1.3 Cocycles

Definition 1.3.1. Suppose $\Gamma \curvearrowright X$ is an action of a group Γ on a set X, and Λ is a group. A **cocycle** for the action into Λ is a map $\alpha : \Gamma \times X \to \Lambda$ such that

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, \gamma_2 x)\alpha(\gamma_2, x),$$

for all $\gamma_1, \gamma_2 \in \Gamma$, and $x \in X$. Two cocycles $\alpha, \beta : \Gamma \times X \to \Lambda$ are **cohomologous** if there is a map $\xi : X \to \Lambda$ such that

$$\alpha(\gamma, x) = \xi(\gamma x)\beta(\gamma, x)\xi(x)^{-1},$$

for all $\gamma \in \Gamma$, $x \in X$. A cocycle α is **trivial** if it is cohomologous to the cocycle which takes constant value $e \in \Lambda$. A cocycle α **untwists** if there is a homomorphism $\delta : \Gamma \to \Lambda$ such that α is cohomologous to the cocycle given by $(\gamma, x) \mapsto \delta(\gamma)$, for all $\gamma \in \Gamma$, and $x \in X$.

The set of all cocycles for the action $\Gamma \curvearrowright X$ with values in Λ is denoted by $Z^1(\Gamma \curvearrowright X; \Lambda)$, the set of trivial cocycles is denoted by $B^1(\Gamma \curvearrowright X; \Lambda)$, and the set of equivalence classes of cohomologous cocycles is denoted by $H^1(\Gamma \curvearrowright X; \Lambda)$. Note that if Λ is abelian then $Z^1(\Gamma \curvearrowright X; \Lambda)$ is an abelian group under pointwise multiplication, $B^1(\Gamma \curvearrowright X; \Lambda)$ is a subgroup and $H^1(\Gamma \curvearrowright X; \Lambda)$ is the quotient group.

Example 1.3.2. If $\alpha : \Gamma \times X \to \Lambda$ is a cocycle for an action $\Gamma \curvearrowright X$ on a group Λ, and if $S \subset \Gamma$ is a generating set, then the cocycle relation implies that the cocycle is completely determined by its values on the set $S \times X \subset \Gamma \times X$. In particular, for the group $\mathbb{Z}$, a cocycle $\alpha : \mathbb{Z} \times X \to \Lambda$ for an action $\mathbb{Z} \curvearrowright X$ is completely determined by the function $x \mapsto \alpha(1, x)$. Moreover, any function $\xi : X \to \Lambda$ determines a cocycle α such that $\alpha(1, x) = \xi(x)$.

For instance if $T : X \to X$ is a bijection and $\mathbb{Z} \curvearrowright X$ by $n \cdot x = T^n(x)$, and if $\xi : X \to \mathbb{R}$ is a function, then the corresponding cocycle

$$\alpha(n, x) = \Sigma_{k=0}^{n-1} \xi \circ T^k(x),$$

for $n > 0$. Hence, if we denote by $S_n : X \to \mathcal{R}$ the function given by

$$S_n(x) = \alpha(n, T^{-n}(x)),$$

then $\frac{1}{n} S_n$ is the average of the functions $\xi \circ T^{-k}$ for $1 \leq k \leq n$.

Example 1.3.3. Suppose Γ and Λ are two groups and consider the space

$$X = \{f : \Gamma \to \Lambda \mid f(e) = e\} \subset \Lambda^\Gamma.$$

We have an action of Γ on this set by

$$(\gamma \cdot f)(x) = f(x\gamma)f(\gamma)^{-1}.$$

Note that the fixed points of this action are precisely the set of homomorphisms from Γ to Λ.

We then have a cocycle for this action $\alpha : \Gamma \times X \to \Lambda$ given by

$$\alpha(\gamma, f) = f(\gamma).$$

Note that this action preserves the subsets of injective, surjective, and bijective maps.

Exercise 1.3.4. Think about the previous example and verify all the claims.

Example 1.3.5. One natural way in which cocycles arise is if we have a set X together with a pair of actions $\Gamma \curvearrowright X$, and $\Lambda \curvearrowright X$ of groups Γ, and Λ.

If the action of Λ is free (if $\lambda \neq e$ then $\lambda x \neq x$ for all $x \in X$), and if the orbits of Γ are contained in the orbits of Λ ($\Gamma x \subset \Lambda x$ for each $x \in X$), then we can define a cocycle $\alpha : \Gamma \times X \to \Lambda$ by setting $\alpha(\gamma, x)$ to be the unique element in Λ such that

$$\gamma x = \alpha(\gamma, x)x.$$

We can verify that α is a cocycle since for each $\gamma_1, \gamma_2 \in \Gamma$ and $x \in X$ we have

$$\alpha(\gamma_1\gamma_2, x)x = \gamma_1\gamma_2 x$$

$$= \alpha(\gamma_1, \gamma_2 x)(\gamma_1 x) = \alpha(\gamma_1, \gamma_2 x)\alpha(\gamma_2, x)x.$$

Since the action $\Lambda \curvearrowright X$ is free this then implies that

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, \gamma_2 x)\alpha(\gamma_2, x).$$

Definition 1.3.6. If $\Gamma \curvearrowright Y$ is an action of a group Γ on a set Y then a **fundamental domain** for the action is a subset $D \subset Y$ such that each orbit of Γ contains exactly one element in D, i.e., for all $x \in Y$ we have $|\Gamma x \cap D| = 1$.

