1.4 Induced representations

Given an action $\Gamma \curvearrowright X$ of a group Γ on a set X, a cocycle $\alpha : \Gamma \times X \to \Lambda$ into a group Λ, and a unitary representation $\pi : \Lambda \to \mathcal{U}(\mathcal{K})$, we obtain an induced representation $\operatorname{Ind}_\Lambda^\alpha \pi : \Gamma \to \mathcal{U}(\ell^2 X \overline{\otimes} \mathcal{K})$ by linearly extending the formula

$$\operatorname{Ind}_\Lambda^\alpha \pi(\gamma)(\delta_x \otimes \eta) = \delta_{\gamma x} \otimes (\pi(\alpha(\gamma, x))\eta).$$

We can easily check that this is a representation since for all $\gamma_1, \gamma_2 \in \Gamma$, $x \in X$, and $\eta \in \mathcal{K}$ we have

$$\operatorname{Ind}_\Lambda^\alpha \pi(\gamma_1\gamma_2)(\delta_x \otimes \eta) = \delta_{\gamma_1\gamma_2 x} \otimes (\pi(\alpha(\gamma_1\gamma_2, x))\eta)$$

$$= \delta_{\gamma_1\gamma_2 x} \otimes (\pi(\alpha(\gamma_1, \gamma_2 x))\pi(\alpha(\gamma_2, x))\eta)$$

$$= \operatorname{Ind}_\Lambda^\alpha \pi(\gamma_1)(\delta_{\gamma_2 x} \otimes (\pi(\alpha(\gamma_2, x))\eta)$$

$$= \operatorname{Ind}_\Lambda^\alpha \pi(\gamma_1) \operatorname{Ind}_\Lambda^\alpha \pi(\gamma_2)(\delta_x \otimes \eta).$$

If $\beta : \Gamma \times X \to \Lambda$ is cocycle which is cohomologous to α, and $\xi : X \to \Lambda$ such that $\alpha(\gamma, x) = \xi(\gamma x)\beta(\gamma, x)\xi(x)^{-1}$, for all $\gamma \in \Gamma$, and $x \in X$, then we obtain a unitary $U_\xi \in \mathcal{U}(\ell^2 X \overline{\otimes} \mathcal{K})$, by linearly extending the formula

$$U_\xi(\delta_x \otimes \eta) = \delta_x \otimes \pi(\xi(x)^{-1})\eta.$$

We then see easily that for all $\gamma \in \Gamma$, $x \in X$, and $\eta \in \mathcal{K}$ we have

$$U_\xi^* \operatorname{Ind}_\Lambda^\beta \pi(\gamma) U_\xi(\delta_x \otimes \eta) = U_\xi^* \operatorname{Ind}_\Lambda^\beta \pi(\gamma)(\delta_x \otimes \pi(\xi(x)^{-1})\eta)$$

$$= U_\xi^*(\delta_{\gamma x} \otimes \pi(\beta(\gamma, x))\pi(\xi(x)^{-1})\eta)$$

$$= \delta_{\gamma x} \otimes \pi(\xi(\gamma x))\pi(\beta(\gamma, x))\pi(\xi(x)^{-1})\eta$$

$$= \operatorname{Ind}_\Lambda^\alpha \pi(\gamma)(\delta_x \otimes \eta).$$

Hence, the representations $\operatorname{Ind}_\Lambda^\beta \pi$ and $\operatorname{Ind}_\Lambda^\alpha \pi$ are isomorphic.

Lemma 1.4.1. *Let $\Gamma \curvearrowright X$ be an action of a group Γ on a set X, let Λ be a group, and suppose $\alpha : \Gamma \times X \to \Lambda$ is a cocycle. If $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ and $\rho : \Lambda \to \mathcal{U}(\mathcal{K})$ are unitary representations such that $\pi \cong \rho$ then $\operatorname{Ind}_\Lambda^\alpha \pi \cong \operatorname{Ind}_\Lambda^\alpha \rho$.*

Proof. If $W : \mathcal{H} \to \mathcal{K}$ is a unitary such that $W\pi(\gamma) = \rho(\gamma)W$ for all $\gamma \in \Gamma$. Then $\operatorname{id} \otimes W : \ell^2 X \otimes \mathcal{H} \to \ell^2 X \otimes K$, and it follows easily that

$$(\operatorname{id} \otimes W) \operatorname{Ind}_\Lambda^\alpha \pi(\gamma) = \operatorname{Ind}_\Lambda^\alpha \rho(\gamma)(\operatorname{id} \otimes W).$$

□

Example 1.4.2. Consider a group Γ acting on itself by left multiplication, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then we obtain cocycles $\alpha, \beta : \Gamma \times \Gamma \to \Gamma$ given by $\alpha(\gamma, x) = \gamma$, and $\beta(\gamma, x) = e$.

