1.5 Invariant vectors

Definition 1.5.1. Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ contains **invariant vectors** if there exists a non-zero vector $\xi \in \mathcal{H}$ such that $\pi(\gamma)\xi = \xi$ for all $\gamma \in \Gamma$. The representation contains **almost invariant vectors** if for each $F \subset \Gamma$, and $\varepsilon > 0$, there exists $\xi \in \mathcal{H}$, such that

$$\|\pi(\gamma)\xi - \xi\| < \varepsilon\|\xi\|, \text{ for all } \gamma \in F.$$

We will also say that a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **ergodic** if it does not contain invariant vectors. In general, if we denote by $\mathcal{H}_0 \subset \mathcal{H}$ the subspace of Γ-invariant vectors, then we say the representation π has **spectral gap** if $\mathcal{H}_0 \neq \mathcal{H}$ and the sub-representation $\pi_{|\mathcal{H}_0^\perp}$ does not contain almost invariant vectors.

Proposition 1.5.2. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c > 0$ such that $\Re(\langle \pi(\gamma)\xi, \xi\rangle) > c\|\xi\|^2$ for all $\gamma \in \Gamma$, then π contains an invariant vector.*

Proof. Let K be the closed convex hull of the orbit $\pi(\Gamma)\xi$. We therefore have that K is Γ-invariant and $\Re(\langle \eta, \xi\rangle) \geq c\|\xi\|^2$ for every $\eta \in K$. Let $\xi_0 \in K$ be the unique element of minimal norm, then since Γ acts isometrically we have that for each $\gamma \in \Gamma$, $\pi(\gamma)\xi_0$ is the unique element of minimal norm for $\pi(\gamma)K = K$, and hence $\pi(\gamma)\xi_0 = \xi_0$ for each $\gamma \in \Gamma$. Since $\xi_0 \in K$ we have that $\Re(\langle \xi_0, \xi\rangle) \neq 0$, and hence $\xi_0 \neq 0$. □

Corollary 1.5.3. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c < \sqrt{2}$ such that $\|\pi(\gamma)\xi - \xi\| < c\|\xi\|$ for all $\gamma \in \Gamma$, then π contains an invariant vector.*

Proof. For each $\gamma \in \Gamma$ we have

$$2\Re(\langle \pi(\gamma)\xi, \xi\rangle) = 2\|\xi\|^2 - \|\pi(\gamma)\xi - \xi\|^2 \geq (2 - c^2)\|\xi\|^2.$$

Hence, we may apply Proposition 1.5.2. □

Lemma 1.5.4. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. Then π contains almost invariant vectors if and only if $\pi^{\oplus n}$ contains almost invariant vector, where $n \geq 1$ is any cardinal number.*

Proof. If π does not contain almost invariant vectors then there exists $c > 0$, and $S \subset \Gamma$ finite, such that for all $\xi \in \mathcal{H}$ we have

$$c\|\xi\|^2 \leq \Sigma_{\gamma \in S}\|\pi(\gamma)\xi - \xi\|^2.$$

If I is a set $|I| = n$, and $\xi_i \in \mathcal{H}$ for $i \in I$, such that $\Sigma_{i\in I}\|\xi_i\|^2 < \infty$, then

$$c\| \oplus_{i\in I} \xi_i\|^2 = \Sigma_{i\in I} c\|\xi_i\|^2$$

$$\leq \Sigma_{i\in I}\Sigma_{\gamma\in S}\|\pi(\gamma)\xi_i - \xi_i\|^2 = \Sigma_{\gamma\in S}\|\pi^{\oplus n}(\gamma)(\oplus_{i\in I}\xi_i) - \oplus_{i\in I}\xi_i\|^2.$$

Hence, $\pi^{\oplus n}$ does not contain almost invariant vectors. The converse is trivial since π is contained in $\pi^{\oplus\infty}$. □

Proposition 1.5.5. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. Then π contains almost invariant vectors if and only if for any finite symmetric set $S \subset \Gamma$ the operator $T_S = \frac{1}{|S|}\Sigma_{\gamma \in S}\pi(\gamma)$ satisfies $\|T_S\| = 1$.*

Proof. If π contains almost invariant vectors then the triangle inequality easily implies that 1 is in the spectrum of T_S, for each finite symmetric set $S \subset \Gamma$.

Conversely, if $S \subset \Gamma$ is a finite symmetric set with $e \in S$ and $\|T_S\| = 1$, then since T_S is self-adjoint either 1 or -1 is contained in its spectrum, however since $e \in S$ it is easy to see that $-1 \notin \sigma(T_S)$, hence for any $\varepsilon > 0$ there exists $\xi \in \mathcal{H}$, $\|\xi\| = 1$ such that

$$|1 - \langle T_S\xi, \xi\rangle| < \varepsilon^2/2|S|.$$

We then have that for each $\gamma \in S$

$$\|\xi - \pi(\gamma)\xi\|^2 = 2|1 - \Re\langle\pi(\gamma)\xi, \xi\rangle|$$

$$\leq 2|S||1 - \langle T_S\xi, \xi\rangle| < \varepsilon^2.$$

Since, ε and S were arbitrary this shows that π contains almost invariant vectors. □

Proposition 1.5.6. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. The following are equivalent:*

(1). *The representation $\pi \otimes \overline{\pi}$ contains invariant vectors.*

(2). *The representation $\pi \otimes \lambda$ contains invariant vectors for some unitary representation $\lambda : \Gamma \to \mathcal{U}(\mathcal{K})$.*

(3). *The representation π contains a finite dimensional sub-representation.*

Proof. (1) $\implies$ (2) is obvious.

To show (2) $\implies$ (3) suppose $\lambda : \Gamma \to \mathcal{U}(\mathcal{K})$ is a unitary representation such that $\pi \otimes \lambda$ contains invariant vectors. Identifying $\mathcal{H}\overline{\otimes}\mathcal{K}$ with the space of Hilbert-Schmidt operators $\mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$ we then have that there exists $T \in \mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$, non-zero, such that $\pi(\gamma)T\overline{\rho}(\gamma^{-1}) = T$, for all $\gamma \in \Gamma$. Then $TT^* \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ is positive, non-zero, compact, and $\pi(\gamma)TT^*\pi(\gamma^{-1}) = TT^*$, for all $\gamma \in \Gamma$. By taking the range of a non-trivial spectral projection of TT^* we then obtain a finite dimensional invariant subspace of π.

(3) $\implies$ (1) follows because if $\mathcal{L} \subset \mathcal{H}$ is a finite dimensional invariant subspace then $\mathrm{Proj}_{\mathcal{L}}$ is a finite rank projection such that $\pi(\gamma)\,\mathrm{Proj}_{\mathcal{L}}\,\pi(\gamma^{-1}) = \mathrm{Proj}_{\mathcal{L}}$, for all $\gamma \in \Gamma$. By identifying $\mathrm{HS}(\mathcal{H}, \mathcal{H})$ with $\mathcal{H}\overline{\otimes}\overline{\mathcal{H}}$, we then obtain a non-zero invariant vector for $\pi \otimes \overline{\pi}$. □

jesse.d.peterson@vanderbilt.edu