1.6 Amenability

Much of the material of this section has been taken from Section 2.6 in the book of Brown and Ozawa [BO08].

Definition 1.6.1. Let Γ be a group, a **Følner net** is a net of non-empty finite subsets $F_i \subset \Gamma$ such that $|F_i \Delta \gamma F_i|/|F_i| \to 0$, for all $\gamma \in \Gamma$.

Note that we do not require that $\Gamma = \cup_i F_i$, nor do we require that F_i are increasing, however, if $|\Gamma| = \infty$ then it is easy to see that any Følner net $\{F_i\}_i$ must satisfy $|F_i| \to \infty$.

Exercise 1.6.2. Let Γ be a group, show that if Γ has a Følner net, then for each finite set $E \subset \Gamma$, and $\varepsilon > 0$, there exists a finite set $F \subset \Gamma$ such that $E \subset F$, $F = F^{-1}$, and $|F \Delta \gamma F|/|F|, |F \Delta F \gamma|/|F| < \varepsilon$.

Definition 1.6.3. A **mean** m on a non-empty set X is a finitely additive probability measure on 2^X, i.e., $m : 2^X \to [0,1]$ such that $m(X) = 1$, and if $A_1, \ldots, A_n \subset X$ are disjoint then $m(\cup_{j=1}^n A_n) = \Sigma_{j=1}^n m(A_n)$.

Given a mean m on X it is possible to define an integral over X just as in the case if m were a measure. We therefore obtain a state $\phi_m \in (\ell^\infty X)^*$ by the formula $\phi_m(f) = \int_X f \, dm$. Conversely, if $\phi \in (\ell^\infty X)^*$ is a state, then restricting ϕ to characteristic functions defines a corresponding mean.

If $\Gamma \curvearrowright X$ is an action, then an invariant mean m on X is a mean such that $m(\gamma A) = m(A)$ for all $A \subset X$. An **approximately invariant mean** is a net of probability measures $\mu_i \in \mathrm{Prob}(X)$, such that $\|\gamma_* \mu_i - \mu_i\|_1 \to 0$, for all $\gamma \in \Gamma$.

Definition 1.6.4. Let Γ be a group, Γ is **amenable** if it Γ has a mean which is invariant under the action of left multiplication, or equivalently, if there is a state on $\ell^\infty \Gamma$ which is invariant under the action of left multiplication.

Amenable groups were first introduced by von Neumann [vN29]. The term amenable was coined by M. M. Day.

Theorem 1.6.5. *Let Γ be a group. The following conditions are equivalent.*

(1). Γ *is amenable.*

(2). Γ *has an approximate invariant mean under the action of left multiplication.*

(3). Γ *has a Følner net.*

(4). *The left regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2 \Gamma)$ contains almost invariant vectors.*

(5). *For any finite symmetric set $S \subset \Gamma$ the operator $T_S = \frac{1}{|S|} \Sigma_{\gamma \in S} \lambda(\gamma)$ does not contain 1 in its spectrum.*

(6). *There exists a state $\Phi \in (\mathcal{B}(\ell^2 \Gamma))^*$ such that $\Phi(\lambda(\gamma) T) = \Phi(T \lambda(\gamma))$ for all $\gamma \in \Gamma$, $T \in \mathcal{B}(\ell^2 \Gamma)$.*

(7). *The continuous action of* Γ *on its Stone-Čech compactification* $\beta\Gamma$ *which is induced by left-multiplication admits an invariant Radon probability measure.*

(8). *Any continuous action* $\Gamma \curvearrowright K$ *on a compact Hausdorff space* K *admits an invariant Radon probability measure.*

Proof. We show (1) $\implies$ (2) using the method of Day [Day57]. Since $\ell^\infty\Gamma = (\ell^1\Gamma)^*$, the unit ball in $\ell^1\Gamma$ is wk*-dense in the unit ball of $(\ell^\infty\Gamma)^* = (\ell^1\Gamma)^{**}$. It follows that $\mathrm{Prob}(\Gamma) \subset \ell^1\Gamma$ is wk*-dense in the state space of $\ell^\infty\Gamma$.