By considering the function $\xi(x) = x$ we see that these cocycles are cohomologous, for every $\gamma, x \in \Gamma$ we have

$$\alpha(\gamma, x) = \gamma = (\gamma x)x^{-1} = \xi(\gamma x)\beta(\gamma, x)\xi(x)^{-1}.$$

We therefore recover Fell's absorption principle by obtaining an isomorphism between the representations $\lambda \otimes \pi = \mathrm{Ind}_\Gamma^\alpha \pi$ and $\lambda \otimes 1_{\mathcal{H}} = \mathrm{Ind}_\Gamma^\beta \pi$.

If Γ is a group, and $\Sigma < \Gamma$ is a subgroup, then we may consider the cocycle $\alpha : \Gamma \times (\Gamma/\Sigma) \to \Sigma$ associated to some fundamental domain for Σ as is described in Example 1.3.8. Hence if $\pi : \Sigma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Σ then we obtain the **induced representation** $\mathrm{Ind}_\Sigma^\Gamma \pi : \Gamma \to \mathcal{U}(\ell^2(\Gamma/\Sigma)\overline{\otimes}\mathcal{H})$ as the induced representation associated to the cocycle α. Since the cohomology class of α does not depend on the fundamental domain, we have that the induced representation is well defined up to unitary equivalence.

Observe that if $\xi_0 \in \mathcal{H}$, then we may consider the vector $\xi' = \delta_\Sigma \otimes \xi_0 \in \ell^2(\Gamma/\Sigma)\overline{\otimes}\mathcal{H}$. We then have that the positive definite function $\phi_{\xi'} : \Gamma \to \mathbb{C}$ is given by

$$\phi_{\xi'}(\gamma) = \begin{cases} \phi_\xi(\gamma) & \text{if } \gamma \in \Sigma; \\ 0 & \text{otherwise.} \end{cases}$$

Also observe that if $\pi = 1_{\mathcal{H}} : \Sigma \to \mathcal{U}(\mathcal{H})$ is the trivial representation then $\mathrm{Ind}_\Sigma^\Gamma \pi = \lambda_{\Gamma/\Sigma} \otimes 1_{\mathcal{H}}$ is a multiple of the quasi-regular representation corresponding to Σ.

Remark 1.4.3. Suppose Y is a set and we have commuting left and right actions $\Gamma \curvearrowright Y \curvearrowleft \Lambda$ of groups Γ and Λ such that the action of Λ is free. If $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ is a unitary representation, and $\alpha : \Gamma \times (Y/\Lambda) \to \Lambda$ is the cocycle coming from a fundamental domain D for $Y \curvearrowleft$ as explained in Example 1.3.7, then we obtain the induced representation $\mathrm{Ind}_\Lambda^\alpha \pi$. Here we will explain an alternate way to obtain this representation which makes no reference to a fundamental domain.

Consider a function $\xi : X \to \mathcal{H}$ which is Λ-equivariant, i.e., $\xi(x\lambda^{-1}) = \pi(\lambda)\xi(x)$, for all $x \in X$, and $\lambda \in \Lambda$. Because, ξ is Λ-equivariant we have that the function $x \mapsto \|\xi(x)\|$ is constant on the Λ-orbits. We may therefore consider the well defined space

$$L^2(Y; \mathcal{H})^\Lambda = \{\xi : Y \to \mathcal{H} \mid \xi \text{ is } \Lambda - \text{equivariant, and } \Sigma_{x\Lambda \in Y/\Lambda} \|\xi(x)\|^2 < \infty\}.$$

Since this space consists of Λ-equivariant functions, this becomes a Hilbert space when it is endowed with the well defined inner-product

$$\langle \xi, \eta \rangle = \Sigma_{x\Lambda \in Y/\Lambda} \langle \xi(x), \eta(x) \rangle.$$

