by considering the finite dimensional subspace $\mathcal{L}$ spanned by $\mathcal{F}$ we have shown that there exists $\gamma \in \Gamma$ such that for all $\xi, \eta \in \mathcal{F}$ we have

$$|\langle \pi(\gamma)\xi, \eta \rangle| \leq \| \operatorname{Proj}_{\mathcal{L}}(\pi(\gamma)\xi)\| \|\eta\| < \varepsilon.$$

□

From the above proposition together with the equivalence between conditions 2) and 3) in Proposition 1.5.6 we obtain the following.

Corollary 1.7.6. *Let Γ be a group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then π is weak mixing if and only if $\pi \otimes \overline{\pi}$ is weak mixing, if and only if $\pi \overline{\otimes} \rho$ is weak mixing for all unitary representations ρ.*

Corollary 1.7.7. *Let Γ be a group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a weak mixing unitary representation. If $\Sigma < \Gamma$ is a finite index subgroup then $\pi_{|\Sigma}$ is also weak mixing.*

Proof. Let $D \subset \Gamma$ be a set of coset representatives for Σ. If $\pi_{|\Sigma}$ is not mixing, then by Proposition 1.7.5 there is a finite dimensional subspace $\mathcal{L} \subset \mathcal{H}$ which is Σ-invariant. We then have that $\Sigma_{\gamma \in D} \pi(\gamma)(\mathcal{L}) \subset \mathcal{H}$ is finite dimensional and Γ-invariant. Hence, again by Proposition 1.7.5, π is not weak mixing. □

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