Let $S \subset \Gamma$, be finite and let $K \subset \oplus_{\gamma\in S}\ell^1\Gamma$ be the wk-closure of the set $\{\oplus_{\gamma\in S}(\gamma_*\mu - \mu) \mid \mu \in \mathrm{Prob}(\Gamma)\}$. From the remarks above, since Γ has a left invariant state on $\ell^\infty\Gamma$, we have that $0 \in K$. However, K is convex and so by the Hahn-Banach Separation Theorem the wk-closure coincides with the norm closure. Thus, for any $\varepsilon > 0$ there exists $\mu \in \mathrm{Prob}(\Gamma)$ such that

$$\Sigma_{\gamma\in S}\|\gamma_*\mu - \mu\|_1 < \varepsilon.$$

We show (2) $\implies$ (3) using the method of Namioka [Nam64]. Let $S \subset \Gamma$ be a finite set, and denote by E_r the characteristic function on the set (r, ∞). If $\mu \in \mathrm{Prob}(\Gamma)$ then we have

$$\Sigma_{\gamma\in S}\|\gamma_*\mu - \mu\|_1 = \Sigma_{\gamma\in S}\Sigma_{x\in\Gamma}|\gamma_*\mu(x) - \mu(x)|$$

$$= \Sigma_{\gamma\in S}\Sigma_{x\in\Gamma}\int_{\mathbb{R}_{\geq 0}} |E_r(\gamma_*\mu(x)) - E_r(\mu(x))|\, dr$$

$$= \Sigma_{\gamma\in S}\int_{\mathbb{R}_{\geq 0}} \Sigma_{x\in\Gamma}|E_r(\gamma_*\mu(x)) - E_r(\mu(x))|\, dr$$

$$= \Sigma_{\gamma\in S}\int_{\mathbb{R}_{\geq 0}} \|E_r(\gamma_*\mu) - E_r(\mu)\|_1\, dr.$$

By hypothesis, if $\varepsilon > 0$ then there exists $\mu \in \mathrm{Prob}(\Gamma)$ such that $\Sigma_{\gamma\in S}\|\gamma_*\mu - \mu\|_1 < \varepsilon$, and hence for this μ we have

$$\Sigma_{\gamma\in S}\int_{\mathbb{R}_{\geq 0}} \|E_r(\gamma_*\mu) - E_r(\mu)\|_1\, dr < \varepsilon = \varepsilon\int_{\mathbb{R}_{\geq 0}} \|E_r(\mu)\|_1\, dr.$$

Hence, if we denote by $F_r \subset \Gamma$ the (finite) support of $E_r(\mu)$, then for some particular $r > 0$ we must have

$$\Sigma_{\gamma\in S}|\gamma F_r \Delta F_r| = \Sigma_{\gamma\in S}\|E_r(\gamma_*\mu) - E_r(\mu)\|_1 < \varepsilon\|E_r(\mu)\|_1 = \varepsilon|F_r|.$$

For (3) $\implies$ (4) just notice that if $F_i \subset \Gamma$ is a Følner net, then $\frac{1}{|F_i|^{1/2}}\Sigma_{x\in F_i}\delta_x \in \ell^2\Gamma$ is a net of almost invariant vectors.

(4) $\iff$ (5) follows from Proposition 1.5.5.

For (4) $\implies$ (6) let $\xi_i \in \ell^2\Gamma$ be a net of almost invariant vectors for λ. We define states Φ_i on $\mathcal{B}(\ell^2\Gamma)$ by $\Phi_i(T) = \langle T\xi_i, \xi_i\rangle$. By wk-compactness of the state space, we may take a subnet and assume that this converges in the weak topology to $\Phi \in \mathcal{B}(\ell^2\Gamma)^*$. We then have that for all $T \in \mathcal{B}(\ell^2\Gamma)$ and $\gamma \in \Gamma$,

$$|\Phi(\lambda(\gamma)T - T\lambda(\gamma))| = \lim_i |\langle(\lambda(\gamma)T - T\lambda(\gamma))\xi_i, \xi_i\rangle|$$

$$= \lim_i |\langle T\xi_i, \lambda(\gamma^{-1})\xi_i\rangle - \langle T\lambda(\gamma)\xi_i, \xi_i\rangle|$$

$$\leq \lim_i \|T\|(\|\lambda(\gamma^{-1})\xi_i - \xi_i\| + \|\lambda(\gamma)\xi_i - \xi_i\|) = 0.$$