We then obtain the induced representation of π on $L^2(Y; \mathcal{H})^\Lambda$ by requiring that for all $\gamma \in \Gamma$ and $x \in Y$ we have

$$(\mathrm{Ind}_\Lambda^\Gamma \pi(\gamma)\xi)(x) = \xi(\gamma^{-1}x).$$

If $D \subset Y$ is a fundamental domain for the action $Y \curvearrowleft \Lambda$, then any function $\xi \in \ell^2 D \overline{\otimes} \mathcal{H} \cong \ell^2(D; \mathcal{H})$ extends uniquely to a Λ-equivariant function $\tilde{\xi} \in \ell^2(Y; \mathcal{H})^\Lambda$ by setting $\tilde{\xi}(x\lambda^{-1}) = \pi(\lambda)\xi(x)$ for each $\lambda \in \Lambda$ and $x \in D$. By identifying Y/Λ with D by the map Φ we then obtain a unitary $W : \ell^2(Y/\Lambda)\overline{\otimes}\mathcal{H} \to \ell^2(Y; \mathcal{H})$. If $\alpha : \Gamma \times (Y/\Lambda) \to \Lambda$ is the cocycle from Example 1.3.7, then unwinding the definitions gives

$$W \operatorname{Ind}_\Lambda^\alpha(\gamma) = \operatorname{Ind}_\Lambda^\Gamma(\gamma) W,$$

for all $\gamma \in \Gamma$.

Lemma 1.4.4. *Let Γ be a group and $\Delta < \Sigma < \Gamma$ subgroups. If $\pi : \Delta \to \mathcal{U}(\mathcal{H})$ is a unitary representation then*

$$\operatorname{Ind}_\Delta^\Gamma \pi \cong \operatorname{Ind}_\Sigma^\Gamma \operatorname{Ind}_\Delta^\Sigma \pi.$$

Proof. Using the equivalence in the previous remark, we may consider the map $W : L^2(\Gamma; L^2(\Sigma; \mathcal{H})^\Delta)^\Sigma \to L^2(\Gamma; \mathcal{H})^\Delta$ given by

$$(Wf)(\gamma) = (f(\gamma))(e).$$

Note that if $\delta \in \Delta$ then we have

$$(Wf)(\gamma\delta^{-1}) = (f(\gamma\delta^{-1}))(e) = (\operatorname{Ind}_\Delta^\Sigma \pi(\delta) f(\gamma))(e)$$

$$= (f(\gamma))(\delta^{-1})$$

$$= \pi(\delta)((f(\gamma))(e)) = \pi(\delta)((Wf)(\gamma)).$$

Also, if $D \subset \Sigma$ is a set of coset representatives for Δ, and $E \subset \Gamma$ is a set of coset representatives for Σ then $ED \subset \Gamma$ is a set of coset representatives for Δ, hence just as above we see that for all $f \in L^2(\Gamma; L^2(\Sigma; \mathcal{H})^\Delta)^\Sigma$ we have

$$\|Wf\|^2 = \Sigma_{(\gamma,\lambda)\in E\times D} \|(Wf)(\gamma\lambda)\|^2$$

$$= \Sigma_{\gamma\in E}\Sigma_{\lambda\in D}\|((Wf)(\gamma))(\lambda)\|^2 = \|f\|^2.$$

Thus, W is a well defined isometry, which is easy to see is a unitary. An easy calculation then shows that for all $\gamma \in \Gamma$ we have

$$\operatorname{Ind}_\Delta^\Gamma \pi(\gamma) W = W \operatorname{Ind}_\Sigma^\Gamma \operatorname{Ind}_\Delta^\Sigma \pi(\gamma).$$

□

Proposition 1.4.5. *If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ and $\rho : \Sigma \to \mathcal{U}(\mathcal{K})$ is a unitary representation of Σ, then*

$$\pi \otimes \operatorname{Ind}_\Sigma^\Gamma \rho \cong \operatorname{Ind}_\Sigma^\Gamma(\pi_{|\Sigma} \otimes \rho).$$

Proof. Consider the map $W : \mathcal{H}\overline{\otimes}L^2(\Gamma;\mathcal{K})^\Sigma \to L^2(\Gamma;\mathcal{H}\overline{\otimes}\mathcal{K})^\Sigma$ such that for each $\gamma \in \Gamma$, $\xi \in \mathcal{H}$, and $f \in L^2(\Gamma;\mathcal{K})^\Sigma$ we have

$$(W(\xi \otimes f))(\gamma) = (\pi(\gamma^{-1})\xi) \otimes f(\gamma).$$

Then, if $\sigma \in \Sigma$ we have

$$\begin{aligned}(W(\xi \otimes f))(\gamma\sigma^{-1}) &= (\pi(\sigma\gamma)\xi) \otimes f(\gamma\sigma^{-1}) \\ &= (\pi \otimes \rho)(\sigma)(W(\xi \otimes f)).\end{aligned}$$