Note that if $D \subset Y$ is a fundamental domain for the action $\Gamma \curvearrowright Y$ then we have that the map from $\Gamma \times D$ to Y given by $(\gamma, x) \mapsto \gamma x$ is sujective. Moreover, if the action $\Gamma \curvearrowright Y$ is free then the map is also injective and hence the space Y decomposes as a disjoint union of orbits

$$Y = \sqcup_{x \in D} \Gamma x.$$

We thus obtain a bijection between the fundamental domain D and the orbit space $\Gamma \backslash X$ given by $x \mapsto \Gamma x$.

Example 1.3.7. Another way in which cocycles arise is if Y is a set and we have a left action of Γ on Y and a right action of Λ such that the actions commute, i.e., for each $\gamma \in \Gamma$, $\lambda \in \Lambda$, and $x \in Y$ we have

$$(\gamma x)\lambda = \gamma(x\lambda).$$

As in Example 1.3.5 we will require the action of Λ to be free.

Since the actions of Γ and Λ commute, the action of Γ on Y passes to the space of Λ orbits Y/Λ. If $D \subset Y$ is a fundamental domain for the action $Y \curvearrowleft \Lambda$

then as mentioned above the map from D to Y/Λ given by $x \mapsto x\Lambda$ is a bijection. We will denote by $\Phi : Y/\Lambda \to D$ the inverse map.

We then obtain a cocycle $\alpha : \Gamma \times (Y/\Lambda) \to \Lambda$ by assigning to each $\gamma \in \Gamma$ and $x\Lambda \in Y/\Lambda$ the value of $\alpha(\gamma, x\Lambda)$ to be the unique element in Λ such that

$$\gamma\Phi(x\Lambda) = \Phi(\gamma x\Lambda)\alpha(\gamma, x\Lambda).$$

If $\gamma_1, \gamma_2 \in \Gamma$ and $x\Lambda \in Y/\Lambda$ then we have

$$\Phi(\gamma_1\gamma_2 x\Lambda)\alpha(\gamma_1\gamma_2, x\Lambda) = \gamma_1\gamma_2\Phi(x\Lambda)$$

$$= \gamma_1\Phi(\gamma_2 x\Lambda)\alpha(\gamma_2, x\Lambda)$$

$$= \Phi(\gamma_1\gamma_2 x\Lambda)\alpha(\gamma_1, \gamma_2 x\Lambda)\alpha(\gamma_2, x\Lambda).$$

Since the action $Y \curvearrowleft \Lambda$ is free this shows that α is indeed a cocycle.

Suppose $D' \subset Y$ is also a fundamental domain for the action $Y \curvearrowleft \Lambda$ and $\beta : \Gamma \times (Y/\Lambda) \to \Lambda$ is the corresponding cocycle. If we let $\Phi' : Y/\Lambda \to D'$ be the corresponding selection map for D', and we define $\xi : Y/\Lambda \to \Lambda$ so that for all $x\Lambda \in Y/\Lambda$ we have

$$\xi(x\Lambda) = \Phi'(x\Lambda)\Phi(x\Lambda)^{-1},$$

then it's easy to see that for each $\gamma \in \Gamma$ and $x\Lambda \in Y/\Lambda$ we have

$$\xi(\gamma x\Lambda)\alpha(\gamma, x\Lambda)\xi(x\Lambda)^{-1} = \beta(\gamma, x\Lambda).$$

Hence, the cohomology class of α is independent of the fundamental domain D.

Example 1.3.8. A special case of the above example to keep in mind is when Γ is a group and $\Sigma < \Gamma$ is a subgroup. Then $\Gamma \curvearrowright \Gamma \curvearrowleft \Sigma$ is a pair of commuting actions given by group multiplication. Thus, we obtain a cocycle (unique up to cohomology) $\alpha : \Gamma \times (\Gamma/\Sigma) \to \Sigma$.

Exercise 1.3.9. Show that in Example 1.3.8 if the action $\Gamma \curvearrowleft \Sigma$ has a fundamental domain $\Gamma_0 \subset \Gamma$ which is a subgroup of Γ, then we have that $\Sigma \lhd \Gamma$ is a normal subgroup and Γ splits as a semidirect product $\Gamma = \Gamma_0 \ltimes \Sigma$. Also, compute the corresponding cocycle for this fundamental domain.

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