For (6) $\implies$ (1), we have a natural embedding $M : \ell^\infty\Gamma \to \mathcal{B}(\ell^2\Gamma)$ as "diagonal matrices", i.e., for a function $f \in \ell^\infty\Gamma$ we have $M_f(\Sigma_{x\in\Gamma}\alpha_x\delta_x) = \Sigma_{x\in\Gamma}\alpha_x f(x)\delta_x$. Moreover, for $f \in \ell^\infty\Gamma$ and $\gamma \in \Gamma$ we have $\lambda(\gamma)M_f\lambda(\gamma^{-1}) = M_{\gamma\cdot f}$. Thus, if $\Phi \in \mathcal{B}(\ell^2\Gamma)^*$ is a state which is invariant under the conjugation by $\lambda(\gamma)$, then restricting this state to $\ell^\infty\Gamma$ gives a state on $\ell^\infty\Gamma$ which is Γ-invariant.

For (1) $\implies$ (7), the map $\beta : \ell^\infty\Gamma \to C(\beta\Gamma)$ which takes a bounded function on Γ to its unique continuous extension on $\beta\Gamma$, is a C^*-algebra isomorphism, which is Γ-equivariant. Hence amenability of Γ implies the existence of a Γ-invariant state on $C(\beta\Gamma)$. The Riesz Representation Theorem then gives an invariant probability measure on $\beta\Gamma$.

For (7) $\iff$ (8), suppose Γ acts continuously on a compact Hausdorff space K, and fix a point $x_0 \in K$. Then the map $f(\gamma) = \gamma x_0$ on Γ extends uniquely to a continuous map $\beta f : \beta\Gamma \to K$, moreover since f is Γ-equivariant, so is βf. If μ is an invariant Radon probability measure for the action on $\beta\Gamma$ then we obtain the invariant Radon probability measure $(\beta f)_*\mu$ on K. Since $\beta\Gamma$ itself is compact, the converse is trivial.

For (7) $\implies$ (1), if there is a Γ-invariant Radon probability measure μ on $\beta\Gamma$, then we obtain an invariant mean m on Γ my setting $m(A) = \mu(\overline{A})$. $\square$

Example 1.6.6. Any finite group is amenable, and any group which is locally amenable (each finitely generated subgroup is amenable) is also amenable. The group of integers $\mathbb{Z}$ is amenable (consider the Følner sequence $F_n = \{1, \ldots, n\}$ for example). From this it follows easily that all finitely generated abelian groups are amenable, and hence all abelian groups are.

It is also easy to see from the definition that subgroups of amenable groups are amenable. A bit more difficult is to show that if $1 \to \Sigma \to \Gamma \to \Lambda \to 1$ is an exact sequence of groups then Γ is amenable if and only if both Σ and Λ are amenable. Thus all solvable groups, and even all nilpotent groups are amenable.

There are also finitely generated amenable groups which cannot be constructed from finite, and abelian groups using only the operations above [Gri84].

Example 1.6.7. Let $\mathbb{F}_2$ be the free group on two generators a, and b. Let A^+ be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a, let A^- be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a^{-1}, let B^+, and B^- be defined analogously, and consider $C = \{e, b, b^2, \ldots\}$. Then we have that

$$\mathbb{F}_2 = A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C)$$

$$= A^+ \sqcup aA^-$$
$$= b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C).$$

If m were a left-invariant mean on $\mathbb{F}_2$ then we would have

$$m(\mathbb{F}_2) = m(A^+) + m(A^-) + m(B^+ \setminus C) + m(B^- \cup C)$$
$$= m(A^+) + m(aA^-) + m(b^{-1}(B^+ \setminus C)) + m(B^- \cup C)$$
$$= m(A^+ \sqcup aA^-) + m(b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C)) = 2m(\mathbb{F}_2).$$

Hence, $\mathbb{F}_2$ and also any group containing $\mathbb{F}_2$ is not-amenable. There are also finitely generated nonamenable groups which do not contain $\mathbb{F}_2$[Ols80], and even finitely presented nonamenable groups which do not contain $\mathbb{F}_2$ [OS02].