Hence, it follows easily that W is a well defined unitary operator. A routine check then shows that

$$\mathrm{Ind}_\Sigma^\Gamma(\pi_{|\Sigma} \otimes \rho)(\gamma)W = W(\pi \otimes \mathrm{Ind}_\Sigma^\Gamma \rho)(\gamma),$$

for all $\gamma \in \Gamma$. □

If Γ, Λ, and Υ are groups, $\Gamma \curvearrowright X$, and $\Lambda \curvearrowright Y$ are actions, and $\alpha : \Gamma \times X \to \Lambda$, and $\beta : \Lambda \times Y \to \Upsilon$ are cocycles then just as we induced representations above we may induce the action $\Gamma \curvearrowright X$ to an action $\Gamma \curvearrowright X \times Y$ by the formula

$$\gamma(x, y) = (\gamma x, \alpha(\gamma, x)y).$$

We then may define the **composition of cocycles** $\beta\alpha : \Gamma \times (X \times Y) \to \Upsilon$ by the formula

$$\beta\alpha(\gamma, (x, y)) = \beta(\alpha(\gamma, x), y).$$

We can verify that this is indeed a cocycle since for all $\gamma_1, \gamma_2 \in \Gamma$, and $(x, y) \in X \times Y$ we have

$$\begin{aligned}\beta\alpha(\gamma_1\gamma_2, (x, y)) &= \beta(\alpha(\gamma_1, \gamma_2 x)\alpha(\gamma_2, x), y) \\ &= \beta(\alpha(\gamma_1, \gamma_2 x), \alpha(\gamma_2, x)y)\beta(\alpha(\gamma_2, x), y) \\ &= \beta\alpha(\gamma_1, (\gamma_2 x, \alpha(\gamma_2, x)y))\beta\alpha(\gamma_2, (x, y)) \\ &= \beta\alpha(\gamma_1, \gamma_2(x, y))\beta\alpha(\gamma_2, (x, y)).\end{aligned}$$

Exercise 1.4.6. If we have an inclusion of groups $\Delta < \Sigma < \Gamma$ and we consider the cocycles $\alpha_{\Sigma<\Gamma}$, $\alpha_{\Delta<\Sigma}$, and $\alpha_{\Delta<\Gamma}$ as described in Example 1.3.8, then show that by identifying the sets $(\Gamma/\Sigma) \times (\Sigma/\Delta)$ and Γ/Δ we obtain an identification of $\alpha_{\Delta<\Sigma}\alpha_{\Sigma<\Gamma}$ and $\alpha_{\Delta<\Gamma}$.

In light of the previous exercise we then have that the following lemma is an extension of Lemma 1.4.4.

Lemma 1.4.7. *Let* Γ, Λ, *and* Υ *be groups,* $\Gamma \curvearrowright X$, *and* $\Lambda \curvearrowright Y$ *be actions, and* $\alpha : \Gamma \times X \to \Lambda$, *and* $\beta : \Lambda \times Y \to \Upsilon$ *be cocycles. If* $\pi : \Upsilon \to \mathcal{U}(\mathcal{H})$ *is a unitary representation, then*

$$\mathrm{Ind}_\Upsilon^{\beta\alpha} \pi \cong \mathrm{Ind}_\Upsilon^{\beta} \mathrm{Ind}_\Lambda^{\alpha} \pi.$$

Proof. If we consider the natural identification $V : \ell^2 X \otimes \ell^2 Y \to \ell^2(X \times Y)$, then it is easy to see that $V \otimes \mathrm{id}_{\mathcal{H}}$ implements an equivalence between the representations $\mathrm{Ind}_{\Upsilon}^{\beta} \mathrm{Ind}_{\Lambda}^{\alpha} \pi$ and $\mathrm{Ind}_{\Upsilon}^{\beta\alpha} \pi$. □

Lemma 1.4.8. *Let* $\Gamma \curvearrowright X$ *be a action of a group* Γ *on a set* X*, let* Λ *be a group, and let* $\alpha : \Gamma \times X \to \Lambda$ *be a cocycle, if* $\pi_i : \Lambda \to \mathcal{U}(\mathcal{H}_i)$*,* $i \in I$ *is a family of unitary representations, then*

$$\mathrm{Ind}_{\Lambda}^{\alpha}(\oplus_i \pi_i) \cong \oplus_{i \in I} \mathrm{Ind}_{\Lambda}^{\alpha}(\pi_i).$$

Proof. This follows easily by considering the natural unitary from $\oplus_{i \in I}(\ell^2 X \otimes \mathcal{H}_i)$ to $\ell^2 X \otimes (\oplus_{i \in I} \mathcal{H}_i)$. □

jesse.d.peterson@vanderbilt.edu