1.6.1 Von Neumann's Mean Ergodic Theorem

Amenable groups allow for nice averaging properties, we give such an example here.

Theorem 1.6.8. *Let Γ be an amenable group with Følner net $F_i \subset \Gamma$, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Let $P_0 \in \mathcal{B}(\mathcal{H})$ be the projection onto the subspace of Γ invariant vectors. Then for each $\xi \in \mathcal{H}$ we have that*

$$\| \frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} \pi(\gamma)\xi - P_0(\xi) \| \to 0.$$

Proof. By considering the vector $\xi - P_0(\xi)$ instead, we may assume that $P_0(\xi) = 0$.

Note that if $\gamma \in \Gamma$, and $\eta \in \mathcal{H}$ then $\eta - \pi(\gamma)\eta$ is orthogonal to the space of invariant vectors. Indeed, if $\zeta \in \mathcal{H}$ is an invariant vector, then $\langle \eta - \pi(\gamma)\eta, \zeta \rangle = \langle \eta, \zeta - \pi(\gamma^{-1})\zeta \rangle = 0$.

Denote by $\mathcal{L}$ the closure of the subspace of $\mathcal{H}$ spanned by vectors of the form $\eta - \pi(\gamma)\eta$ for $\eta \in \mathcal{H}$, $\gamma \in \Gamma$. Then if $\zeta \in \mathcal{L}^\perp$ we have that $0 = \langle \zeta, \eta - \pi(\gamma)\eta \rangle = \langle \zeta - \pi(\gamma^{-1})\zeta, \eta \rangle$ for all $\gamma \in \Gamma$, and $\eta \in \mathcal{H}$, thus $\zeta - \pi(\gamma)\zeta = 0$, for all $\gamma \in \Gamma$. Therefore we have shown that $\mathcal{L}^\perp$ is precisely the space of invariant vectors.

Fix $\gamma_0 \in \Gamma$, and $\eta \in \mathcal{H}$, then if $\xi = \eta - \pi(\gamma_0)\eta$ we have that

$$\| \frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} \pi(\gamma)\xi \| = \| \frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} (\pi(\gamma)\eta - \pi(\gamma\gamma_0)\eta) \|$$
$$\leq (|F_i \Delta \gamma_0 F_i| / |F_i|) \|\eta\| \to 0.$$

Since $\frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} \pi(\gamma) \in \mathcal{B}(\mathcal{H})$ is always a contraction we may then pass to the closure of the span to conclude that for all $\xi \in \mathcal{L}$ we have

$$\| \frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} \pi(\gamma)\xi \| \to 0.$$

□

In the case when the representation π is ergodic, $\Gamma = \mathbb{Z}$, and the we consider the Følner sequence $F_n = \{0, -1, \dots, -n+1\}$, the above theorem then gives the following corollary.

Corollary 1.6.9. *Let $\pi : \mathbb{Z} \to \mathcal{U}(\mathcal{H})$ be a unitary representation. If π is ergodic, then for each $\xi \in \mathcal{H}$ we have that*

$$\lim_{n\to\infty} \| \frac{1}{n} \Sigma_{k=0}^{n-1} \pi(k)\xi \| = 0.$$

We remark that from the perspective of Example 1.3.2 another possible generalization of Corollary 1.6.9 could be given in terms of cocycles. This perspective would then lead to Corollary 3.3 in [CTV07] the proof of which is quite similar to the proof of von Neumann's Ergodic Theorem given above.

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